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# Novel Stability Criteria of T-S Fuzzy Hopfield Neural Networks with Time-varying Delays and Uncertainties

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***Abstract*—the problem of asymptotic stability for Takagi-Sugeno (T-S) fuzzy Hopfield neural networks with time-varying delays is studied in this paper. Based on the Lyapunov functional method, new delay-dependent stability criteria are derived in terms of Linear Matrix Inequalities (LMIs) that can be calculated easily by the LMI Toolbox in MATLAB. The proposed approach does not involve free weighting matrices and can provide less conservative results than some existing ones. Besides, numerical examples are given to show the effectiveness of the proposed approach.**

***Keywords—asymptotic stability; T-S fuzzy model; Hopfield neural networks; time-varying delay; uncertainty***

## I. INTRODUCTION

Hopfield neural networks (HNNs) were first introduced by Hopfield [1]. The dynamic behavior of HNNs has been widely studied due to their potential applications in signal processing, combinatorial optimization and pattern recognition [2]-[4]. These applications are mostly dependent on the stability of the equilibrium of neural networks. Thus, the stability analysis is a necessary step for the design and applications of neural networks. Sometimes, neural networks have to be designed such that there is only global stable equilibrium. For example, when a neural network is applied to solve the optimization problem, it must have unique equilibrium which is globally stable.

Both in biological and artificial neural networks, the interactions between neurons are generally asynchronous which inevitably result in time delays. Time-delay is often the main factor of instability and poor performance of neural network systems [5]. Therefore, lots of efforts have been made on stability analysis of neural networks with time-varying delays in recent years [6]-[9]. The free-weighting matrix method was proposed to investigate the delay-dependent stability [10], and some less conservative delay-dependent stability criteria for systems with time-varying delay were presented [11]-[16]. However, Researchers have realized that too many slack variables introduced will make the system synthesis complicated, lead to a significant increase in the computational burden, and cannot result in less conservative results indeed [17]-[19]. In practical systems, there always are some uncertain elements, and these uncertainties may come from unknown internal or external noise, environmental influence, and so on. Hence, robust stability analysis for uncertain time-delay systems has been the focus of intensive research in recent years [10], [12], [20].

It is well-known that the T-S fuzzy models have been very important in academic research and practical applications, and the fuzzy logic theory has shown to be an efficient method to dealing with the analysis and synthesis issues for complex nonlinear systems [21]-[24]. Very recently, some results have been produced in the study of stability analysis of T-S fuzzy Hopfield neural networks systems with time-varying delays [25]-[27], To the best of our knowledge, the robust stability problem for uncertain fuzzy HNNs with time-varying interval delays has not been fully investigated, which remains as an open and challenging issue.

In this paper, the problem of stability analysis for T-S fuzzy HNNs with time-varying delays is considered. Based on Jensen integral inequality and some important Lemma, new sufficient conditions are derived in terms of LMIs. By constructing a Lyapunov-Krasovskii function without free-weighting matrices approach, the proposed criteria in this paper are much less conservative than some existing results. Numerical examples are given to show the applicability of the obtained results. The rest of this paper is arranged as follows. Section Ⅱ gives problem statement and some preliminaries used in later sections. Section Ⅲ presents our main results. Section Ⅳ provides the numerical examples and Section Ⅴ concludes the paper.

## II. PROBLEM STATEMENT AND PRELIMINARIES

Throughout this paper, $\mathbf{R}^n$ and $\mathbf{R}^{n\times m}$ denote the $n$-dimensional Euclidean space and the set of all $n\times m$ real matrices, respectively. A real symmetric matrix $X>0(\geq 0)$ denotes $X$ being a positive definite (positive semi-definite) matrix, and $X>(\geq)Y$ means $X-Y\ >(\geq)0$ . The superscript $T$ denotes the transpose. The notation * represents the symmetric terms in a symmetric matrix. $I$ is used to denote an identity matrix with proper dimension.

The model of Hopfield neural networks can be expressed as follows:

$$\dot{y}(t)=-Ay(t)+Bg(y(t))+J \tag{1}$$

where $y(t)=(y_1(t),y_2(t),\cdots,y_n(t))^T\in\mathbf{R}^n$ is the

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state vector associated with the neurons, $A = diag(a_1, a_2, \cdots, a_n) > 0$ is a positive diagonal matrix, $B = (b_{ij})_{n\times n} \in \mathbf{R}^{n\times n}$ is the interconnection matrix, $g(y(t)) = (g_1(y_1(t)), g_2(y_2(t)), \cdots, g_n(y_n(t)))^T$ is the neuron activation function vector, $J = (J_1, J_2, \cdots, J_n)^T$ is a constant external input vector.

