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Performance Analysis of LVI-Based PDNN Applied to Real-Time Solution of Time-Varying Quadratic Programming

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***Abstract*—This paper illustrates theoretical analysis and simulative verification on the performance of the linear-variational-inequality based primal-dual neural network (LVI-PDNN), which was designed originally for static quadratic programming (QP) problem solving but is now applied to time-varying QP problem solving. It is theoretically proved that the LVI-PDNN for solving the time-varying QP problem subject to equality, inequality and bound constraints simultaneously could only approximately approach the time-varying theoretical solution, instead of converging exactly. In other words, the steady-state error of the real-time solution can not decrease to zero. In order to better evaluate the time-varying situation, we investigate the upper bound of such an error and the global exponential convergence rate for the LVI-PDNN approaching its loose error bound. Computer simulations further substantiate the performance analysis of the LVI-PDNN exploited for real-time solution of the time-varying QP problem.**

I. Time-Varying QP of Interest

Quadratic programming (QP) problems play a significant role in mathematical optimization, and have been theoretically analyzed [1][2] and extensively applied to plenty of scientific fields; e.g., optimal controller design, power-scheduling, digital signal processing, and robot-arm motion planning [3][4]. In the past, researchers usually handle optimization problems only subject to one or two kinds of constraints [5]. In addition, some QP problems are just investigated based on static coefficients (or to say, constant coefficient matrices and vectors) [6], which may not applicative for time-varying cases. Motivated by real-time engineering applications in robotics [5][7], the general time-varying QP (TVQP) in this paper is presented as follows.

$$\text{minimize} \quad x^{\mathrm{T}}(t)W(t)x(t)/2 + q^{\mathrm{T}}(t)x(t), \tag{1}$$

$$\text{subject to} \quad J(t)x(t) = d(t), \tag{2}$$

$$A(t)x(t) \leq b(t), \tag{3}$$

$$\xi^{-}(t) \leq x(t) \leq \xi^{+}(t), \tag{4}$$

where Hessian matrix $W(t) \in R^{n\times n}$ is smoothly time-varying, positive-definite and symmetric at any time instant $t \in [0, +\infty)$. Besides, coefficient matrices $J(t) \in R^{m\times n}$ and $A(t) \in R^{k\times n}$ as well as coefficient vectors $q(t) \in R^n$, $\xi^{-}(t) \in R^n$, $\xi^{+}(t) \in R^n$, $d(t) \in R^m$ and $b(t) \in R^k$ are all assumed smoothly time-varying. In time-varying QP (1)-(4), unknown vector $x(t) \in R^n$ is to be solved in real time $t \in [0, +\infty)$.

II. General Solutions to Static QP

To solve the fundamental static QP problem, a lot of methods/algorithms have been proposed [1][2]. In general, there are two common solutions to such a QP problem. The first one is the numerical algorithms performed on digital computers and it has been widely used to solve small-scale static QP problems. However, when it comes to large-scale real-time applications, in view of its serial-processing nature, such numerical algorithms may result in decline of the performances [8]. Usually, the minimal arithmetic operations are proportional to the cube of Hessian matrix dimension n, which is computationally expensive. As for the second general type of solution, the application of parallel processing has influenced the algorithmic developments [9][10]. Thus, various dynamic and analog solvers have been developed and investigated with the in-depth research of recurrent neural network (RNN). Owing to its parallel distributed nature and convenience of hardware implementation, the neural-dynamic approach is now regarded as one of the powerful alternatives to real-time computation of QP problems [11][12].

A number of neural-dynamic models have been proposed, however, according to our previous work [6], the early neural models [13][14] contain finite penalty parameters and generate approximate solutions only. Besides, Lagrange neural network has premature defect when applied to inequality-constrained QP problems. To always obtain optimal/exact solutions, traditional primal-dual neural networks were proposed based on the Karush-Kuhn-Tacker condition and projection operator [6]. However, due to minimizing the duality gap by gradient descent methods, the dynamic equations of such primal-dual network are usually complicated, even containing high-order nonlinear terms [5]. To reduce implementation and computation complexities, dual neural network was developed for solving strictly-convex static QP problems with simple piecewise linearity and global convergence to the optimal solutions [5][7].

III. Generalized LVI-PDNN Solution to TVQP

To overcome the less favorable properties/phenomena of the aforementioned neural models for solving the quadratic programming problems, a primal-dual neural network model designed based on linear variational inequality (LVI) has been developed with simple pricewise linear dynamics [9][15].

Since the LVI-based primal-dual neural network (LVI-PDNN) does not entail any matrix inversion, matrix-matrix multiplication or high-order nonlinear computation which are embodied in other researches with expensive $O(n^3)$ operations, it might reduce the implementation and computation complexities as compared with other recurrent neural models [5].

