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Synchronization Control of Hybrid-Coupled Heterogeneous Complex Networks

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***Abstract*— This paper is concerned with the problem of synchronization control for the delayed hybrid-coupled heterogeneous network with stochastic disturbances. To begin with, the open-loop control is imposed on the whole network, based on which the pinning adaptive control and the impulsive control are introduced to synchronize the whole network to an arbitrary objective trajectory. Furthermore, by employing stochastic analysis techniques and the improved Halanay inequality, some easy-to-verify sufficient conditions are derived to guarantee the asymptotic/exponential synchronization in the mean square of the complex network under study. Numerical example of a directed network is illustrated to demonstrate the applicability and efficiency of the proposed theoretical results.**

I. INTRODUCTION

SYNCHRONIZATION of complex networks, known as a special kind of collective behaviors, has received notable attentions in the past few decades [1], [2], [3], [4]. Generally, the dynamical networks under synchronization studies are coupled linearly and instantaneously, and all the nodes in the network are governed by the same dynamical model when decoupled. These restrictions obviously does not match the practical cases in the real world, where the network nodes may evolve in different dynamical equations, and the delays do exist either in the individual nodes dynamics (called decoupling delays) or during the signal transmission processes from nodes to nodes (called coupling delays) in the form of constants, time-varying or distributed ones [5], [6]. Meanwhile, stochastic disturbances are inevitable occurring in the accurate modeling of the real systems [7], [8], [9]. Therefore, synchronization of delayed hybrid-coupled heterogeneous networks with stochastic disturbances is of great interesting to be taken into account for modeling the real complex dynamical networks [10], [11].

For the complex dynamical network, under proper coupling strengths, the synchronization phenomenon occurs spontaneously if the nodes of the complex dynamical network have a common synchronization manifold. However, in some networks such as heterogeneous network (network with nonidentical nodes), such kind of synchronization manifold does not exist. In order to achieve the expected synchronization characteristic, extra controllers have to be added on the network nodes. Recently, great efforts have been devoted to the investigation of synchronization control problem for the heterogeneous networks. For example, by simple pinning control technique, cluster synchronization has been considered in [12], [13] for the community networks with nonidentical nodes. Via combining the open-loop control and the adaptive strategy as well as the impulsive effects, a dynamical network with nonidentical nodes has been synchronized to any given smooth goal dynamics [14]. By using only one impulsive controller [15], the authors proved that the network can be pinned to any prescribed state if the underlying graph of the network has spanning trees. In [16], the authors studied the synchronization control problem of impulsive dynamical networks under a single impulsive controller or a single negative state-feedback controller. For more related works, see [17], [18] and the references cited therein. It should be noticed that in all the above mentioned literatures, dynamics of the isolated nodes are governed by the systems independent of time delays.

The synchronization problem of the complex network with nonidentical delayed nodes has been considered by pinning control in [19] and in [20] under adaptive coupling strengths. The authors in [21] investigated the synchronization control problem for the hybrid-coupled heterogeneous network by pinning control, pinning adaptive control, and impulsive control. In [22], the synchronization for time-delayed complex networks with adaptive coupling weights and feedback gains was studied under pinning strategy. Based on the Lyapunov stability theory and stochastic analysis techniques, the exponential synchronization problem of coupled neural networks with stochastic noise perturbations was investigated by intermittent control [23]. The authors in [24] considered the synchronization control problem for the stochastic dynamical networks with nonlinear coupling by pinning impulsive control. For more related works, see [25], [26], [27].

In this contribution, we make further investigations for the synchronization control problem of the hybrid-coupled heterogeneous network by considering the stochastic disturbances. Each decoupled node is governed by a different delayed dynamical system, and coupling delays in the discrete-time/distributed forms are also considered. The main contributions of this paper can be summarized from the following aspects. Firstly, by designing the pinning adaptive controller and the impulsive controller, the hybrid-coupled complex networks are synchronized asymptotically/exponentially in the mean square to the objective system. To the best of our knowledge, the pinning synchronization control of such

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hybrid-coupled heterogeneous network has not been explored up to date. Secondly, reduced-order matrix conditions are derived for hybrid-coupled network which different from the previous works with LMIs or full-order matrix verified conditions when dealing with time delays and pinning control problems. Thirdly, based on the improved Halanay inequality, the synchronized impulsive control become more efficiency and less conservative.

Notations. Throughout this paper, the Kronecker product of matrices A and B is denoted as $A \otimes B$ and $\|\cdot\|$ is the Euclidean norm of a vector or its induced norm of a matrix. The abbreviation A_s of matrix A represents the matrix $\frac{1}{2}(A+A^T)$. For any matrix $M \in \mathbb{R}^{N\times N}$, M_l denotes the minor matrix of M by removing its first l $(1 \le l \le N)$ row-column pairs from M [28]. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\ge 0}, \mathcal{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\ge 0}$ satisfying the usual conditions, and $\mathbb{E}\{\cdot\}$ stands for the mathematical expectation operator with respect to the given probability measure $\mathcal{P}$. Let $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ be the minimal and maximal eigenvalues of matrix A, respectively.

II. Preliminaries and model description

Consider a hybrid coupled dynamical network consisting of N nonidentical nodes with stochastic disturbances, which is described as follows

$$\begin{aligned} \mathrm{d}x_i(t) =& \Big[f_i(t, x_i(t), x_i(t-\sigma_i(t))) + c_0 \sum_{j=1}^{N} G_{ij}^{(0)} \Gamma_0 x_j(t) \\ &+ c_1 \sum_{j=1}^{N} G_{ij}^{(1)} \Gamma_1 x_j(t-\tau_1(t)) \\ &+ c_2 \sum_{j=1}^{N} G_{ij}^{(2)} \Gamma_2 \int_{t-\tau_2(t)}^{t} x_j(s)\mathrm{d}s + u_i(t)\Big]\mathrm{d}t \\ &+ h_i(t, x_1(t), \ldots, x_N(t))\mathrm{d}\omega_i(t), \quad t \ge 0 \qquad (1) \end{aligned}$$