Hopfield neural networks with time-varying delays have been widely investigated in recent years, and several stability criteria have been obtained [13-15], for instances. Meanwhile, the stability of Hopfield neural networks with uncertainties has received much research attention [23], for example. The model of Hopfield neural networks with time-varying delays can be described as follows:

$$\begin{aligned}\dot{y}(t) = &-(A+\Delta A)y(t) + (B+\Delta B)g(y(t)) \\ &+(C+\Delta C)g(y(t-d(t))) + J \end{aligned} \tag{2}$$

It is reasonable to assume that the neural network (2) has only one equilibrium point, denoted by $y^* = (y_1^*, y_2^*, \cdots, y_n^*)^T$. By making a transformation $x(t) = y(t) - y^*$, system (2) can be rewritten as

$$\begin{aligned}\dot{x}(t) = &-(A+\Delta A)x(t) + (B+\Delta B)f(x(t)) \\ &+(C+\Delta C)f(x(t-d(t)))\end{aligned}$$

where $f(x(t)) = (f_1(x_1(t)), f_2(x_2(t)), \cdots, f_n(x_n(t)))^T$, and $f_i(x_i(t)) = g_i(x_i(t) + y_i^*) - g_i(y_i^*)$, $f(x(t-d(t))) = (f_1(x_1(t-d(t))), f_2(x_2(t-d(t))), \cdots, f_n(x_n(t-d(t))))^T$ and $f_i(x_i(t-d(t))) = g_i(x_i(t-d(t)) + y_i^*) - g_i(y_i^*)$.

In this brief, we will consider the following HNNs with uncertainties represented by a T-S fuzzy model, and the $i$th rule of the T-S fuzzy model is of the following form:

Plant rule $i$:

$$\begin{aligned}&IF\quad z_1(t)\ is\ M_1^i\ and\ z_2(t)\ is\ M_2^i, \cdots, and\ z_n(t)\ is\ M_n^i \\ &\textbf{THEN}\ \dot{x}(t) = -(A_i+\Delta A_i)x(t) + (B_i+\Delta B_i)f(x(t)) \\ &\qquad\qquad +(C_i+\Delta C_i)f(x(t-d(t))) \\ &x(t) = \varphi(t), t\in[-h_2, 0], i = 1,2,\cdots q,\end{aligned} \tag{3}$$

where $M_j^i (j = 1,2,\cdots,n)$ is the fuzzy set, $z(t) = [z_1(t), z_2(t), \cdots, z_n(t)]$ is the premise variable vector, $x(t) \in \mathbf{R}^n$ is the system state variable, the time delay $0 \le h_1 \le d(t) \le h_2$ is the time-varying delay with an upper bound of $h_2$, $\dot{d}(t) \le \mu$ and $q$ is the number of *IF-THEN* rules. $\Delta A_i$, $\Delta B_i$ and $\Delta C_i$ are unknown matrices that represent the time-varying parameter uncertainties and are assumed to be of the form

$$[\Delta A_i\ \Delta B_i\ \Delta C_i] = H_i F_i(t)[E_{1i}\ E_{2i}\ E_{3i}] \tag{4}$$

where $H_i$, $E_{1i}$, $E_{2i}$ and $E_{3i}$ are known real constant matrices, and $F_i(t)$ denotes unknown time-varying matrix functions. It is assumed that all elements $F_i(t)$ are Lebesgue measurable satisfying $F_i^T(t)F_i(t) \le I$.

Using a standard fuzzy inference method, the system (1) is inferred as follows:

$$\begin{aligned}\dot{x}(t) = \sum_{i=1}^{q}\mu_i(z(t))[&-(A_i+\Delta A_i)x(t) + (B_i+\Delta B_i)f(x(t)) \\ &+(C_i+\Delta C_i)f(x(t-d(t)))]\end{aligned} \tag{5}$$

where

$$\mu_i(z(t)) = \frac{w_i(z(t))}{\sum_{j=1}^{q} w_j(z(t))}, w_i(z(t)) = \prod_{j=1}^{n} M_j^i(z(t)), \tag{6}$$

from the fuzzy sets theory, we have $\mu_i(z(t)) \ge 0$, $\sum_{i=1}^{q}\mu_i(z(t)) = 1$ for all $t$.

To obtain our main results, we introduce the following lemmas.