Based on the conversion of such a QP problem to an LVI and then to a system of piecewise linear equations, the LVI-PDNN solving the time-varying QP problem depicted in (1)-(4) can be generalized with its dynamics as follows [8][11][15]:

$$\dot{y}(t) = \gamma(I + H^{\mathrm{T}}(t))\left(\mathcal{P}_\Omega(y(t) - (H(t)y(t) + p(t))) - y(t)\right), \tag{5}$$

where design parameter $\gamma > 0$, being the reciprocal of a capacitance parameter, should be set as large as the hardware would permit, or selected appropriately for experimental and/or simulative purposes [6][11][16]. The coefficients are defined as

$$H(t) = \begin{bmatrix} W(t) & -J^{\mathrm{T}}(t) & A^{\mathrm{T}}(t) \\ J(t) & 0 & 0 \\ -A(t) & 0 & 0 \end{bmatrix}, \; p(t) = \begin{bmatrix} q(t) \\ -d(t) \\ b(t) \end{bmatrix}. \tag{6}$$

Besides, the primal-dual decision vector $y(t)$ as well as its lower and upper bounds are defined respectively as

$$y(t) = \begin{bmatrix} x(t) \\ u(t) \\ v(t) \end{bmatrix}, \; \varsigma^-(t) = \begin{bmatrix} \xi^-(t) \\ -\varpi 1_{\mathrm{v}} \\ 0 \end{bmatrix}, \; \varsigma^+(t) = \begin{bmatrix} \xi^+(t) \\ +\varpi 1_{\mathrm{v}} \\ +\varpi 1_{\mathrm{v}} \end{bmatrix},$$

where

- constant $\varpi \gg 0$ is sufficiently large to represent and replace $+\infty$ numerically for implementation purposes, and 1_{v} denotes an appropriately-dimensioned vector of ones;
- $x(t) \in [\xi^-(t), \xi^+(t)]$ is evidently the original decision variable vector used in primal QP (1)-(4);
- $u(t) \in R^m$ denotes the dual decision variable vector defined for equality constraint (2);
- $v(t) \in R^k$ denotes the dual decision variable vector defined for inequality constraint (3).

The vector-input vector-valued projection operator [11] $\mathcal{P}_\Omega(z(t)) = [\mathcal{P}_\Omega(z_1(t)), \mathcal{P}_\Omega(z_2(t)), \ldots, \mathcal{P}_\Omega(z_{n+m+k}(t))]^{\mathrm{T}}$ projects from $z(t) \in R^{n+m+k}$ onto set $\Omega = \{\mathbf{z}(t) | \varsigma^-(t) \leq \mathbf{z}(t) \leq \varsigma^+(t)\}$, with its scalar-input scalar-valued processing element $\mathcal{P}_\Omega(y_i(t))$ being defined as

$$\mathcal{P}_\Omega(z_i(t)) = \begin{cases} \varsigma^-(t), & z_i(t) < \varsigma^-(t) \\ z_i(t), & \varsigma^-(t) \leq z_i(t) \leq \varsigma^+(t), \\ \varsigma^+(t), & z_i(t) > \varsigma^+(t) \end{cases}$$

where $i = 1, 2, \ldots, (n + m + k)$.

It is worth pointing out that such an LVI-PDNN was designed originally for the static QP problem solving, but is now applied to the time-varying problem (1)-(4) solving. The LVI-PDNN (5) may thus generate a considerably large solution error, as reflected in Fig. 1. Facing this less favorable phenomenon, the authors have been interested in the problems inside and investigated the performance of the LVI-PDNN solver applied to time-varying QP (1)-(4).

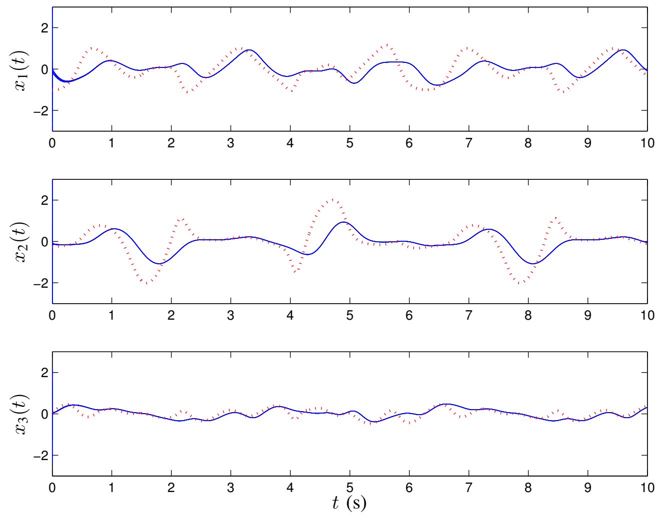


Fig. 1. Solution of time-varying QP problem (1)-(4) by LVI-PDNN (5) with $\gamma = 100$, where theoretical solution $x^*(t)$ is denoted by dotted curves

The main contributions of this paper thus lie as follows.

