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# Oscillation Analysis of the Solutions for a Four Coupled FHN Network Model with Delays

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***Abstract*—In this paper, the existence of oscillatory solutions for a four coupled FHN network model with delays is investigated. Some theorems to determine the oscillatory solutions for the system are obtained. The practical criteria for selecting the parameters in this network are provided. Computer simulations are also given to illustrate the effectiveness of the results.**

## I. Introduction

The FitzHugh-Nagumo model (FHN) is a simplified version of the Hodgkin and Huxley model of an excitable-oscillatory membrane which is very adequate for emulation of various biological systems from the dynamic behavior of spiking neurons to the information processing in the brain via neural firing. For the following FHN system with time delay feedback driven by two periodic signals:

$$\begin{aligned} \varepsilon x'(t) &= x(t) - \frac{1}{3}x^3(t) - y(t) \\ y'(t) &= -x(t) + a + f\cos(\omega t) + F\cos(\Omega t) \\ &\quad + k(y(t-\tau) - y(t)) \end{aligned} \tag{1}$$

Hu et al. have investigated the phenomenon of vibrational multi-resonance in system (1). The authors found that the quasi-periodic and periodic vibrational resonances in the system can be induced by the time-delay feedback [1]. With respect to system (1), a coupled FHN system may be able to better describe the dynamic characteristic such as its synchronization and stability. Therefore, the dynamic properties of various coupled FHN systems have recently been investigated by many researchers for better understanding about their collective behavior [2-7]. In 2009, Wang et al. have investigated a coupled FHN neural model as follows [2]:

$$\begin{aligned} v_1'(t) &= -v_1^3(t) + av_1(t) - w_1(t) \\ &\quad + c_1\tanh(v_2(t-\tau)) \\ w_1'(t) &= v_1(t) - b_1 w_1(t) \\ v_2'(t) &= -v_2^3(t) + av_2(t) - w_2(t) \\ &\quad + c_2\tanh(v_1(t-\tau)) \\ w_2'(t) &= v_2(t) - b_2 w_2(t) \end{aligned} \tag{2}$$

where $a, b_i, c_i (i = 1, 2)$ are constants. The authors found that time delay can control the occurrence of bifurcation in the coupled FHN neural model and synchronization is sometimes related to bifurcation transition. Fan and Hong introduced a second time delay in model (2) as the followng [3]:

$$\begin{aligned} x_1'(t) &= -x_1^3(t) + ax_1(t) - x_2(t) \\ &\quad + c_1\tanh(x_3(t-\tau_1)) \\ x_2'(t) &= x_1(t) - b_1 x_2(t) \\ x_3'(t) &= -x_3^3(t) + ax_3(t) - x_4(t) \\ &\quad + c_2\tanh(x_1(t-\tau_2)) \\ x_4'(t) &= x_3(t) - b_2 x_4(t) \end{aligned} \tag{3}$$

Defining $\tau = \tau_1 + \tau_2$ as a parameter, the authors have shown that there is a critical value for this parameter, and a Hopf bifurcation occurs. The oscillations induced by the Hopf bifurcation appeared when the parameter passed through the critical value. Zhen and Xu generalized models (2) and (3) to a three coupled FHN neurons with time delay as follows [5]:

$$\begin{aligned} u_1' &= -\frac{1}{3}u_1^3 + cu_1^2 + du_1 - u_2 + \alpha u_1^2 \\ &\quad + \beta[f(u_3(t-\tau)) + f(u_5(t-\tau))] \\ u_2' &= \varepsilon(u_1 - bu_2) \\ u_3' &= -\frac{1}{3}u_3^3 + cu_3^2 + du_3 - u_4 + \alpha u_3^2 \\ &\quad + \beta[f(u_1(t-\tau)) + f(u_5(t-\tau))] \\ u_4' &= \varepsilon(u_3 - bu_4) \\ u_5' &= -\frac{1}{3}u_5^3 + cu_5^2 + du_5 - u_6 + \alpha u_5^2 \\ &\quad + \beta[f(u_1(t-\tau)) + f(u_3(t-\tau))] \\ u_6' &= \varepsilon(u_5 - bu_6) \end{aligned} \tag{4}$$

where $\alpha$ and $\beta$ represent the synaptic strength of the self-connection and of the neighborhood-interaction, respectively, $f(x)$ is a sufficiently smooth sigmoid amplification function. In system (4), Zhen and Xu considered a quadratic term as the self-connection function to simulate the influence of the chemical synaptic coupling which does not alter the stability of the resting state of system (4). For this system the authors have discussed the Bautin bifurcation which is also known as the generalized Hopf bifurcation. Bautin bifurcation arises at the transition between sub-and super-critical Hopf bifurcations has the property of appearing two limit circles for the parameters near Bautin bifurcation values. The parameters conditions for the Bautin bifurcation of the synchronous solution of system (4) by applying the Bautin bifurcation theorem of