**Lemma 1** [28] Let $M, P, Q$ be the given matrices such that $Q > 0$, then

$$\begin{bmatrix} P & M^T \\ M & -Q \end{bmatrix} < 0 \Leftrightarrow P + M^T Q^{-1} M < 0$$

**Lemma 2** [29] For any constant matrix $M \in \mathbf{R}^{m\times m}$, $M = M^T > 0, \gamma > 0$ is a scalar, $\omega : \mathbf{R} \to \mathbf{R}^m$ is a vector function, then the following inequality holds:

$$\left(\int_0^\gamma \omega(s)ds\right)^T M \left(\int_0^\gamma \omega(s)ds\right) \le \gamma \int_0^\gamma \omega^T(s) M \omega(s) ds$$

**Lemma 3** [18] For any scalars $W_1 \ge 0, W_2 \ge 0, d(t)$ is a continuous function and satisfies $h_1 < d(t) < h_2$, then

$$\frac{W_1}{d(t)-h_1} + \frac{W_2}{h_2 - d(t)} \ge \min\left\{\frac{3W_1 + W_2}{h_2 - h_1}, \frac{W_1 + 3W_2}{h_2 - h_1}\right\}$$

**Lemma 4** [30] Assume that $\Psi$, $M$ and $E$ are real matrices with appropriate dimensions and $F(t)$ is a matrix function satisfying $F^T(t)F(t) \le I$. Then, $\Psi + MF(t)N + [MF(t)N]^T < 0$ holds if and only if there exists a scalar $\varepsilon > 0$ satisfying $\Psi + \varepsilon^{-1}MM^T + \varepsilon N^T N < 0$.

## III. Main Results

### A. Time-varying Delay Systems without Uncertainties

To discuss asymptotically stable for the system (5), we first consider the case without uncertainties, that is $\Delta A_i$ =0, $\Delta B_i$ =0 and $\Delta C_i$ =0. For this case, the asymptotically stable condition is obtained in the following theorem.

**Theorem 1.** For given scalars $0 \le h_1 < h_2$ and $h_{12} = h_2 - h_1$, system (5) is asymptotically stable if there exist matrices $P > 0$ , $Q_i > 0 (i = 1,2,3)$ , $R_1 > 0$ , $R_2 > 0$ with appropriate dimensions such that the following LMIs hold:

$$\Phi_{i,j} = \begin{bmatrix} \Phi_{00i} + \Phi_{0j} & \Phi_{12i} & \Phi_{13i} \\ * & -R_1 & 0 \\ * & * & -R_2 \end{bmatrix}, i = 1,2,\cdots q, j = 1,2,3,4. \tag{7}$$

where

$$\Phi_{00i} = \begin{bmatrix} \Psi_{00i} & 0 & 0 & 0 & PB_i & PC_i \\ * & -Q_1 & 0 & 0 & 0 & 0 \\ * & * & -Q_2 & 0 & 0 & 0 \\ * & * & * & -(1-\mu)Q_3 & 0 & 0 \\ * & * & * & * & 0 & 0 \\ * & * & * & * & * & 0 \end{bmatrix} \tag{8}$$

$$\Psi_{00i} = -A_i^T P - PA_i + Q_1 + Q_2 + Q_3 \tag{9}$$

$$\Phi_{12i} = \begin{bmatrix} -\sqrt{h_1} A_i^T R_1 \\ 0 \\ 0 \\ 0 \\ \sqrt{h_1} B_i^T R_1 \\ \sqrt{h_1} C_i^T R_1 \end{bmatrix} \tag{10}$$

$$\Phi_{13i} = \begin{bmatrix} -\sqrt{h_{12}} A_i^T R_2 \\ 0 \\ 0 \\ 0 \\ \sqrt{h_{12}} B_i^T R_{12} \\ \sqrt{h_{12}} C_i^T R_{12} \end{bmatrix} \tag{11}$$

$$\Phi_{01} = \begin{bmatrix} -\frac{R_1}{h_1} & 0 & 0 & \frac{R_1}{h_1} & 0 & 0 \\ * & -\frac{3R_1}{h_1} - \frac{3R_2}{h_{12}} & 0 & \frac{3R_1}{h_1} + \frac{3R_2}{h_{12}} & 0 & 0 \\ * & * & -\frac{R_2}{h_{12}} & \frac{R_2}{h_{12}} & 0 & 0 \\ * & * & * & -\frac{4R_1}{h_1} - \frac{4R_2}{h_{12}} & 0 & 0 \\ * & * & * & * & 0 & 0 \\ * & * & * & * & * & 0 \end{bmatrix} \tag{12}$$