- In this paper, the less favorable phenomenon of LVI-PDNN solving the time-varying QP problem subject to equality, inequality and bound constraints simultaneously is pointed out formally and systematically. This research expands the QP formulation to the most general case. The results show that the conventional LVI-PDNN as a system can not solve the time-varying QP problem exactly. In other words, there always exists a steady-state solution error between the LVI-PDNN solution and the time-varying theoretical solution.
- This paper investigates and analyzes the performance of the LVI-PDNN applied to the time-varying QP problem solving. Theoretical analysis is provided rigorously for estimating the steady-state solution error bound and the exponential convergence rate when the LVI-PDNN approaching the loose outer value of the error bound (i.e., loose bound).
- Illustrative simulation examples substantiate well the theoretical analysis and results. In addition, we discuss the significance of design parameter γ and solution-variation rate ζ in the LVI-PDNN solving the time-varying QP problem.

IV. Lemmas about LVI-PDNN Solving Static QP

Summarizing the analysis results of [3][9][11], the following lemmas are presented about the global exponential convergence of the LVI-PDNN applied to solving static QP.

Lemma 1 (LVI-PDNN Convergence): Starting from any initial state $y(0) \in R^{m+n+k}$, the LVI-PDNN state vector $y(t)$ is convergent to an equilibrium point y^*, of which the first n elements constitute the optimal solution x^* to the QP problem. Besides, the exponential convergence is achieved, as there exists a constant $\varrho > 0$ such that

$$\|y - \mathcal{P}_\Omega(y - (Hy + p))\|_2^2 \geq \varrho \|y - y^*\|_2^2, \tag{7}$$

where $\|\cdot\|_2$ denotes the two norm of a vector.

Proof: It can be generalized from [6][7] as well as references therein and thereafter by using Lyapunov function candidate $\|y(t)-y^*(t)\|_2^2$ and projection-related inequalities. Note that parameter ϱ above is presented only for analysis purposes, and there is no need to know its exact value.

Lemma 2 (Important Inequality): The following inequality holds true for the LVI-PDNN solving the static QP [6]:

$$
\begin{aligned}
(y-y^*)^{\mathrm{T}}(I+H^{\mathrm{T}})(y-\mathcal{P}_\Omega(y-(Hy+p))) \geq \\
\|y-\mathcal{P}_\Omega(y-(Hy+p))\|_2^2+\|y-y^*\|_H^2.
\end{aligned}
\tag{8}
$$

Proof: Please see Eq. (25) of [6].

Lemma 3 (Cauchy Inequality): Given $a_k, b_k \in R$, with $k=1,2,\ldots,n$, the following inequality holds true:

$$
\left(\sum_{k=1}^{n} a_k b_k\right)^2 \leq \left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right).
$$

V. Time-Varying Performance Analysis

In Section III, we have found that the less favorable phenomenon occurs in the LVI-PDNN solver for time-varying QP solving. In this section, we explore the performance of LVI-PDNN (5) and present a theoretical analysis on the error bound and the exponential convergence rate.

A. Tight Error Bound

When we apply LVI-PDNN model (5) to handling the time-varying QP problem, the following theorem about its steady-state solution-error bound can be derived.

Theorem 1: Assume that the solution-variation rate is uniformly bounded as $\|\mathrm{d}(y^*(t))/\mathrm{d}t\|_2 \leq \zeta$, $\forall t \in [0,\infty)$, $\exists 0 \leq \zeta < \infty$. Starting with any initial state $y(0) \in R^{n+m+k}$, the steady-state solution error of LVI-PDNN (5) is upper bounded tightly as

$$
\lim_{t\to\infty} \sup \|y(t)-y^*(t)\|_2 \leq \frac{\zeta}{\gamma\varrho}. \tag{9}
$$

Proof: For LVI-PDNN (5), let us define solution error $e(t) = y(t)-y^*(t) \in R^{n+m+k}$; in other words, $e(t)$ denotes the difference between the LVI-PDNN solution $y(t)$ and the theoretical optimal solution $y^*(t)$. Then we have $y(t)=e(t)+y^*(t)$ and its time-derivative equation $\dot{y}(t)=\dot{e}(t)+\dot{y}^*(t)$.

Consequently, the LVI-PDNN (5) is transformed into the following dynamic equation in terms of $e(t)$ and $y(t)$:

$$
\begin{aligned}
&\dot{e}(t)=\gamma(I+H^{\mathrm{T}}(t))\cdot \\
&(\mathcal{P}_\Omega(y(t)-(H(t)y(t)+p(t)))-y(t))-\dot{y}^*(t),
\end{aligned}
\tag{10}
$$

where initial state $e(0)=y(0)-y^*(0)$. To analyze (10) as well as LVI-PDNN (5), we first define a Lyapunov function [17] candidate $\varepsilon(t)=\|e(t)\|_2^2/2$, and evidently $\varepsilon(t)$ is positive-definite in view of $\varepsilon(t)=(e^{\mathrm{T}}(t)e(t))/2>0$ for $e(t)\neq 0$ and $\varepsilon(t)=0$ for $e(t)=0$ only.