where $i = 1, 2, \ldots, N$ and $x_i(t) = (x_{i1}, x_{i2}, \ldots, x_{in})^T \in \mathbb{R}^n$ is the state variable of the ith node at time t. The function $f_i(\cdot) : \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, denoting the local dynamics of the ith node, is continuous and capable of performing abundant dynamical behaviors such as chaos, periodic orbits, equilibrium. $\sigma_i(t)$ is the uncoupled time-varying delay of node i, $\tau_1(t)$ and $\tau_2(t)$ are the discrete delay and the distributed delay of the coupled terms. Γ_0, Γ_1 and $\Gamma_2 \in \mathbb{R}^{n\times n}$ are the inner coupling matrices and it is assumed that $\Gamma_0 = \Gamma_0^T > 0$. The positive constants c_0, c_1 and c_2 are the corresponding coupling strengths. The coupling matrices $G^{(k)} = (G_{ij}^{(k)})_{N\times N}$ $(k = 0, 1, 2)$ are defined to satisfy: $G_{ij}^{(k)} \ge 0$ $(i \ne j)$ and $G_{ii}^{(k)} = -\sum_{j=1, j\ne i}^{N} G_{ij}^{(k)}$. Generally, $G^{(0)}$, $G^{(1)}$ and $G^{(2)}$ are asymmetric matrices which may be different from each other. $u_i(t) \in \mathbb{R}^n$ is the control input imposed on the ith node. Matrix function $h_i(t, x_1(t), \ldots, x_N(t)) \in \mathbb{R}^{n\times n}$ satisfies $h_i(t, v, \ldots, v) = 0$ for any $v \in \mathbb{R}^n$, and $\omega_i(t) = (\omega_{i1}(t), \omega_{i2}(t), \ldots, \omega_{in}(t))^T$ is a Brownian motion defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\ge 0}, \mathcal{P})$. It is further assumed that $\omega_i(t)$ and $\omega_j(t)$ are independent process of one another for $i \ne j$, and time-varying delays satisfy the following constraints.

Assumption 1: There exist constants σ_i, $\mu_i (i = 1,2,\ldots,N)$ and τ_k, ρ_k $(k = 1, 2)$ such that $0 \le \sigma_i(t) \le \sigma_i$, $\dot{\sigma}_i(t) \le \mu_i < 1$ and $0 \le \tau_k(t) \le \tau_k$, $\dot{\tau}_k(t) \le \rho_k < 1$.

Remark 1: It should be noted that the uncoupled delay $\sigma_i(t)$, and the coupled delays $\tau_1(t)$ and $\tau_2(t)$ are all time-varying, while they are assumed to be equal in [6], or to be constants in [29]. Usually, the inner coupling matrices are assumed to be diagonal and positive definite, while in network (1), Γ_0 is required to be only positive definite, and no restrictions are made on Γ_1 and Γ_2. Furthermore, the coupling matrices $G^{(k)}$ $(k = 0, 1, 2)$ need not to be symmetric or irreducible, which is more consistent with the realistic networks.

The system (1) is supplemented with initial condition given by

$$x_i(t) = \varphi_i(t) \in L^2_{\mathcal{F}_0}([-\tau^*, 0], \mathbb{R}^n)$$

where $L^2_{\mathcal{F}_0}([-\tau^*, 0], \mathbb{R}^n)$ represents the set of all $\mathcal{F}_0$-measurable $\mathcal{C}([-\tau^*, 0], \mathbb{R}^n)$-valued random variables satisfying $\sup_{-\tau^*\le s\le 0} \mathbb{E}\{\|\varphi_i(s)\|\} < \infty$, where $\tau^* = \max_{1\le i\le N}\{\sigma_i, \tau_1, \tau_2\}$, and $\mathcal{C}([-\tau^*, 0], \mathbb{R}^n)$ denotes the family of all continuous $\mathbb{R}^n$-valued functions $\varphi_i(s)$ on $[-\tau^*, 0]$ with the norm $\|\varphi_i\| = \sup_{-\tau^*\le s\le 0} \varphi_i^T(s)\varphi_i(s)$.

In the above complex networks, the nodes are nonidentical, which means they are described by different dynamical equations, the complete synchronization of the network may not be achieved due to the lack of a common manifold. Under such cases, many existing synchronization criteria for complex networks with identical nodes are not effective anymore. It is therefore, the aim of this paper to design appropriate control schemes such that the hybrid-coupled complex network with nonidentical nodes can be synchronized to *any common objective dynamics* $\{s(t) \in \mathbb{R}^n |\ t \in [0, \infty)\}$ *which is continuously derivative*.

Definition 1: The hybrid-coupled complex network (1) with nonidentical nodes is said to be globally asymptotically synchronizable in the mean square to the goal trajectory $s(t)$ if the following discriminant relations

$$\lim_{t\to\infty} \mathbb{E}\{\|x_i(t) - s(t)\|^2\} = 0, \qquad i = 1, 2 \ldots, N$$

hold for all initial functions, where $x_i(t)$ is the solution of the controlled closed-loop network (1).

Assumption 2: For the nonlinear dynamical functions $f_i(t, x_i(t), x_i(t-\sigma_i(t))) : \mathbb{R}^+ \times\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ $(i = 1, 2, \ldots, N)$, the uniform semi-Lipschitzian conditions hold with respect to the time $t \in \mathbb{R}^+$. That is, there exist positive constants $\theta_i > 0$ and $\gamma_i > 0$ such that

$$\begin{aligned} &[x_i(t) - y_i(t)]^T [f_i(t, x_i(t), x_i(t-\sigma_i(t))) \\ &\qquad - f_i(t, y_i(t), y_i(t-\sigma_i(t)))] \\ &\le \theta_i [x_i(t) - y_i(t)]^T [x_i(t) - y_i(t)] \\ &\quad + \gamma_i [x_i(t-\sigma_i(t)) - y_i(t-\sigma_i(t))]^T \\ &\quad \times [x_i(t-\sigma_i(t)) - y_i(t-\sigma_i(t))] \end{aligned}$$

hold for all $i = 1, 2, \ldots, N$, $x_i(t)$, $y_i(t) \in \mathbb{R}^n$ and $t \in \mathbb{R}^+$.

Lemma 1: [30] Let $\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_n$, $\beta_1 \geq \beta_2 \geq \ldots \geq \beta_n$ and $\gamma_1 \geq \gamma_2 \geq \ldots \geq \gamma_n$ be eigenvalues of matrices A, B and $A+B$, respectively, where A and B are symmetric matrices in $\mathbb{R}^{n\times n}$. Then one has $\alpha_i+\beta_n \leq \gamma_i \leq \alpha_i+\beta_1$ $(i = 1, 2, \ldots, n)$.