$$\Phi_{02} = \begin{bmatrix} -\frac{R_1}{h_1} & 0 & 0 & \frac{R_1}{h_1} & 0 & 0 \\ * & -\frac{3R_1}{h_1} - \frac{R_2}{h_{12}} & 0 & \frac{3R_1}{h_1} + \frac{R_2}{h_{12}} & 0 & 0 \\ * & * & -\frac{3R_2}{h_{12}} & \frac{3R_2}{h_{12}} & 0 & 0 \\ * & * & * & -\frac{4R_1}{h_1} - \frac{4R_2}{h_{12}} & 0 & 0 \\ * & * & * & * & 0 & 0 \\ * & * & * & * & * & 0 \end{bmatrix} \tag{13}$$

$$\Phi_{03} = \begin{bmatrix} -\frac{3R_1}{h_1} & 0 & 0 & \frac{3R_1}{h_1} & 0 & 0 \\ * & -\frac{R_1}{h_1} - \frac{3R_2}{h_{12}} & 0 & \frac{R_1}{h_1} + \frac{3R_2}{h_{12}} & 0 & 0 \\ * & * & -\frac{R_2}{h_{12}} & \frac{R_2}{h_{12}} & 0 & 0 \\ * & * & * & -\frac{4R_1}{h_1} - \frac{4R_2}{h_{12}} & 0 & 0 \\ * & * & * & * & 0 & 0 \\ * & * & * & * & * & 0 \end{bmatrix} \tag{14}$$

$$\Phi_{04} = \begin{bmatrix} -\frac{3R_1}{h_1} & 0 & 0 & \frac{3R_1}{h_1} & 0 & 0 \\ * & -\frac{R_1}{h_1} - \frac{R_2}{h_{12}} & 0 & \frac{R_1}{h_1} + \frac{R_2}{h_{12}} & 0 & 0 \\ * & * & -\frac{3R_2}{h_{12}} & \frac{3R_2}{h_{12}} & 0 & 0 \\ * & * & * & -\frac{4R_1}{h_1} - \frac{4R_2}{h_{12}} & 0 & 0 \\ * & * & * & * & 0 & 0 \\ * & * & * & * & * & 0 \end{bmatrix} \tag{15}$$

Proof. Choose a Lyapunov-Krasovskii functional candidate as follows:

$$V(x_t)=V_1(x_t)+V_2(x_t)+V_3(x_t)+V_4(x_t)$$

where

$$V_1(x_t)=x^T(t)Px(t)$$

$$V_2(x_t)=\int_{t-h_1}^{t}x^T(s)Q_1x(s)ds+\int_{t-h_2}^{t}x^T(s)Q_2x(s)ds$$

$$V_3(x_t)=\int_{t-d(t)}^{t}x^T(s)Q_3x(s)ds$$

$$V_4(x_t)=\int_{h_1}^{0}\int_{t+\theta}^{t}\dot{x}^T(s)R_1\dot{x}(s)dsd\theta+\int_{-h_2}^{-h_1}\int_{t+\theta}^{t}\dot{x}^T(s)R_2\dot{x}(s)dsd\theta$$

Then, the time derivative of $V(x_t)$ along the trajectory of system (5) yields

$$\dot{V}_1(x_t)=2x^T(t)P\dot{x}(t) \tag{16}$$

$$\dot{V}_2(x_t)=x^T(t)(Q_1+Q_2)x(t)-x^T(t-h_1)Q_1x(t-h_1)-x^T(t-h_2)Q_2x(t-h_2) \tag{17}$$

$$\dot{V}_3(x_t)=x^T(t)Q_3x(t)-(1-\mu)x^T(t-d(t))Q_3x(t-d(t)) \tag{18}$$

$$\dot{V}_4(x_t)=\dot{x}^T(t)(h_1R_1+h_{12}R_2)\dot{x}(t)-\int_{t-h_1}^{t}\dot{x}^T(s)R_1\dot{x}(s)ds-\int_{t-h_2}^{t-h_1}\dot{x}^T(s)R_2\dot{x}(s)ds \tag{19}$$

By using Lemma 2 and Lemma 3, we have

$$\begin{aligned}&-\int_{t-h_1}^{t}\dot{x}^T(s)R_1\dot{x}(s)ds\\&=-\int_{t-d(t)}^{t}\dot{x}^T(s)R_1\dot{x}(s)ds-\int_{t-h_1}^{t-d(t)}\dot{x}^T(s)R_1\dot{x}(s)ds\\&\le-\{[(\int_{t-d(t)}^{t}\dot{x}(s)ds)^TR_1\int_{t-d(t)}^{t}\dot{x}(s)ds]/d(t)\\&+[(\int_{t-h_1}^{t-d(t)}\dot{x}(s)ds)^TR_1\int_{t-h_1}^{t-d(t)}\dot{x}(s)ds]/(h_1-d(t))\}\\&\le-\max\{\frac{W_1+3W_2}{h_1},\frac{3W_1+W_2}{h_1}\}\end{aligned} \tag{20}$$