With the lemmas presented in Section IV, we derive the time-derivative of $\varepsilon(t)$ along the trajectories of (5) and (10) as follows (with argument t omitted for presentation convenience).

$$
\begin{aligned}
\dot{\varepsilon} &= \frac{\mathrm{d}\varepsilon}{\mathrm{d}t} = \frac{\mathrm{d}\|e(t)\|_2^2/2}{\mathrm{d}t} = e^{\mathrm{T}}\frac{\mathrm{d}e}{\mathrm{d}t} = e^{\mathrm{T}}\dot{e} \\
&= e^{\mathrm{T}}(\gamma(I+H^{\mathrm{T}})(\mathcal{P}_\Omega(y-(Hy+p))-y)-\dot{y}^*) \\
&= e^{\mathrm{T}}\gamma(I+H^{\mathrm{T}})(\mathcal{P}_\Omega(y-(Hy+p))-y)-e^{\mathrm{T}}\dot{y}^* \\
&= -\gamma e^{\mathrm{T}}(I+H^{\mathrm{T}})(y-\mathcal{P}_\Omega(y-(Hy+p)))-e^{\mathrm{T}}\dot{y}^* \\
&\leq -\gamma\|y-\mathcal{P}_\Omega(y-(Hy+p))\|_2^2-\gamma\|e\|_H^2-e^{\mathrm{T}}\dot{y}^* \\
&\leq -\gamma\varrho\|e\|_2^2-\gamma\|e\|_H^2-e^{\mathrm{T}}\dot{y}^* \\
&\leq -\gamma\varrho\|e\|_2^2-e^{\mathrm{T}}\dot{y}^* \leq -\gamma\varrho\|e\|_2^2+\zeta\|e\|_2 \\
&= -\|e\|_2(\gamma\varrho\|e\|_2-\zeta).
\end{aligned}
\tag{11}
$$

To understand the aforementioned proof procedure (11) better, the two important inequalities are emphasized.

- Generalizing from Lemmas 1 and 2, we have

$$
\begin{aligned}
&e^{\mathrm{T}}\gamma(I+H^{\mathrm{T}})(\mathcal{P}_\Omega(y-(Hy+p))-y) \\
&= -\gamma e^{\mathrm{T}}(I+H^{\mathrm{T}})(y-\mathcal{P}_\Omega(y-(Hy+p))) \\
&\leq -\gamma\|y-\mathcal{P}_\Omega(y-(Hy+p))\|_2^2-\gamma\|e\|_H^2 \\
&\leq -\gamma\varrho\|e\|_2^2-\gamma\|e\|_H^2 \\
&\leq -\gamma\varrho\|e\|_2^2.
\end{aligned}
\tag{12}
$$

- Generalizing from Lemma 3, we have

$$
-e^T\dot{y}^* \leq \zeta\|e\|_2. \tag{13}
$$

During the time evolution of $e(t)$ depicted in (11), it falls into one of the following three situations: (i) $\gamma\varrho\|e\|_2-\zeta>0$; (ii) $\gamma\varrho\|e\|_2-\zeta=0$; and (iii) $\gamma\varrho\|e\|_2-\zeta<0$.

- If in the time interval $[t_0,t_1)$ the trajectory of error system (10) is in the first situation [i.e., $\|e\|_2>\zeta/(\gamma\varrho)$], then $\dot{\varepsilon}(t)<0$ which implies that $e(t)$ approaches $0\in R^{n+m+k}$ [i.e., $y(t)$ approaches $y^*(t)$] as time evolves.

- If at any time t the trajectory of error system (10) is in the second situation [i.e., $\|e\|_2=\zeta/(\gamma\varrho)$, a so-called ball surface], then $\dot{\varepsilon}(t)\leq 0$ which implies that $e(t)$ approaches $0\in R^{n+m+k}$ [i.e., $y(t)$ approaches $y^*(t)$] or stays on the ball surface with $\|e\|_2=\zeta/(\gamma\varrho)$ [i.e., $\|y(t)-y^*(t)\|_2=\zeta/(\gamma\varrho)$], in view of $\dot{\varepsilon}(t)\leq 0$ containing sub-situations $\dot{\varepsilon}(t)<0$ and $\dot{\varepsilon}(t)=0$, respectively. Simply put, for this situation, $e(t)$ will not go outside the ball of $\zeta/(\gamma\varrho)$.

- For any time t at which the system trajectory falls into the third situation [i.e., $\|e\|_2<\zeta/(\gamma\varrho)$, inside the ball], it follows from (10) that $\dot{\varepsilon}(t)$ is less than a positive scalar (containing sub-situations $\dot{\varepsilon}(t)\leq 0$ and $\dot{\varepsilon}(t)>0$), and thus the distance $\|e(t)\|_2$ between $y(t)$ and $y^*(t)$ may not decrease again. Now let us analyze the worst case, i.e., $\dot{\varepsilon}(t)>0$: it is readily known that $\varepsilon(t)$ and $\|e(t)\|_2$ would increase, which increases $\gamma\varrho\|e\|_2-\zeta$ as well, as time evolves. So, there must exist a certain time instant t_2 such that $\gamma\varrho\|e\|_2-\zeta=0$, which returns to the second situation, i.e., $\dot{\varepsilon}(t)\leq 0$, and the worst is $\|e\|_2=\zeta/(\gamma\varrho)$.