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delay differential equations have been given. Since for each neuron the synaptic strength of the self-connection $\alpha$ and the neighborhood-interaction $\beta$ are the same, the authors pointed out that the dynamics of system (4) is completely characterized by the following system:

$$\begin{aligned} u_1' &= -\frac{1}{3}u_1^3 + (c+\alpha)u_1^2 + du_1 - u_2 \\ &\quad +2\beta f(u_1(t-\tau)) \\ u_2' &= \varepsilon(u_1 - bu_2) \end{aligned} \tag{5}$$

where $[u_1, u_2]^T$ represents a completely synchronous solution of system (4). The Bautin bifurcation of synchronous solution for this neural system (5) is investigated. However, generally speaking, the synaptic strength of self-connection, neighborhood-interaction for each neuron and the time delays are different. In this paper, we discuss the following four coupled FHN network model:

$$\begin{aligned} u_1' &= -\frac{1}{3}u_1^3 + c_1u_1^2 + d_1u_1 - u_2 + \alpha_1u_1^2 \\ &\quad +\beta_1[f(u_3(t-\tau_3)) + f(u_5(t-\tau_5)) \\ &\quad +f(u_7(t-\tau_7))] \\ u_2' &= \varepsilon_1(u_1 - b_1u_2) \\ u_3' &= -\frac{1}{3}u_3^3 + c_2u_3^2 + d_2u_3 - u_4 + \alpha_2u_3^2 \\ &\quad +\beta_2[f(u_1(t-\tau_1)) + f(u_5(t-\tau_5)) \\ &\quad +f(u_7(t-\tau_7))] \\ u_4' &= \varepsilon_2(u_3 - b_2u_4) \\ u_5' &= -\frac{1}{3}u_5^3 + c_3u_5^2 + d_3u_5 - u_6 + \alpha_3u_5^2 \\ &\quad +\beta_3[f(u_1(t-\tau_1)) + f(u_3(t-\tau_3)) \\ &\quad +f(u_7(t-\tau_7))] \\ u_6' &= \varepsilon_3(u_5 - b_3u_6) \\ u_7' &= -\frac{1}{3}u_7^3 + c_4u_7^2 + d_4u_7 - u_8 + \alpha_4u_7^2 \\ &\quad +\beta_4[f(u_1(t-\tau_1)) + f(u_3(t-\tau_3)) \\ &\quad +f(u_5(t-\tau_5))] \\ u_8' &= \varepsilon_4(u_7 - b_4u_8) \end{aligned} \tag{6}$$

where $b_i, c_i, d_i, \alpha_i, \beta_i, \varepsilon_i (i = 1, 2, 3, 4)$ are constants. $\tau_j > 0 (j = 1, 3, 5, 7)$ represent the time delays in signal transmission. System (6) can be rewritten as the follows:

$$\begin{aligned} u_1' &= [-\frac{1}{3}u_1^2 + (c_1+\alpha_1)u_1 + d_1]u_1 - u_2 \\ &\quad +\beta_1[f(u_3(t-\tau_3)) + f(u_5(t-\tau_5)) \\ &\quad +f(u_7(t-\tau_7))] \\ u_2' &= \varepsilon_1u_1 - \gamma_1u_2, \\ u_3' &= [-\frac{1}{3}u_3^2 + (c_2+\alpha_2)u_3 + d_2]u_3 - u_4 \\ &\quad +\beta_2[f(u_1(t-\tau_1)) + f(u_5(t-\tau_5)) \\ &\quad +f(u_7(t-\tau_7))], \\ u_4' &= \varepsilon_2u_3 - \gamma_2u_4, \\ u_5' &= [-\frac{1}{3}u_5^2 + (c_3+\alpha_3)u_5 + d_3]u_5 - u_6 \\ &\quad +\beta_3[f(u_1(t-\tau_1)) + f(u_3(t-\tau_3)) \\ &\quad +f(u_7(t-\tau_7))], \\ u_6' &= \varepsilon_3u_5 - \gamma_3u_6. \\ u_7' &= [-\frac{1}{3}u_7^2 + (c_4+\alpha_4)u_7 + d_4]u_7 - u_8 \\ &\quad +\beta_4[f(u_1(t-\tau_1)) + f(u_3(t-\tau_3)) \\ &\quad +f(u_5(t-\tau_5))], \\ u_8' &= \varepsilon_4u_7 - \gamma_4u_8. \end{aligned} \tag{7}$$