where

$$W_1=\begin{bmatrix}x(t)\\x(t-d(t))\end{bmatrix}^T\begin{bmatrix}R_1&-R_1\\ *&R_1\end{bmatrix}\begin{bmatrix}x(t)\\x(t-d(t))\end{bmatrix}$$

$$W_2=\begin{bmatrix}x(t-d(t))\\x(t-h_1)\end{bmatrix}^T\begin{bmatrix}R_1&-R_1\\ *&R_1\end{bmatrix}\begin{bmatrix}x(t-d(t))\\x(t-h_1)\end{bmatrix}$$

and

$$\begin{aligned}&-\int_{t-h_2}^{t-h_1}\dot{x}^T(s)R_2\dot{x}(s)ds\\&=-\int_{t-d(t)}^{t-h_1}\dot{x}^T(s)R_2\dot{x}(s)ds-\int_{t-h_2}^{t-d(t)}\dot{x}^T(s)R_2\dot{x}(s)ds\\&\le-\{[\int_{t-d(t)}^{t-h_1}\dot{x}(s)ds)^TR_2\int_{t-d(t)}^{t-h_1}\dot{x}(s)ds]/(d(t)-h_1)\\&+[\int_{t-h_2}^{t-d(t)}\dot{x}(s)ds)^TR_2\int_{t-h_2}^{t-d(t)}\dot{x}(s)ds]/(h_2-d(t))\}\\&\le-\max\{\frac{W_3+3W_4}{h_{12}},\frac{3W_3+W_4}{h_{12}}\}\end{aligned} \tag{21}$$

where

$$W_3=\begin{bmatrix}x(t-h_1)\\x(t-d(t))\end{bmatrix}^T\begin{bmatrix}R_2&-R_2\\ *&R_2\end{bmatrix}\begin{bmatrix}x(t-h_1)\\x(t-d(t))\end{bmatrix}$$

$$W_4=\begin{bmatrix}x(t-d(t))\\x(t-h_2)\end{bmatrix}^T\begin{bmatrix}R_2&-R_2\\ *&R_2\end{bmatrix}\begin{bmatrix}x(t-d(t))\\x(t-h_2)\end{bmatrix}$$

It can be shown from (7),(15)-(20) and Lemma 1 that

$$\dot{V}(x_t)=\dot{V}_1(x_t)+\dot{V}_2(x_t)+\dot{V}_3(x_t)+\dot{V}_4(x_t)$$

$$\le\sum_{i=1}^{q}\mu_i(z(t))\xi^T(t)\Phi_{i,j}\xi^T(t)\le 0,\quad j=1,2,3,4.$$

where $\xi^T(t)=$
$[x(t)\ x(t-h_1)\ x(t-h_2)\ x(t-d(t)\ f(x(t))\ f(x(t-d(t)))]$

Hence, system (5) is asymptotically stable. This completes the proof.

When there is no fuzzy and no uncertainties in (5), the system is reduced to

$$\dot{x}(t)=-Ax(t)+Bf(x(t))+Cf(x(t-d(t))) \tag{22}$$

**Corollary 1** For given scalars $0\le h_1<h_2$ and $h_{12}=h_2-h_1$, system (22) is asymptotically stable if there exist matrices $P>0$, $Q_i>0(i=1,2,3)$, $R_1>0$, $R_2>0$ with appropriate dimensions such that the following LMIs hold:

$$\Phi_j=\begin{bmatrix}\Phi_{00}+\Phi_{0j}&\Phi_{12}&\Phi_{13}\\ *&-R_1&0\\ *&*&-R_2\end{bmatrix},\ j=1,2,3,4. \tag{23}$$

where

$$\Phi_{00}=\begin{bmatrix}\Psi_{00}&0&0&0&PB&PC\\ *&-Q_1&0&0&0&0\\ *&*&-Q_2&0&0&0\\ *&*&*&-(1-\mu)Q_3&0&0\\ *&*&*&*&0&0\\ *&*&*&*&*&0\end{bmatrix}$$

$$\Psi_{00} = -A^T P - PA + Q_1 + Q_2 + Q_3$$

$\Phi_{0j}$ are defined as in theorem **1**.

*B. Time-varying Delay Systems with Uncertainty*

Now, we shall discuss the feasible robust stability criteria for time-varying delay systems with uncertainty.