For a better understanding about the above analysis, Fig. 2 is presented. Summarizing the above three situations, the steady-state solution error of LVI-PDNN (5) is upper bounded

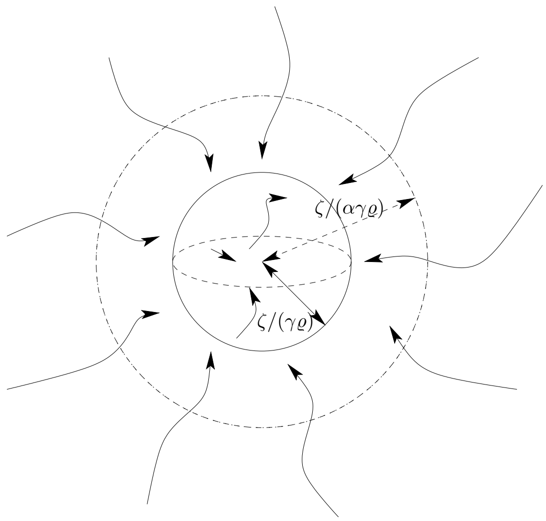


Fig. 2. Solution error $e(t)$ of LVI-PDNN (5) globally converges to the ball of $\zeta/(\gamma\varrho)$

by $\zeta/(\gamma\varrho)$); in mathematics,

$$\lim_{t\to\infty} \sup \|e(t)\|_2 = \lim_{t\to\infty} \sup \|y(t) - y^*(t)\|_2 \leq \zeta/(\gamma\varrho).$$

The proof is thus completed.

B. Exponential Convergence Rate

Theorem 1 in the preceding subsection presents a tight steady-state solution-error bound $\zeta/(\gamma\varrho)$ of LVI-PDNN (5). Evidently, it follows from (11), its analysis and Fig. 2 that the solution error $e(t)$ of LVI-PDNN (5) globally converges to the ball of $\zeta/(\gamma\varrho)$, which is asymptotic in nature [i.e., $\lim_{t\to\infty} \sup \|y(t) - y^*(t)\|_2 \leq \zeta/(\gamma\varrho)$]. However, the asymptotic convergence (AC) may not be good enough in practice, as it takes infinitely long time to reach the ball. So, in this section, we investigate (via the following theorem) the global exponential convergence rate and finite convergence time of LVI-PDNN (5) to a relatively loose error bound of $\zeta/(\alpha\gamma\varrho)$ with $0 < \alpha < 1$ chosen by LVI-PDNN users.

Theorem 2: Consider that the solution-variation rate is uniformly bounded as $\|\mathrm{d}(y^*(t))/\mathrm{d}t\|_2 \leq \zeta, \forall t \in [0, \infty), \exists 0 \leq \zeta < \infty$. Starting with any initial state $y(0) \in R^{n+m+k}$, the solution error $\|y(t) - y^*(t)\|_2$ of LVI-PDNN (5) is globally exponentially convergent to or stays within the error bound $\zeta/(\alpha\gamma\varrho)$, where $\alpha \in (0, 1)$. Besides, the exponential convergence rate is $(1-\alpha)\gamma\varrho$, and the convergence time t_{c} of LVI-PDNN (5) to a relatively loose error bound of $\zeta/(\alpha\gamma\varrho)$ is

$$t_{\mathrm{c}} = \begin{cases} \frac{\ln(\alpha\gamma\varrho\|e(0)\|_2/\zeta)}{(1-\alpha)\gamma\varrho}, & e(0) \geq \zeta/(\alpha\gamma\varrho) \\ 0. & e(0) \leq \zeta/(\alpha\gamma\varrho) \end{cases}$$

Proof: Following the proof of Theorem 1, we now show the global exponential convergence to a relatively loose error bound $\zeta/(\alpha\gamma\varrho)$ with $\alpha \in (0, 1)$. That is, (11) is rewritten as

$$\begin{aligned} \dot{\varepsilon}(t) &\leq -\gamma\varrho\|e\|_2^2 + \zeta\|e\|_2 \\ &= -(1-\alpha)\gamma\varrho\|e\|_2^2 + (-\alpha\gamma\varrho\|e\|_2^2 + \zeta\|e\|_2), \end{aligned} \tag{14}$$

where parameter $\alpha \in (0, 1)$ is termed as "a loosing ratio". Evidently, on the right-hand side of (14), the first term $-(1-\alpha)\gamma\varrho\|e\|_2^2 \leq 0$. In addition, for solution error $e(t)$ satisfying $-\alpha\gamma\varrho\|e\|_2^2 + \zeta\|e\|_2 \leq 0$ [i.e., $\|e(t)\|_2 \geq \zeta/(\alpha\gamma\varrho)$, outside or on the surface of new ball $\zeta/(\alpha\gamma\varrho)$ depicted in Fig. 2 by a dash-dotted circle], the second term on the right-hand side of (14) is dropped. Thus,