Lemma 2: [31] For a diagonal matrix $D = \mathrm{diag}\{d_1, \ldots, d_l, 0, \ldots, 0\} \in \mathbb{R}^{N\times N}$ with $d_i > 0$ $(i = 1, 2, \ldots, l; 1 \leq l \leq N)$ and a symmetric matrix $M \in \mathbb{R}^{N\times N}$, let

$$M - D = \begin{bmatrix} A - \tilde{D} & B \\ B^T & M_l \end{bmatrix}$$

where M_l is the minor matrix of M by removing its first l row-column pairs, A and B are matrices with appropriate dimensions, $\tilde{D} = \mathrm{diag}\{d_1, d_2, \ldots, d_l\}$. If $d_i > \lambda_{\max}(A - BM_l^{-1}B^T)$ $(i = 1, 2, \ldots, l)$, then $M - D < 0$ is equivalent to $M_l < 0$.

Lemma 3: For any matrix $M > 0$, scalars $a < b$, and vector $x(t)$ with appropriate dimension, we have

$$(b-a)\int_a^b x^T(s)Mx(s)\mathrm{d}s \geq \Big(\int_a^b x(s)\mathrm{d}s\Big)^T M\Big(\int_a^b x(s)\mathrm{d}s\Big).$$

III. Main Results

A. Pinning adaptive control scheme

In this subsection, the pinning adaptive synchronization of network (1) will be investigated. Considering the fact that the objective dynamics $s(t)$ is differentiable, the pinning adaptive controller is designed as follows which is composed both by the open-loop control and by the adaptive feedback control:

$$\begin{aligned} u_i(t) =&\dot{s}(t) - f_i(t, s(t), s(t - \sigma_i(t))) \\ &- d_i(t)\Gamma_0(x_i(t) - s(t)) \end{aligned} \tag{2}$$

where $d_i(0) = 0$ and

$$\dot{d}_i(t) = \begin{cases} \alpha_i(x_i(t) - s(t))^T\Gamma_0(x_i(t) - s(t)), \\ \qquad\qquad \alpha_i > 0, \quad i = 1, 2, \ldots, l; \\ 0, \qquad\qquad i = l+1, l+2, \ldots, N. \end{cases}$$

By letting the synchronization error $e_i(t) = x_i(t) - s(t)$, $h_i(t, x(t)) = h_i(t, x_1(t), \ldots, x_N(t))$, and $h_i(t, s(t)) = h_i(t, s(t), \ldots, s(t))$, one can derive the following error system:

$$\begin{aligned} \mathrm{d}e_i(t) =&\Big[g_i(t, e_i(t), e_i(t - \sigma_i(t))) + c_0\sum_{j=1}^N G_{ij}^{(0)}\Gamma_0 e_j(t) \\ &+ c_1\sum_{j=1}^N G_{ij}^{(1)}\Gamma_1 e_j(t - \tau_1(t)) \\ &+ c_2\sum_{j=1}^N G_{ij}^{(2)}\Gamma_2\int_{t-\tau_2(t)}^t e_j(s)\mathrm{d}s \\ &- d_i(t)\Gamma_0 e_i(t)\Big]\mathrm{d}t + \tilde{h}_i(t, e(t))\mathrm{d}\omega_i(t), \end{aligned} \tag{3}$$

where $g_i(t, e_i(t), e_i(t - \sigma_i(t))) = f_i(t, e_i(t) + s(t), e_i(t - \sigma_i(t)) + s(t - \sigma_i(t))) - f_i(t, s(t), s(t - \sigma_i(t)))$, $\tilde{h}_i(t, e(t)) = h_i(t, x(t)) - h_i(t, s(t))$, and $e(t) = (e_1^T(t), \ldots, e_N^T(t))^T$.

Here, for the noise intensity function $\tilde{h}_i(\cdot)$, the following assumption is made.

Assumption 3: There exist nonnegative constants L_{ij}, $i, j = 1, 2, \ldots, N$, such that

$$\mathrm{trace}[\tilde{h}_i^T(t, e(t))\tilde{h}_i(t, e(t))] \leq \sum_{j=1}^N L_{ij}e_j^T(t)e_j(t).$$

To conclude that the hybrid-coupled heterogeneous network (1) under the control input (2) is globally asymptotically synchronizable (to the goal trajectory $s(t)$) in the mean square sense, one just need to prove that the error system (3) is globally asymptotically stable in the mean square.

Before stating the main results, the following notations are introduced:

$$\begin{aligned} \beta_1 =&\frac{1}{1-\rho_1}\Big(\frac{c_1}{2} + \varepsilon_1\Big), \qquad \beta_2 = \frac{\tau_2}{1-\rho_2}\Big(\frac{c_2}{2} + \varepsilon_2\Big), \\ \kappa_1 =&\frac{c_1\lambda_{\max}(\Gamma_1\Gamma_1^T)\lambda_{\max}((G^{(1)}G^{(1)^T})_l)}{2\lambda_{\min}(\Gamma_0)}, \\ \kappa_2 =&\frac{c_2\lambda_{\max}(\Gamma_2\Gamma_2^T)\lambda_{\max}((G^{(2)}G^{(2)^T})_l)}{2\lambda_{\min}(\Gamma_0)}, \\ \sigma =&\frac{\max\limits_{1\leq i\leq N}\{\theta_i + \frac{\gamma_i}{1-\mu_i} + \frac{1}{2}\sum_{j=1}^N L_{ji}\} + \beta_1 + \beta_2\tau_2}{\lambda_{\min}(\Gamma_0)} \\ &+ \kappa_1 + \kappa_2, \\ M =&\frac{\max\limits_{1\leq i\leq N}\{\theta_i + \frac{\gamma_i}{1-\mu_i} + \frac{1}{2}\sum_{j=1}^N L_{ji}\} + \beta_1 + \beta_2\tau_2}{\lambda_{\min}(\Gamma_0)}I_N \\ &+ \frac{c_1\lambda_{\max}(\Gamma_1\Gamma_1^T)}{2\lambda_{\min}(\Gamma_0)}G^{(1)}G^{(1)^T} \\ &+ \frac{c_2\lambda_{\max}(\Gamma_2\Gamma_2^T)}{2\lambda_{\min}(\Gamma_0)}G^{(2)}G^{(2)^T} + c_0G_s^{(0)}, \\ M - D^* =& \begin{bmatrix} \hat{A} - \hat{D}^* & \hat{B} \\ \hat{B}^T & M_l \end{bmatrix}, \end{aligned}$$

where ε_1 and ε_2 are small positive constants, $D^* = \mathrm{diag}\{d_1^*, \ldots, d_l^*, 0, \ldots, 0\} \in \mathbb{R}^{N\times N}$ and $\hat{D}^* = \mathrm{diag}\{d_1^*, \ldots, d_l^*\}$, M_l is the minor matrix of M by removing its first l $(1 \leq l < N)$ row column pairs, $\hat{A}$ and $\hat{B}$ are matrices with appropriate dimensions.