**Theorem 2.** For given scalars $h_2 > h_1 \geq 0$ ( $h_{12} = h_2 - h_1$ ) and $\varepsilon > 0$ , the system(5) is robust stability if there exist matrices $P > 0$, $Q_i > 0 (i = 1,2,3)$, $R_1 > 0$, $R_2 > 0$ of appropriate dimensions and scalar $\varepsilon > 0$ such that the following LMIs hold:

$$\begin{bmatrix} \Phi_{i,j} & M_i & N_i^T \\ * & -\varepsilon I & 0 \\ * & * & -\varepsilon I \end{bmatrix} < 0, i = 1,2,\cdots,q, \ j = 1,2,3,4 \tag{24}$$

where $\Phi_{i,j}$ is defined in (7), and

$$M_i = \begin{bmatrix} PH_i \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \sqrt{h_1} H_i^T R_1^T \\ \sqrt{h_{12}} H_i^T R_2^T \end{bmatrix}, \ N_i^T = \begin{bmatrix} -E_{A_i}^T \\ 0 \\ 0 \\ 0 \\ E_{B_i}^T \\ E_{C_i}^T \\ 0 \\ 0 \end{bmatrix}$$

**Proof.** Assume that inequalities (24) hold, from Lemma 1 and Lemma 4,

$$\Phi_{i,j} + M_i F(t) N_i + [M_i F(t) N_i]^T < 0$$
$$(i = 1,2,\cdots,q; \ j = 1,2,3,4)$$

hold.

When $A_i = A_i + H_i F(t) E_{A_i}$, $B_i = B_i + H_i F(t) E_{B_i}$ and $C_i = C_i + H_i F(t) E_{C_i}$ in (7), it can be verified that $\Phi_{i,j} + M_i F(t) N_i + [M_i F(t) N_i]^T$ are exactly the left-hand side of (7).Hence, system (5) is robust stability from theorem 1.

When there is no fuzzy in (5), the system is reduced to

$$\begin{aligned} \dot{x}(t) = &-(A + \Delta A)x(t) + (B + \Delta B) f(x(t)) \\ &+ (C + \Delta C) f(x(t - d(t))) \end{aligned} \tag{25}$$

**Corollary 2** For given scalars $0 \leq h_1 < h_2$ and $h_{12} = h_2 - h_1$ , system (25) is asymptotically stable if there exist matrices $P > 0$ , $Q_i > 0 (i = 1,2,3)$ , $R_1 > 0$ , $R_2 > 0$ with appropriate dimensions and scalar $\varepsilon > 0$ such that the following LMIs hold:

$$\begin{bmatrix} \Phi_j & M & N^T \\ * & -\varepsilon I & 0 \\ * & * & -\varepsilon I \end{bmatrix} < 0, \ j = 1,2,3,4 , \tag{26}$$

where $\Phi_j$ is defined in (23), and

$$M = \begin{bmatrix} PH \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \sqrt{h_1} H^T R_1^T \\ \sqrt{h_{12}} H^T R_2^T \end{bmatrix}, \ N^T = \begin{bmatrix} -E_A^T \\ 0 \\ 0 \\ 0 \\ E_B^T \\ E_C^T \\ 0 \\ 0 \end{bmatrix}$$

## IV. NUMERICAL EXAMPLES

In this section, three numerical examples are given to illustrate the effectiveness of the proposed methods.

***Example 1*** In this example, we consider the DNNs (5) with

$$A_1 = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}, B_1 = \begin{bmatrix} -0.4 & 0.3 \\ 0.4 & -0.6 \end{bmatrix}, C_1 = \begin{bmatrix} -0.5 & 0.1 \\ -0.2 & -0.5 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}, B_2 = \begin{bmatrix} -0.9 & 0.4 \\ 0.5 & -0.7 \end{bmatrix}, C_2 = \begin{bmatrix} -0.7 & 0.6 \\ -0.3 & -0.1 \end{bmatrix}$$

$\mu = 2$ , $\Delta A_i = \Delta B_i = \Delta C_i = 0$ , $i = 1,2,$

The time-varying delays are taken as $d(t) = 0.1 + \sin^2 t$ and the activation function is described by $f(x) = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$ , the membership function is $\mu_1(z(t)) = \sin^2 x_1$ , $\mu_2(z(t)) = \cos^2 x_1$ , using MATLAB LMI Toolbox to solve the LMIs in theorem 1, some positive definite feasible matrices are given as follows

$$P = \begin{bmatrix} 0.3777 & -0.1747 \\ -0.1747 & 1.2997 \end{bmatrix}, Q_1 = \begin{bmatrix} 0.3915 & -0.2512 \\ -0.2512 & 0.9871 \end{bmatrix}$$

$$Q_2 = \begin{bmatrix} 0.3077 & -0.2371 \\ -0.2371 & 1.0812 \end{bmatrix}, Q_3 = \begin{bmatrix} 0.0263 & -0.0238 \\ -0.0238 & 0.1124 \end{bmatrix}$$

$$R_1 = \begin{bmatrix} 0.5338 & -0.2606 \\ -0.2606 & 0.5518 \end{bmatrix}, R_2 = \begin{bmatrix} 0.1196 & -0.0438 \\ -0.0438 & 0.4276 \end{bmatrix}$$

and the state trajectories of the systems with different initial conditions are showed as follows (Figs. 1-3)