$$\begin{aligned} &\dot{\varepsilon}(t) \leq -(1-\alpha)\gamma\varrho\|e(t)\|_2^2 = -2(1-\alpha)\gamma\varrho\varepsilon(t), \\ &\varepsilon(t) \leq \exp(-2(1-\alpha)\gamma\varrho t)\varepsilon(0), \\ &\|e(t)\|_2 \leq \exp(-(1-\alpha)\gamma\varrho t)\|e(0)\|_2, \quad \forall t \in [0, t_{\mathrm{c}}] \end{aligned}$$

where the exponential convergence rate is $(1-\alpha)\gamma\varrho$, and the convergence time $t_{\mathrm{c}} = \ln(\alpha\gamma\varrho\|e(0)\|_2/\zeta)/((1-\alpha)\gamma\varrho)$ as

$$\begin{aligned} \exp(-(1-\alpha)\gamma\varrho t)\|e(0)\|_2 &= \zeta/(\alpha\gamma\varrho), \\ (1-\alpha)\gamma\varrho t_{\mathrm{c}} &= \ln(\alpha\gamma\varrho\|e(0)\|_2/\zeta). \end{aligned}$$

Note that, for error $e(t)$ entering into the new ball of $\zeta/(\alpha\gamma\varrho)$ [i.e., $\|e(t)\|_2 < \zeta/(\alpha\gamma\varrho)$], such an $e(t)$ can never leave the ball. This is in view of the first situation analysis of (11) in the proof of Theorem 1: $\dot{\varepsilon} < 0$ for any $e(t)$ outside the small ball of $\zeta/(\gamma\varrho)$. So is the situation with $\|e(0)\|_2 < \zeta/(\alpha\gamma\varrho)$, of which the resultant $e(t)$ trajectory can never leave the ball of $\zeta/(\alpha\gamma\varrho)$.

Thus, from (14) and the above analysis, defining the loosing ratio $\alpha \in (0, 1)$ and convergence time $t_{\mathrm{c}} = \ln(\alpha\gamma\varrho\|e(0)\|_2/\zeta)/((1-\alpha)\gamma\varrho)$, we have

$$\|e(t)\|_2 \leq \begin{cases} \exp\left(-(1-\alpha)\gamma\varrho t\right)\|e(0)\|_2, & \forall t \in [0, t_{\mathrm{c}}] \\ \zeta/(\alpha\gamma\varrho), & \forall t \in [t_{\mathrm{c}}, \infty) \end{cases}$$

for $e(0) \geq \zeta/(\alpha\gamma\varrho)$; and $\|e(t)\|_2 \leq \zeta/(\alpha\gamma\varrho)$, $t \in [0, \infty)$ for $e(0) \leq \zeta/(\alpha\gamma\varrho)$. Note that, even in the worst case, the exponential convergence rate is $(1-\alpha)\gamma\varrho$. The proof is thus completed.

VI. Simulation and Verification

As Section V proves, the steady-state solution error does not decrease to zero when the LVI-PDNN model is exploited for solving time-varying QP problem (1)-(4); instead, the solution error exponentially converges to a loose bound. In this section, for illustration and comparison, we present a general and illustrative example to substantiate the aforementioned theoretical analysis as well as the influence of parameter γ and solution-variation rate ζ on the LVI-PDNN exploited for solving the time-varying QP problem with the following coefficients ($n = 3$, $m = 2$ and $k = 1$):

$$W(t) = \begin{bmatrix} 2\cos\omega t + 22 & \cos\omega t - 2 & 3\sin\omega t + 6 \\ \cos\omega t - 2 & \cos 2\omega t + 12 & \sin\omega t \\ 3\sin\omega t + 6 & \sin\omega t & \cos 3\omega t + 8 \end{bmatrix},$$

$$q(t) = \left[\sin 3\omega t, \quad \cos 3\omega t, \quad -\cos 2\omega t\right]^{\mathrm{T}},$$

$$J(t) = \begin{bmatrix} 2\sin 4\omega t & \cos\omega t & \sin\omega t + 4 \\ \cos\omega t & 0.5\sin\omega t & \sin 2\omega t \end{bmatrix},$$

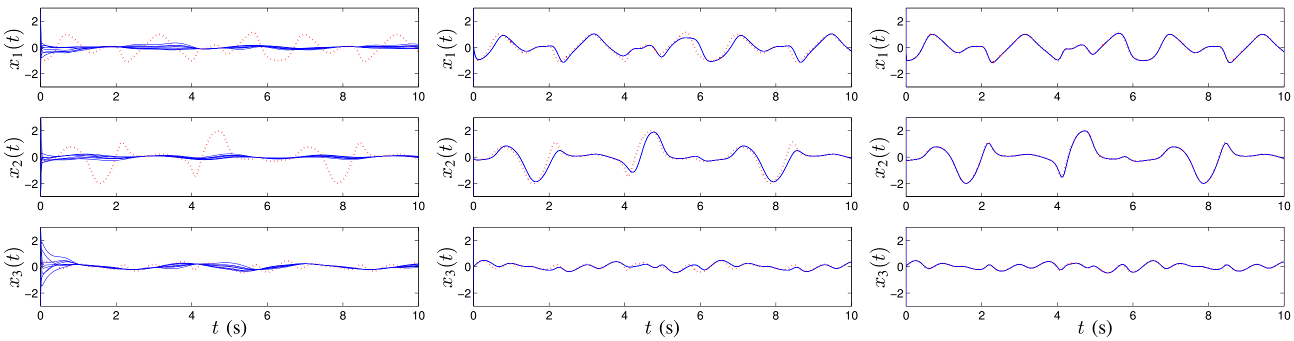
$$d(t) = \left[\sin 2\omega t, \quad -\cos 4\omega t\right]^{\mathrm{T}},$$