Theorem 1: Under Assumptions 1-3, the hybrid-coupled network (1) under the pinning adaptive control law (2) is globally asymptotically synchronizable in the mean square sense to the objective trajectory $s(t)$ if there exist positive constants ε_1 and ε_2 such that the following condition holds:

$$\sigma + c_0\lambda_{\max}((G_s^{(0)})_l) < 0. \tag{4}$$

Proof: Consider the Lyapunov functional candidate

$$V(t) = V_1(t) + V_2(t) + V_3(t) + V_4(t)$$

with

$$\begin{aligned} V_1(t) =&\frac{1}{2}\sum_{i=1}^N e_i^T(t)e_i(t) \\ &+ \sum_{i=1}^N \frac{\gamma_i}{1-\mu_i}\int_{t-\sigma_i(t)}^t e_i^T(\xi)e_i(\xi)\mathrm{d}\xi, \end{aligned}$$

$$V_2(t) = \beta_1 \sum_{i=1}^{N} \int_{t-\tau_1(t)}^{t} e_i^T(\xi) e_i(\xi) \mathrm{d}\xi,$$

$$V_3(t) = \beta_2 \sum_{i=1}^{N} \int_{t-\tau_2(t)}^{t} \int_{\xi}^{t} e_i^T(\eta) e_i(\eta) \mathrm{d}\eta \mathrm{d}\xi,$$

$$V_4(t) = \sum_{i=1}^{N} \frac{1}{2\alpha_i} (d_i(t) - d_i^*)^2,$$

in which $d_i^* > 0$ $(i = 1, 2, \ldots, l)$ are bounded constants to be determined later and $d_i^* = 0$ for $i = l+1, l+2, \ldots, N$; α_i $(i = l+1, l+2, \ldots, N)$ are any nonzero positive constants.

Taking the time derivative of $V_1(t)$ along the trajectories of the error system (3) and by the $It\hat{o}$ differential formula [32], we have

$$\begin{aligned}
\mathrm{d}V_1 \le & \Big\{ \sum_{i=1}^{N} \theta_i e_i^T(t) e_i(t) + \sum_{i=1}^{N} \gamma_i e_i^T(t-\sigma_i(t)) e_i(t-\sigma_i(t)) \\
& + c_0 \sum_{i=1}^{N} \sum_{j=1}^{N} G_{ij}^{(0)} e_i^T(t) \Gamma_0 e_j(t) \\
& + c_1 \sum_{i=1}^{N} \sum_{j=1}^{N} G_{ij}^{(1)} e_i^T(t) \Gamma_1 e_j(t-\tau_1(t)) \\
& + c_2 \sum_{i=1}^{N} \sum_{j=1}^{N} G_{ij}^{(2)} e_i^T(t) \Gamma_2 \int_{t-\tau_2(t)}^{t} e_j(\xi) \mathrm{d}\xi \\
& + \sum_{i=1}^{N} [\frac{\gamma_i}{1-\mu_i} e_i^T(t) e_i(t) - d_i(t) e_i^T(t) \Gamma_0 e_i(t)] \\
& - \sum_{i=1}^{N} \gamma_i e_i^T(t-\sigma_i(t)) e_i(t-\sigma_i(t)) \\
& + \frac{1}{2} \sum_{i=1}^{N} \mathrm{trace}\Big[\big(\tilde{h}_i(t, e(t))\big)^T \big(\tilde{h}_i(t, e(t))\big) \Big] \Big\} \mathrm{d}t \\
& + \sum_{i=1}^{N} e_i^T(t) \tilde{h}_i(t, e(t)) \mathrm{d}\omega_i(t) \\
\le & \Big\{ e^T(t) \big[\Theta \otimes I_n + c_0 G_s^{(0)} \otimes \Gamma_0 - D(t) \otimes \Gamma_0 \big] e(t) \\
& + c_1 e^T(t) (G^{(1)} \otimes \Gamma_1) e(t-\tau_1(t)) \\
& + c_2 e^T(t) (G^{(2)} \otimes \Gamma_2) \int_{t-\tau_2(t)}^{t} e(\xi) \mathrm{d}\xi \\
& + \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} L_{ij} e_j^T(t) e_j(t) \Big\} \mathrm{d}t \\
& + \sum_{i=1}^{N} e_i^T(t) \tilde{h}_i(t, e(t)) \mathrm{d}\omega_i(t) \\
\le & \Big\{ e^T(t) \big[\Theta \otimes I_n + c_0 G_s^{(0)} \otimes \Gamma_0 - D(t) \otimes \Gamma_0 \big] e(t) \\
& + e^T(t) \Big[\frac{c_1}{2} (G^{(1)} {G^{(1)}}^T) \otimes (\Gamma_1 \Gamma_1^T) \\
& + \frac{c_2}{2} (G^{(2)} {G^{(2)}}^T) \otimes (\Gamma_2 \Gamma_2^T) \Big] e(t) \\
& + \frac{c_1}{2} e^T(t-\tau_1(t)) e(t-\tau_1(t)) \\
& + \frac{c_2}{2} \Big(\int_{t-\tau_2(t)}^{t} e(\xi) \mathrm{d}\xi \Big)^T \Big(\int_{t-\tau_2(t)}^{t} e(\xi) \mathrm{d}\xi \Big) \Big\} \mathrm{d}t \\
& + \sum_{i=1}^{N} e_i^T(t) \tilde{h}_i(t, e(t)) \mathrm{d}\omega_i(t),
\end{aligned}$$

where $\Theta = \mathrm{diag}\,\{\theta_1 + \frac{\gamma_1}{1-\mu_1} + L_1, \theta_2 + \frac{\gamma_2}{1-\mu_2} + L_2, \ldots, \theta_N + \frac{\gamma_N}{1-\mu_N} + L_N\}$, with $L_i = \frac{1}{2}\sum_{j=1}^{N} L_{ji}$, $i = 1, 2, \ldots, N$; $D(t) = \mathrm{diag}\{d_1(t), d_2(t), \ldots, d_l(t), 0, \ldots, 0\}$.