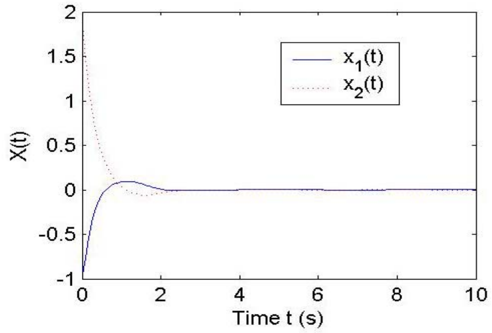


Fig. 1. The state trajectories with initial state $x(0)=\begin{bmatrix}-1 & 2\end{bmatrix}^T$

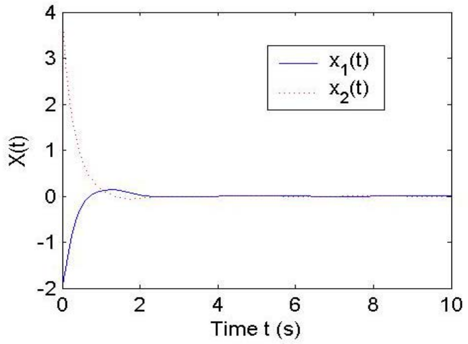


Fig. 2. The state trajectories with initial state $x(0)=\begin{bmatrix}-2 & 4\end{bmatrix}^T$

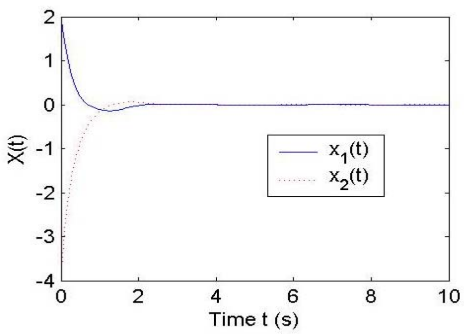


Fig. 3. The state trajectories with initial state $x(0)=\begin{bmatrix}2 & -4\end{bmatrix}^T$

Figs.1-3 show that the state trajectories of the systems are converging to zero with different initial state, that is to say, system (5) is asymptotically stable when theorem 1 holds.

***Example 2*** In this example, we consider the DNNs (22) and corollary 1 with

$$A=\begin{bmatrix}2 & 0\\ 0 & 2\end{bmatrix},\ B=\begin{bmatrix}0.7 & 0.8\\ -0.5 & 0.3\end{bmatrix},$$

$$C=\begin{bmatrix}0.2 & 0.2\\ 0.1 & -0.2\end{bmatrix},$$

$$\mu=2.$$

The activation function is described by $f(x)=\dfrac{e^x-e^{-x}}{e^x+e^{-x}}$, the maximum allowable upper bound of $h_2$ with given $h_1$ is showed in table Ⅰ.

TABLE I. MAXIMUM ALLOWABLE UPPER BOUND OF $h_2$ WITH GIVEN $h_1$

| $h_1$ | **0.0001** | **0.001** | **0.01** | **0.05** | **0.1** |
|---|---|---|---|---|---|
| Muralisankar et al. [15] | <16.17 | <16.17 | <16.17 | <16.17 | 16.1614 |
| Wu et al.[20] | <11.08 | <11.08 | <11.08 | <11.08 | 11.0727 |
| Corollary 1 | 16.4021 | 15.8384 | 14.1293 | 11.3221 | 9.4548 |

According to the table Ⅰ, this example shows that our results are better than those results discussed in[15,20] when $h_1$ is small enough, although free-weighting matrix approach is adopted in [15,20].