$$A(t) = \left[0.5\cos\omega t + 2 \quad \sin\omega t + 1 \quad \sin 4\omega t + 1\right],$$

$$b(t) = 1.5\cos\omega t + 8,$$

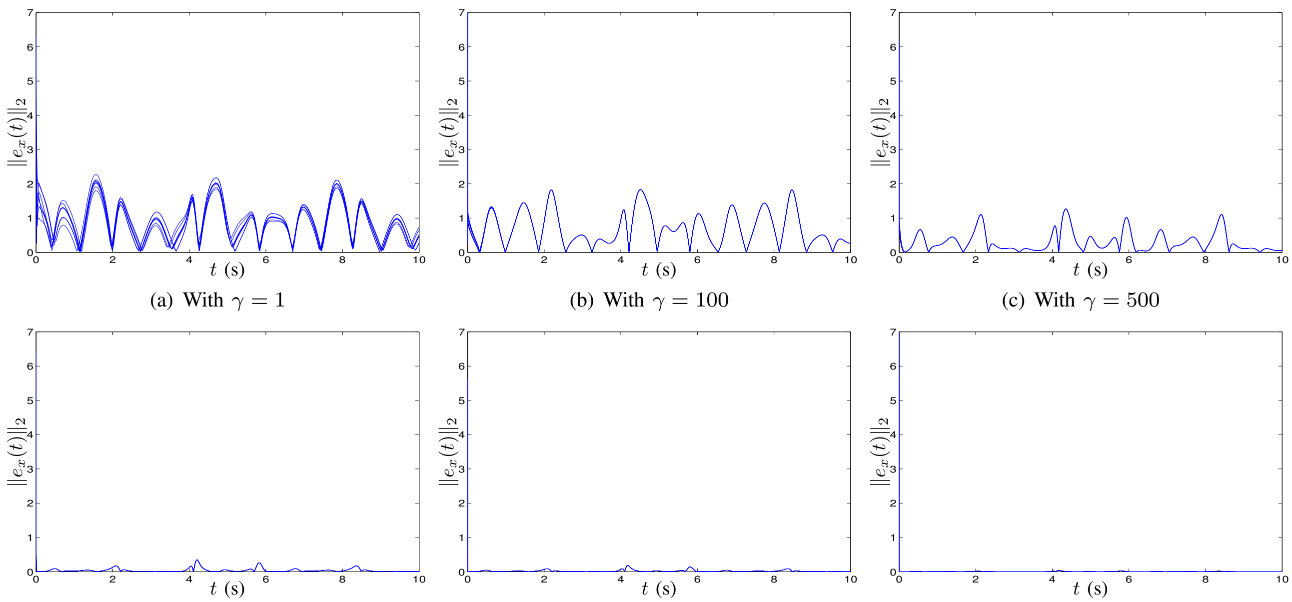
$$\xi^-(t) = \left[\cos 4\omega t - 6, \quad \sin\omega t - 6, \quad \sin(\omega t + 2) - 6\right]^{\mathrm{T}},$$

$$\xi^+(t) = \left[\cos 4\omega t + 6, \quad \sin\omega t + 6, \quad \sin(\omega t + 2) + 6\right]^{\mathrm{T}},$$



(a) With $\gamma = 1$ (b) With $\gamma = 500$ (c) With $\gamma = 5000$

Fig. 3. γ effects on real-time solution of time-varying QP problem (1)-(4) subject to equality, inequality and bound constraints simultaneously via LVI-PDNN (5) with $\omega = 1$