where $\gamma_i = \varepsilon_i b_i (i = 1, 2, 3, 4)$. It must be emphasized that it is hard to deal with system (7) using a bifurcating approach since the delays $\tau_1, \tau_3, \tau_5$, and $\tau_7$ are different positive constants. On the one hand, to find a completely synchronous solution of system (7) is difficult when the $\alpha_i$ and $\beta_i (i = 1, 2, 3, 4)$ are different. On the other hand, to find the bifurcating parameter is problematical when $\tau_j (j = 1, 3, 5, 7)$ are different. In this paper, we use Chafee's criterion to discuss the oscillatory behavior of the solutions for system (7) [8]: For a class of time delay system which has a unique unstable equilibrium point, and all solutions of this system are bounded. Then the system generals a limit cycle, namely an oscillatory solution. This system (7) can indeed obey to Chafee's criterion we refer reader to [9, appendix] for more information on this point. In the sequel, we will provide some restrictive conditions which are easy to check to ensure the existence of oscillatory solutions.

We first assume that $f_j(u_j(t-\tau_j)) (j = 1, 3, 5, 7)$ are continuous monotone bounded functions, satisfying

$$\lim_{u_j \to 0} \frac{f_j(u_j(t))}{u_j(t)} = l_j, f_j(0) = 0 \quad j = 1, 3, 5, 7. \tag{8}$$

For example, the activation functions $\tanh(u_i(t))$, $\arctan(u_i(t))$, and $\frac{1}{2}(|u_i(t)+1| - |u_i(t)-1|)$ satisfy condition (8). From this assumption (8), the linearization of system (7) about the zero point leads to the following:

$$\begin{aligned} u_1' &= d_1u_1 - u_2 + \beta_1[l_3(u_3(t-\tau_3)) \\ &\quad + l_5(u_5(t-\tau_5)) + l_7(u_7(t-\tau_7))] \\ u_2' &= \varepsilon_1 u_1 - \gamma_1 u_2 \\ u_3' &= d_2u_3 - u_4 + \beta_2[l_1(u_1(t-\tau_1)) \\ &\quad + l_5(u_5(t-\tau_5)) + l_7(u_7(t-\tau_7))] \\ u_4' &= \varepsilon_2 u_3 - \gamma_2 u_4 \\ u_5' &= d_3u_5 - u_6 + \beta_3[l_1(u_1(t-\tau_1)) \\ &\quad + l_3(u_3(t-\tau_3)) + f(u_7(t-\tau_7))] \\ u_6' &= \varepsilon_3 u_5 - \gamma_3 u_6 \\ u_7' &= d_4u_7 - u_8 + \beta_4[l_1(u_1(t-\tau_1)) \\ &\quad + l_3(u_3(t-\tau_3)) + l_5(u_5(t-\tau_5))] \\ u_8' &= \varepsilon_4 u_7 - \gamma_4 u_8 \end{aligned} \tag{9}$$

The matrix form of system (9) is the follows:

$$U'(t) = PU(t) + QU(t-\tau) \tag{10}$$

where $U(t) = [(u_1(t), u_2(t), ..., u_8(t))]^T$, $U(t-\tau) = [(u_1(t-\tau_1), 0, u_3(t-\tau_3), 0, u_5(t-\tau_5), 0, u_7(t-\tau_7), 0)]^T$.

$$P = (p_{ij})_{8\times 8} = \begin{pmatrix} d_1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \varepsilon_1 & -\gamma_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & d_2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \varepsilon_2 & -\gamma_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & d_3 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \varepsilon_3 & -\gamma_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & d_4 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \varepsilon_4 & -\gamma_4 \end{pmatrix},$$

$$Q = (q_{ij})_{8\times 8} = \begin{pmatrix} 0 & 0 & \beta_1 l_3 & 0 & \beta_1 l_5 & 0 & \beta_1 l_7 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \beta_2 l_1 & 0 & 0 & 0 & \beta_2 l_5 & 0 & \beta_2 l_7 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \beta_3 l_1 & 0 & \beta_3 l_3 & 0 & 0 & 0 & \beta_3 l_7 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \beta_4 l_1 & 0 & \beta_4 l_3 & 0 & \beta_4 l_5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

## II. Preliminaries

*Lemma 1* Suppose that $b_i > 0, 0 < \varepsilon_i \ll 1, d_i < 0, 3(c_i + \alpha_i)^2 + 4d_i < 0 (i = 1,2,3,4)$, then each solution of system (6) (or (7)) is bounded.