***Example 3*** In this example, we consider the DNNs (25) and corollary 2 with

$$A=\begin{bmatrix}2 & 0\\ 0 & 2\end{bmatrix},\ B=\begin{bmatrix}0.7 & 0.8\\ -0.5 & 0.3\end{bmatrix},$$

$$C=\begin{bmatrix}0.2 & 0.2\\ 0.1 & -0.2\end{bmatrix},\quad E_A=E_B=E_c=\begin{bmatrix}0.02 & 0\\ 0 & 0.02\end{bmatrix},$$

$$F(t)=\begin{bmatrix}\sin t & 0\\ 0 & \cos t\end{bmatrix}, H=I,\ \mu=2,$$

the activation function is described by $f(x)=\dfrac{e^x-e^{-x}}{e^x+e^{-x}}$, the maximum allowable upper bound of $h_2$ with given $h_1$ is showed in table Ⅱ.

TABLE II. MAXIMUM ALLOWABLE UPPER BOUND OF $h_2$ WITH GIVEN $h_1$

| $h_1$ | **0.01** | **0.05** | **0.1** | **0.15** | **0.2** |
|---|---|---|---|---|---|
| Corollary 2 | 0.3224 | 0.2833 | 0.2177 | 0.1998 | 0.2194 |

The time-varying delays are taken as $d(t)=0.05+0.2\sin^2 t$, using MATLAB LMI Toolbox to solve the LMIs in corollary 2, some positive definite feasible matrices are given as follows:

$$P=\begin{bmatrix}0.0364 & 0.0039\\ 0.0039 & 0.0371\end{bmatrix}, Q_1=\begin{bmatrix}0.0621 & 0.0213\\ 0.0213 & 0.0600\end{bmatrix},$$

$$Q_2=\begin{bmatrix}0.0272 & 0.0090\\ 0.0090 & 0.0263\end{bmatrix}, Q_3=\begin{bmatrix}0.0025 & 0.0009\\ 0.0009 & 0.0021\end{bmatrix},$$

$$R_1=\begin{bmatrix}0.1196 & 0.0157\\ 0.0157 & 0.1450\end{bmatrix},$$

$$R_2=\begin{bmatrix}0.0372 & 0.0012\\ 0.0012 & 0.0493\end{bmatrix},$$

and the state trajectories of the systems with different initial conditions are showed as follows(Figs. 4-6)

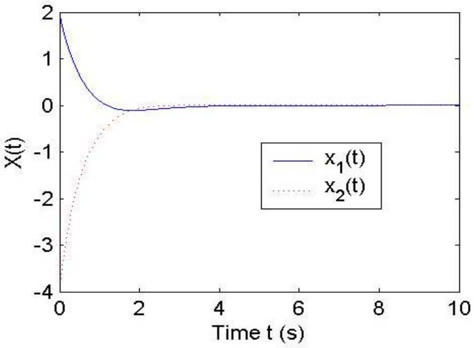


Fig. 4. The state trajectories with initial state $x(0) = \begin{bmatrix} 2 & -4 \end{bmatrix}^T$

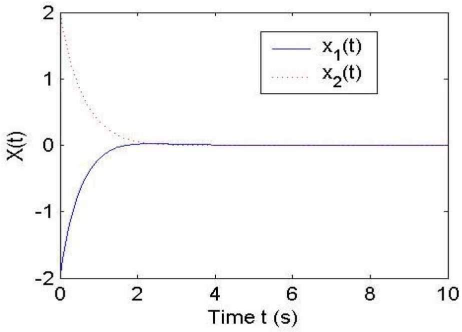


Fig. 5. The state trajectories with initial state $x(0) = \begin{bmatrix} -2 & 2 \end{bmatrix}^T$

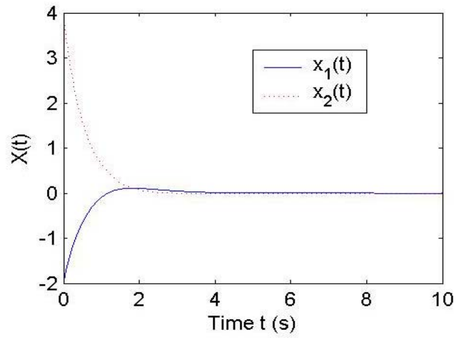


Fig. 6. The state trajectories with initial state $x(0) = \begin{bmatrix} -2 & 4 \end{bmatrix}^T$

Table Ⅱ shows the maximum allowable upper bound of time-delay with given the allowable lower bound. From figs.4-6, it can be seen that the state trajectories of the systems are converging to zero with different initial state, that is to say, system (5) is robust stable when corollary 2 holds.

## V. CONCLUSION

In this paper, we present improved criteria to test robust stability of delay HNNs with time-varying delay and uncertainties. The obtained stability conditions are expressed with LMIs. By comparing the experimental results from numerical examples, it is demonstrated the improvement of our proposed criteria over some existing ones.

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