Similarly, calculating the derivatives of $V_2(t)$, $V_3(t)$, and $V_4(t)$, by Lemma 3, one can obtain

$$\begin{aligned}
\mathrm{d}V_2 \le & \Big\{ \beta_1 e^T(t) e(t) - \beta_1 (1-\rho_1) \\
& \times e^T(t-\tau_1(t)) e(t-\tau_1(t)) \Big\} \mathrm{d}t,
\end{aligned} \tag{5}$$

$$\begin{aligned}
\mathrm{d}V_3 \le & \Big\{ \beta_2 \sum_{i=1}^{N} \Big[\tau_2 e_i^T(t) e_i(t) - (1-\rho_2) \\
& \times \int_{t-\tau_2(t)}^{t} e_i^T(\eta) e_i(\eta) \mathrm{d}\eta \Big] \Big\} \mathrm{d}t \\
\le & \Big[\beta_2 \tau_2 e^T(t) e(t) - \frac{\beta_2(1-\rho_2)}{\tau_2} \\
& \times \Big(\int_{t-\tau_2(t)}^{t} e(\xi) \mathrm{d}\xi \Big)^T \Big(\int_{t-\tau_2(t)}^{t} e(\xi) \mathrm{d}\xi \Big) \Big] \mathrm{d}t
\end{aligned} \tag{6}$$

and

$$\mathrm{d}V_4 = \Big[e^T(t) ((D(t) - D^*) \otimes \Gamma_0) e(t) \Big] \mathrm{d}t. \tag{7}$$

From the above inequalities, it is easy to have

$$\begin{aligned}
& \frac{\mathrm{d}\mathbb{E}\{V(t)\}}{\mathrm{d}t} \\
\le & e^T(t) \Big[(\Theta + (\beta_1 + \beta_2 \tau_2) I_N) \otimes I_n \\
& + (c_0 G_s^{(0)} - D^*) \otimes \Gamma_0 \Big] e(t) \\
& + e^T(t) \Big[\frac{c_1}{2} (G^{(1)} {G^{(1)}}^T) \otimes (\Gamma_1 \Gamma_1^T) \\
& + \frac{c_2}{2} (G^{(2)} {G^{(2)}}^T) \otimes (\Gamma_2 \Gamma_2^T) \Big] e(t) \\
& + \Big[\frac{c_1}{2} - \beta_1 (1-\rho_1) \Big] e^T(t-\tau_1(t)) e(t-\tau_1(t)) \\
& + \Big[\frac{c_2}{2} - \frac{\beta_2(1-\rho_2)}{\tau_2} \Big] \\
& \times \Big(\int_{t-\tau_2(t)}^{t} e(\xi) \mathrm{d}\xi \Big)^T \Big(\int_{t-\tau_2(t)}^{t} e(\xi) \mathrm{d}\xi \Big) \\
\le & e^T(t) \Big[(\Theta + (\beta_1 + \beta_2 \tau_2) I_N) \otimes I_n \\
& + (c_0 G_s^{(0)} - D^*) \otimes \Gamma_0 \Big] e(t) \\
& + e^T(t) \Big[\frac{c_1}{2} (G^{(1)} {G^{(1)}}^T) \otimes (\Gamma_1 \Gamma_1^T) \\
& + \frac{c_2}{2} (G^{(2)} {G^{(2)}}^T) \otimes (\Gamma_2 \Gamma_2^T) \Big] e(t) \\
\le & e^T(t) [(M - D^*) \otimes \Gamma_0] e(t).
\end{aligned}$$

By Lemma 1, one has $\lambda_{\max}(M_l) \le \sigma + c_0 \lambda_{\max}((G_s^{(0)})_l) < 0$, which indicates that $M_l < 0$. By selecting $d_i^* > \lambda_{\max}(\hat{A} - \hat{B} M_l^{-1} \hat{B}^T)$ $(i = 1, 2, \ldots, l)$, and together with Lemma 2, it follows that $M - D^* < 0$. Thus, we have $\mathrm{d}\mathbb{E}\{V(t)\}/\mathrm{d}t < 0$ for $e(t) \neq 0$, which indicates that $\mathbb{E}\{\|e_i(t)\|\}^2 \to 0$ $(i = 1, 2, \ldots, N)$. Therefore, the pinning controlled network (1) can be globally asymptotically synchronized to the objective trajectory in the mean square sense. ■

Remark 2: In [29], synchronization has been studied for the hybrid-coupled complex dynamical networks with constant time delays by utilizing the pinning adaptive control method. Compared with [29], we consider the pinning adaptive synchronization control problem for the hybrid-coupled *heterogeneous* dynamical networks with *time-varying* delays.

B. Impulsive control scheme

In this subsection, impulsive control strategy will be utilized to synchronize the hybrid-coupled complex network (1). The closed-loop stochastic impulsive control network is given as follows:

$$
\begin{cases}
\mathrm{d}x_i(t) = \Big[f_i(t, x_i(t), x_i(t-\sigma_i(t))) \\
\qquad + c_0 \sum_{j=1}^{N} G_{ij}^{(0)} \Gamma_0 x_j(t) \\
\qquad + c_1 \sum_{j=1}^{N} G_{ij}^{(1)} \Gamma_1 x_j(t-\tau_1(t)) \\
\qquad + c_2 \sum_{j=1}^{N} G_{ij}^{(2)} \Gamma_2 \int_{t-\tau_2(t)}^{t} x_j(s)\mathrm{d}s \\
\qquad + \dot{s}(t) - f_i(t, s(t), s(t-\sigma_i(t))) \Big] \mathrm{d}t \\
\qquad + h_i(t, x_1(t), \ldots, x_N(t))\mathrm{d}\omega_i(t), \ t \neq t_k, \\
\Delta x_i(t_k) = H_k\left(x_i(t_k) - s(t_k)\right), \ k = 1, 2, \ldots
\end{cases}
\tag{8}
$$

where $i = 1, 2, \ldots N$, $\Delta x_i(t_k) = x_i(t_k^+) - x_i(t_k)$, $x_i(t_k^+) = \lim_{h\to 0^+} x_i(t_k + h)$, $x_i(t_k) = x_i(t_k^-) = \lim_{h\to 0^-} x_i(t_k + h)$, and the impulsive time sequence $\{t_k\}_{k=1}^{+\infty}$ satisfies $0 = t_0 < t_1 < \ldots < t_k < \ldots$ with $\lim_{k\to\infty} t_k = +\infty$. $H_k \in \mathbb{R}^{n\times n}(k = 1, 2, \ldots)$ are the impulsive control gain matrices.

By letting $e_i(t) = x_i(t) - s(t)$, the error system is derived as follows:

$$
\begin{cases}
\mathrm{d}e_i(t) = \Big[g_i(t, e_i(t), e_i(t-\sigma_i(t))) \\
\qquad + c_0 \sum_{j=1}^{N} G_{ij}^{(0)} \Gamma_0 e_j(t) \\
\qquad + c_1 \sum_{j=1}^{N} G_{ij}^{(1)} \Gamma_1 e_j(t-\tau_1(t)) \\
\qquad + c_2 \sum_{j=1}^{N} G_{ij}^{(2)} \Gamma_2 \int_{t-\tau_2(t)}^{t} e_j(s)\mathrm{d}s \Big] \mathrm{d}t \\
\qquad + \tilde{h}_i(t, e(t))\mathrm{d}\omega_i(t), \quad t \neq t_k \\
e_i(t_k^+) = (I_n + H_k)e_i(t_k);
\end{cases}
\tag{9}
$$

where $k = 1, 2, \ldots$ and $i = 1, 2, \ldots N$.