(a) With $\gamma = 1$ (b) With $\gamma = 100$ (c) With $\gamma = 500$

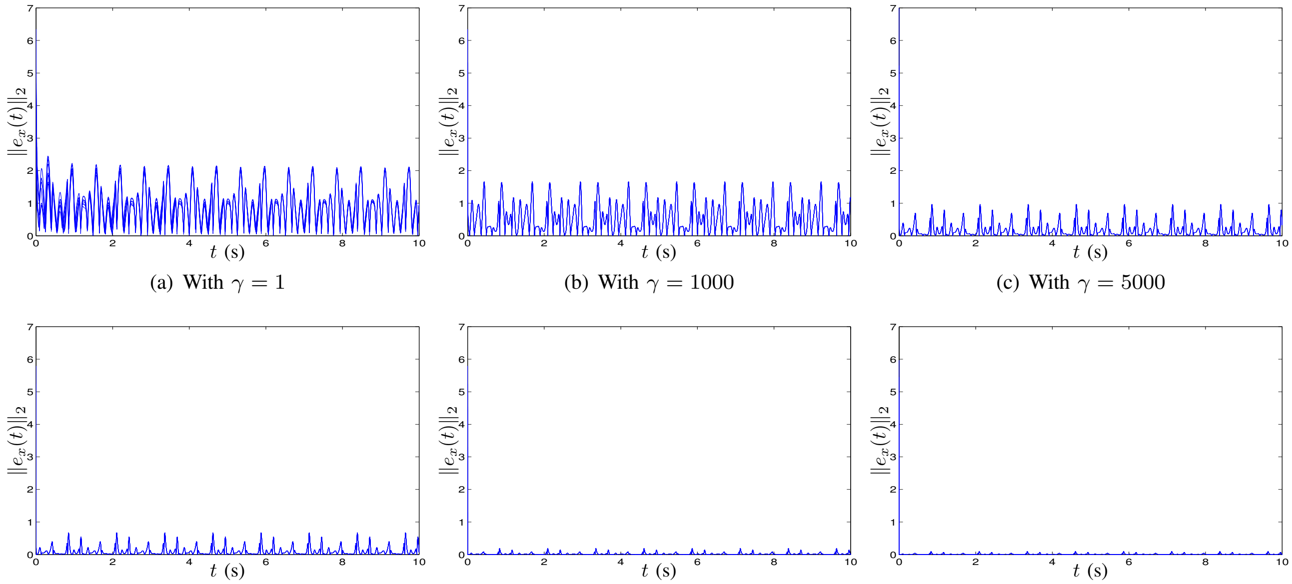
(d) With $\gamma = 5000$ (e) With $\gamma = 10000$ (f) With $\gamma = 50000$

Fig. 4. γ effects on actual error bound of LVI-PDNN solution to time-varying QP problem (1)-(4) with $\omega = 1$

where variation rate ω in the coefficient matrices and vectors is related to the solution-variation rate ζ (which is in the theoretical analysis and results of Section V).

To provide the intuitive understanding of the time-varying QP problem (1)-(4) solving synthesized by LVI-PDNN (5) , Fig. 3 shows the real-time convergence of the above problem solving with parameter $\omega = 1$ used and with parameter γ varying from 1 to 500 and to 5000, in addition to Fig. 1 with $\gamma = 100$. When the value of γ is small relatively, starting with eight randomly generated initial states $x(0) = [x_1(0), x_2(0), x_3(0)]^{\mathrm{T}}$, in which every entry is initialized within $(-5, 5)$, the LVI-PDNN (corresponding to blue solid curves) does not match well with the time-varying theoretical solution (denoted by red dotted curves) in the figures. Besides, LVI-PDNN works better as the value of γ increases.

Moreover, let us consider the actual error bounds denoted by $e_x(t) = \|x(t) - x^*(t)\|_2$ of time-varying QP solving via LVI-PDNN (5) with $\omega = 1$, where $x^*(t)$ is generated by using MATLAB function "QUADPROG" via the hypothesis of short-time invariance in a point-wise manner [3][5][6]. As shown in each subfigure of Fig. 4, starting with eight randomly generated initial states $x(0)$, the solution errors $e_x(t)$ of the LVI-PDNN always converge to some error bound. By increasing design parameter γ from 1 to 5000, the two norm of such a steady-state solution-error bound decreases from roughly 2.284 to 0.323. If γ continues to increase to 50000, such a steady-state solution-error bound decreases to roughly 0. In addition, we increase variation rate ω from 1 to 5 to simulate the convergence performance of LVI-PDNN again. Observed from each subfigure of Fig. 5, the solution errors $e_x(t)$ converge to some error bound as well. Besides, the upper bound of the steady-state solution errors in the case of $\omega = 5$ is with larger oscillation (compared to the case of $\omega = 1$). Furthermore, in the case of $\omega = 5$, by increasing γ from 1 to 5000, the two norm of the maximum steady-state solution error decreases from 2.437 to 0.964. The above results are reasonable and consistent with the presented theoretical analysis and results, because the theoretical solution of time-varying QP problem (1)-(4) changes faster when ω becomes larger, making the time-varying QP problem more difficult to be solved through the LVI-PDNN.



(a) With $\gamma = 1$ (b) With $\gamma = 1000$ (c) With $\gamma = 5000$

(d) With $\gamma = 10000$ (e) With $\gamma = 50000$ (f) With $\gamma = 100000$

Fig. 5. γ effects on actual error bound of LVI-PDNN solution to time-varying QP problem (1)-(4) with $\omega = 5$

VII. Conclusions

This paper has illustrated the performance analysis of LVI-based primal-dual neural network (LVI-PDNN) exploited to real-time solution of time-varying quadratic programming (QP) subject to equality, inequality and bound constraints simultaneously. Theoretical analysis presented in this paper has led to the fact that the LVI-PDNN model solving the time-varying QP can not converge to its theoretical solution exactly. The factors that influence the performance of LVI-PDNN have been verified by means of comparing the results of illustrative simulations (which have substantiated the performance of such an LVI-PDNN model when solving the time-varying QP problem).

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