The following lemma is crucial when deriving the synchronization criterion for the impulsive controlled network (8).

Lemma 4: [33] Let $\tilde{c}_1, \tilde{c}_2 > 0$, $q \geq 0$, $\tau > 0$, $d_k > 0$ $(k = 1, 2, \ldots)$ and p be constants, and the nonnegative function $V(t, x(t))$ is defined on $[-\tau, \infty] \times \mathbb{R}^n$ which is continuous on $(t_{k-1}, t_k] \times \mathbb{R}^n$ for $k = 1, 2, \ldots$ with

$$
\begin{cases}
\tilde{c}_1 x^T x \leq V(t, x) \leq \tilde{c}_2 x^T x, \quad t \geq t_0 - \tau \\
\mathbb{E}\big\{\mathcal{L}V(t, x(t))\big\} \leq p\mathbb{E}\big\{V(t, x(t))\big\} \\
\quad + q \sup_{t-\tau\leq s\leq t} \mathbb{E}\big\{V(s, x(s))\big\}, \quad t \geq t_0, \qquad t \neq t_k, \\
\mathbb{E}\big\{V(t_k^+, x(t_k^+))\big\} \leq d_k \mathbb{E}\big\{V(t_k, x(t_k))\big\}, \quad k = 1, 2, \ldots
\end{cases}
$$

where $\mathcal{L}V(t, x)$ is an infinitesimal operator on the function $V(t, x(t))$. If there exists constant β such that

$$
\frac{\ln d_k}{t_k - t_{k-1}} < \beta \qquad \text{and} \qquad p + dq + \beta < 0
$$

hold for $k = 1, 2, \ldots$; then the zero solution of (9) is exponentially stable in the mean square with λ as the exponential convergence rate, where $d = \sup_{1\leq k\leq +\infty}\{\exp(\beta(t_k - t_{k-1})), 1/\exp(\beta(t_k - t_{k-1}))\}$, and λ is the unique positive root of the equation $\lambda + p + dqe^{\lambda\tau} + \beta = 0$.

Theorem 2: Under Assumptions 1-3, the impulsive controlled hybrid-coupled network (8) is globally exponentially synchronizable to the objective trajectory $s(t)$ in the mean square if there exist constants $\epsilon_1 \neq 0$, $\epsilon_2 \neq 0$ and β such that for $k = 1, 2, \ldots$:

$$
\frac{\ln d_k}{t_k - t_{k-1}} < \beta \qquad \text{and} \qquad p + \tilde{d}q + \beta < 0 \tag{10}
$$

where

$$
\begin{aligned}
q &= 2 \max_{1\leq i\leq N}\{\gamma_i\} + c_1/\epsilon_1^2 + c_2(\tau_2/\epsilon_2)^2, \\
\tilde{d} &= \sup_{1\leq k\leq +\infty}\{\exp(\beta(t_k - t_{k-1})), 1/\exp(\beta(t_k - t_{k-1}))\}, \\
d_k &= \lambda_{\max}\big((I_n + H_k)^T(I_n + H_k)\big) > 0, \\
p &= 2\Big[\max_{1\leq i\leq N}\big\{\theta_i + \frac{1}{2}\sum_{j=1}^{N} L_{ji}\big\} + c_0\lambda_{\max}(G_s^{(0)} \otimes \Gamma_0) \\
&\quad + \frac{c_1\epsilon_1^2}{2}\lambda_{\max}(G^{(1)}{G^{(1)}}^T)\lambda_{\max}(\Gamma_1\Gamma_1^T) \\
&\quad + \frac{c_2\epsilon_2^2}{2}\lambda_{\max}(G^{(2)}{G^{(2)}}^T)\lambda_{\max}(\Gamma_2\Gamma_2^T)\Big].
\end{aligned}
$$

Proof: Consider the following Lyapunov function candidate:

$$
V(t, e(t)) = \frac{1}{2}\sum_{i=1}^{N} e_i^T(t)e_i(t).
$$

For $t \in (t_{k-1}, t_k)$ $(k = 1, 2, \ldots)$, by the *Itô* differential formula, one has

$$
\begin{aligned}
\mathcal{L}V(t) \leq &\sum_{i=1}^{N}\theta_i e_i^T(t)e_i(t) + c_0 e^T(t)(G^{(0)} \otimes \Gamma_0)e(t) \\
&+ c_1 e^T(t)(G^{(1)} \otimes \Gamma_1)e(t-\tau_1(t)) \\
&+ c_2 e^T(t)(G^{(2)} \otimes \Gamma_2)\Big(\int_{t-\tau_2(t)}^{t} e(s)\mathrm{d}s\Big)
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{i=1}^{N}\gamma_i e_i^T(t-\sigma_i(t))e_i(t-\sigma_i(t))\\
&+\frac{1}{2}\sum_{i=1}^{N}\mathrm{trace}\Big[\big(\tilde{h}_i(t,e(t))\big)^T\big(\tilde{h}_i(t,e(t))\big)\Big]\\
\le& e^T(t)\Big[\tilde{\Theta}\otimes I_n+c_0G^{(0)}\otimes\Gamma_0\\
&+\frac{c_1\epsilon_1^2}{2}(G^{(1)}{G^{(1)}}^T)\otimes(\Gamma_1\Gamma_1^T)\\
&+\frac{c_2\epsilon_2^2}{2}(G^{(2)}{G^{(2)}}^T)\otimes(\Gamma_2\Gamma_2^T)\Big]e(t)\\
&+\sum_{i=1}^{N}\gamma_i e_i^T(t-\sigma_i(t))e_i(t-\sigma_i(t))\\
&+\frac{c_1}{2\epsilon_1^2}e^T(t-\tau_1(t))e(t-\tau_1(t))\\
&+\frac{c_2\tau_2}{2\epsilon_2^2}\int_{t-\tau_2(t)}^{t}e^T(s)e(s)\mathrm{d}s
\end{aligned}
$$

where $\tilde{\Theta} = \mathrm{diag}\{\theta_1+L_1,\theta_2+L_2,\ldots,\theta_N+L_N\}$ with $L_i = \frac{1}{2}\sum_{j=1}^{N}L_{ji}$, $i=1,2,\ldots,N$. Thus, it is easy to derive that $\mathbb{E}[\mathcal{L}V(t,x(t))] \le p\mathbb{E}[V(t,x(t))] + q\sup_{t-\tau\le s\le t}\mathbb{E}[V(s,x(s))]$ for $t\neq t_k$.

On the other hand, for $t=t_k$ $(k=1,2,\ldots)$, we have

$$
\begin{aligned}
&\mathbb{E}\big\{V(t_k^+,x(t_k^+))\big\}\\
&=\frac{1}{2}\sum_{i=1}^{N}\mathbb{E}\big\{e_i^T(t_k^+)e_i(t_k^+)\big\}\\
&=\frac{1}{2}\sum_{i=1}^{N}\mathbb{E}\big\{e_i^T(t_k)[(I+H_k)^T(I+H_k)]e_i(t_k)\big\}\\
&\le d_k\big\{V(t_k^+,x(t_k^+))\big\}.
\end{aligned}
$$

From Lemma 4 and the conditions in (10), it follows that the zero solution of error system (9) is globally exponentially stable in the mean square with λ as the exponential convergence rate, where λ is the unique positive root of the following equation: $\lambda+p+\tilde{d}qe^{\lambda\tau^*}+\beta=0$, which implies that the hybrid-coupled complex network (1) can be synchronized under the impulsive control scheme. ∎

Remark 3: In [14], the impulsive synchronization problem has been considered for the heterogeneous networks without delays. While in our present study, the impulsive synchronization for the *delayed hybrid-coupled* heterogeneous networks is investigated, meanwhile, the improved Halanay inequality has been utilized in the proof of Theorem 2. Different from the classical Halanay inequality which has been usually resorted to for handling the impulsive differential equations, here the restriction $-p>q>0$ for coefficients p and q is not required anymore, and the main reason is that the impulses here have played a key role.

IV. NUMERICAL SIMULATIONS

This section provides a simulation example to verify the effectiveness of the proposed pinning adaptive control and impulsive control schemes for the hybrid-coupled complex networks.

Example 1: Consider a directed network consisting of 16 nonidentical nodes described by the following coupled neural networks:

$$
\begin{aligned}
\mathrm{d}x_i(t)=&\Big[-Cx_i(t)+A_kf(x_i(t))+B_kf(x_i(t-\sigma_k(t)))\\
&+c_0\sum_{j=1}^{16}G_{ij}^{(0)}\Gamma_0x_j(t)\\
&+c_1\sum_{j=1}^{16}G_{ij}^{(1)}\Gamma_1x_j(t-\tau_1(t))\\
&+c_2\sum_{j=1}^{16}G_{ij}^{(2)}\Gamma_2\int_{t-\tau_2(t)}^{t}x_j(s)\mathrm{d}s\Big]\mathrm{d}t\\
&+h_i(x_1,\ldots,x_{16})\mathrm{d}\omega_i(t),
\end{aligned}
\tag{11}
$$

where $i=1,2,\ldots,16$, $x_i(t)=(x_{i1}(t),x_{i2}(t))^T$ is the state variable of the ith node; the matrix $C=\mathrm{diag}\{1,1\}$, and the lower index $k=1$ when $1\le i\le 8$ and $k=2$ when $9\le i\le 16$ with

$$
A_1=\begin{bmatrix}2.2 & -0.11\\ -5.3 & 3.1\end{bmatrix},\quad B_1=\begin{bmatrix}-1.8 & -0.2\\ -0.15 & -2.4\end{bmatrix},
$$
$$
A_2=\begin{bmatrix}0.2 & -0.2\\ -0.4 & 0.3\end{bmatrix},\quad B_2=\begin{bmatrix}-4 & -5\\ -2 & -5\end{bmatrix}.
$$

The coupling strengths are taken as $c_0=32$, $c_1=10$ and $c_2=8$; the coupling structure matrix $G^{(0)}$ is determined by the directed network in Fig. 1, meanwhile $G^{(1)}=0.25G^{(0)}$, $G^{(2)}=0.15G^{(0)}$; the inner coupling matrices are chosen as

$$
\Gamma_0=\begin{bmatrix}6 & 1\\ 1 & 4\end{bmatrix},\ \Gamma_1=\begin{bmatrix}0.2 & 0.3\\ 0 & 0.3\end{bmatrix},\ \Gamma_2=\begin{bmatrix}0.1 & 0\\ 0.4 & 0.1\end{bmatrix}.
$$

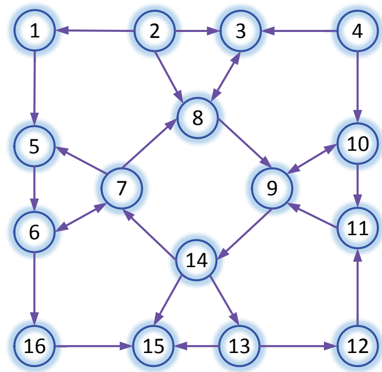

Fig. 1. The directed network with 16 nodes.

Set the nonlinear activation function in system (11) to be $f(x_i(t)) = (\tan(x_{i1}(t)),\tan(x_{i2}(t)))^T$, and $h_i(x_1,\ldots,x_N) = 0.4\mathrm{diag}\{x_{i,1}-x_{i+1,1},x_{i,2}-x_{i+1,2}\}$, where $i=1,\ldots,16$ with $x_{17}=x_1$. By some calculation, it is easy to get that the dynamics of the delayed neural networks satisfy Assumption 1 with $\max_{i=1,2}\{\theta_i\}=5.3$ and $\max_{i=1,2}\{\gamma_i\}=8$. The time-varying delays $\sigma_1(t)=\sigma_2(t)=e^t/(1+e^t)$, $\tau_1(t)=0.15e^t/(1+e^t)$ and $\tau_2(t)=$

$0.25e^t/(1+e^t)$, it is easy to know that Assumption 2 is satisfied with $\mu_1 = \mu_2 = 0.25$, $\tau_2 = 0.25$, $\rho_1 = 0.0375$ and $\rho_2 = 0.0625$. Meanwhile, for $i = 1, \ldots, 16$,

$$\mathrm{trace}\big((\tilde{h}_i e(t))^T(\tilde{h}_i e(t))\big) \leq 0.32\big(\|e_i(t)\|^2 + \|e_{i+1}(t)\|^2\big)$$

with $e_{17}(t) = e_1(t)$. Thus, the Assumption 3 is satisfied.

The objective synchronization system is described as follows:

$$\dot{s}(t) = -\tilde{C}s(t) + Af(s(t)) + Bf(s(t-\delta(t))) \tag{12}$$

where $\delta(t) = e^t/(1+e^t)$, $f(\cdot)$ is defined as in (11) and

$$\tilde{C} = \begin{bmatrix} 1.2 & 0 \\ 0 & 1 \end{bmatrix}, \ A = \begin{bmatrix} 3 & -0.3 \\ 4 & 5 \end{bmatrix}, \ B = \begin{bmatrix} -1.4 & 0.1 \\ 0.3 & -8 \end{bmatrix}.$$

The initial functions for $x(t)$ are taken randomly on the interval $[-1, 0]$ and $s(t) \equiv (0.2, 0.5)^T$ for $t \in [-1, 0]$, respectively, for the hybrid-coupled complex network (11) and the objective synchronization system (12). In the following, we will analyze the synchronization of the hybrid-coupled network (11) under the pinning adaptive control and impulsive control schemes.

Firstly, the pinning adaptive control strategy is considered. Checking the network finds that nodes 2 and 4 are zero in-degrees nodes, so they should be pinned first. Then rearranging the network nodes according to the maximum degree-differences of the network [29], and setting $\varepsilon_1 = 0.02$, $\varepsilon_2 = 0.04$, it is found that $l_{\min} = 3$, i.e, at least 3 nodes should be pinned to reach the synchronization aim, so the node 14 is pinned. By calculation, $\lambda_{\max}((G_s^0)_l) = -0.2289$, $\sigma = 7.1099$, $\lambda_{\max}(\hat{A} - \hat{B}M_l^{-1}\hat{B}^T) = 6.6963$. We set $d_i^* = 7.0$ $(i = 1, 2, \ldots, 3)$, then the condition $\sigma + c_0\lambda_{\max}((G_s^0)_l) = -0.2150 < 0$ is satisfied. Theorem 1 ensures that the whole hybrid-coupled network (11) can be synchronized to the given goal trajectory in the mean square, the evolutions of the synchronization error system and the pinning feedback gains are shown, respectively, in Fig. 2 and Fig. 3.

Secondly, we discuss the impulsive control method. By setting the impulsive gain matrices $H_k \equiv H = \mathrm{diag}\{-0.15, -0.15\}$, constants $\epsilon_1 = 0.3$, $\epsilon_2 = 0.5$ and $\beta = -200$, $t_k - t_{k-1} = \Delta t = 0.001$, through some calculation, we get $d_k = d = 0.7225$, $p = 16.0906$, $\tilde{d} = 1.2214$, $q = 132.6667$. It can be verified that the conditions in (10) hold. Therefore, Theorem 2 ensures that the hybrid-coupled complex network (11) can be synchronized to the given goal trajectory under the impulsive control scheme. The corresponding synchronization error trajectory is given in Fig. 4.

V. Conclusions

This paper has investigated the synchronization control problem of the delayed hybrid-coupled heterogeneous network with stochastic disturbances, where the linear couplings include both the discrete time-varying case and the distributed delay form. The inner coupling matrices are not necessary to be diagonal, and the outer coupling matrices are

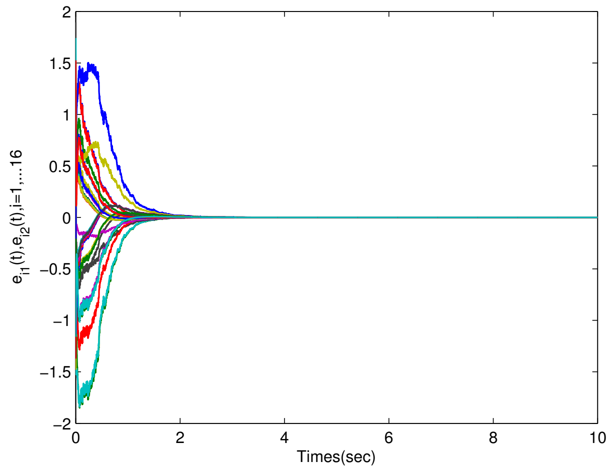

Fig. 2. Synchronization error trajectories for the system (11) under the pinning adaptive control scheme.

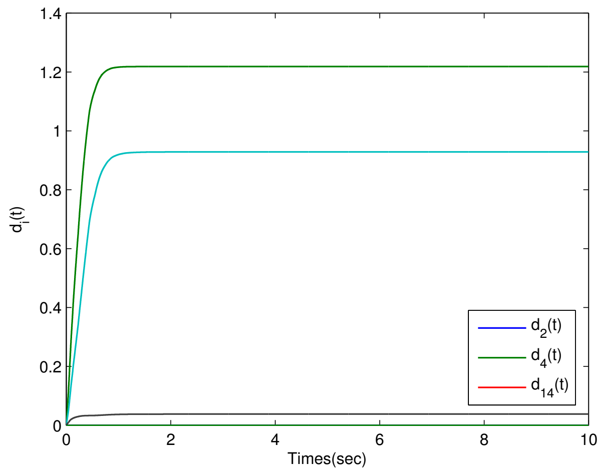

Fig. 3. The evolution of the pinning adaptive feedback gains for the system (11).

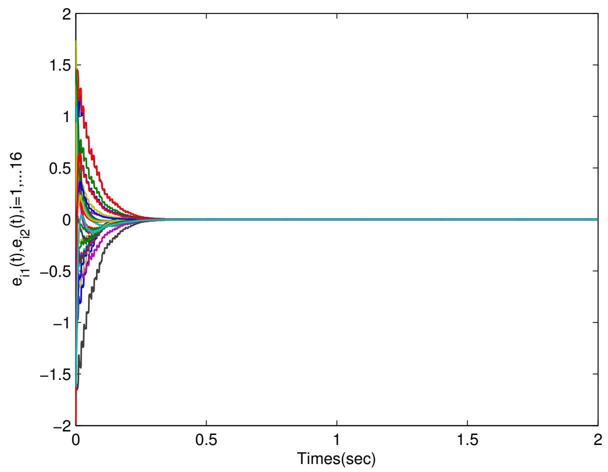

Fig. 4. Synchronization error trajectories for the system (11) under the impulsive control scheme.

just to be diffusive without restrictions on their symmetry or irreducibility. Two kinds of control schemes including pinning adaptive control and impulsive control are utilized to synchronize the whole dynamical network to the objective synchronization system.

As is well known, when pinning control the complex networks, the most challenging problems are what kinds of nodes should be pinned and what is the minimum number of the pinned nodes. Up to now, some effective pinning schemes have been proposed for the directed/undirected complex networks, as reported in the literature [28], [29], [34], [35], [36], [37].

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