*Proof* Noting that $f_j(u_j(t-\tau_j)) (j = 1,3,5,7)$ are continuous monotone bounded functions, therefore, there exist $N_j > 0$ such that $|f_j(u_j(t-\tau_j))| \le N_j (j = 1,3,5,7)$. Since $d_i < 0, 3(c_i+\alpha_i)^2 + 4d_i < 0 (i = 1,2,3,4)$, this implies that there exist constants $k_i > 0$ such that for any values $u_i$ we have

$$\begin{aligned} & -\frac{1}{3}u_{2i-1}^2 + (c_i+\alpha_i)u_{2i-1} + d_i \\ = & -\frac{1}{3}[u_{2i-1} - \frac{3}{2}(c_i+\alpha_i)]^2 + \frac{3}{4}(c_i+\alpha_i)^2 + d_i \\ \le & -k_i < 0 \quad (i = 1,2,3,4). \end{aligned} \tag{11}$$

Noting that $\gamma_i = \varepsilon_i b_i > 0$ $(i = 1,2,3,4)$, from (7) when $u_i(t) \ge 0$ we get $|u_i(t)| = u_i(t) (i = 1,2,...,8)$, and $\frac{d|u_1(t)|}{dt} = \frac{du_1(t)}{dt} \le -k_1u_1 - u_2 + |\beta_1|(N_3+N_5+N_7) \le -k_1|u_1| + |u_2| + |\beta_1|(N_3+N_5+N_7)$. While as $u_i(t) < 0$ we get $|u_i(t)| = -u_i(t) (i = 1,2,...,8)$. and $\frac{d|u_1(t)|}{dt} = -\frac{du_1(t)}{dt} \le -k_1(-u_1) - (-u_2) + |\beta_1|(N_3+N_5+N_7) \le -k_1|u_1| + |u_2| + |\beta_1|(N_3+N_5+N_7)$. Similarly, we always have

$$\begin{aligned} \frac{d|u_1(t)|}{dt} &\le -k_1|u_1| + |u_2| + |\beta_1|(N_3+N_5+N_7) \\ \frac{d|u_2(t)|}{dt} &\le -\gamma_1|u_2| + \varepsilon_1|u_1|, \\ \frac{d|u_3(t)|}{dt} &\le -k_2|u_3| + |u_4| + |\beta_2|(N_1+N_5+N_7) \\ \frac{d|u_4(t)|}{dt} &\le -\gamma_2|u_4| + \varepsilon_2|u_3|, \\ \frac{d|u_5(t)|}{dt} &\le -k_3|u_5| + |u_6| + |\beta_3|(N_1+N_3+N_7) \\ \frac{d|u_6(t)|}{dt} &\le -\gamma_3|u_6| + \varepsilon_3|u_5|, \\ \frac{d|u_7(t)|}{dt} &\le -k_4|u_7| + |u_8| + |\beta_4|(N_1+N_3+N_5) \\ \frac{d|u_8(t)|}{dt} &\le -\gamma_4|u_8| + \varepsilon_4|u_7| \end{aligned} \tag{12}$$

Since system (12) is a first order linear system of equations with constant coefficients, the eigenvalues of system (12) can be obtained by setting $v_i = |u_i| (i = 1,2,...,8)$, and first considering

$$\begin{aligned} v_1' &= -k_1v_1 + v_2 + |\beta_1|(N_3+N_5+N_7) \\ v_2' &= -\gamma_1 v_2 + \varepsilon_1 v_1, \end{aligned} \tag{13}$$

Since $|\beta_1|(N_3+N_5+N_7)$ is a constant, thus we only discuss the associated homogeneous system with (13) as follows

$$\begin{aligned} v_1' &= -k_1v_1 + v_2 \\ v_2' &= -\gamma_1 v_2 + \varepsilon_1 v_1, \end{aligned} \tag{14}$$

The eigenvalues of system (14) is the follows:

$$\lambda_{11,12} = \frac{-(k_1+\gamma_1) \pm \sqrt{(k_1+\gamma_1)^2 - 4k_1\varepsilon_1 - 4\varepsilon_1}}{2} \tag{15}$$

Noting that $k_1 > 0, 0 < \varepsilon_1 \ll 1, b_1 > 0$, thus $-(k_1+\gamma_1) < 0$ and $\lambda_{11,12} < 0$ if $(k_1+\gamma_i)^2 - 4k_1\varepsilon_1 - 4\varepsilon_1 > 0$, or $\lambda_{11,12}$ are complex numbers with $Re\lambda_{11,12} < 0$ if $(k_1+\gamma_1)^2 - 4k_1\varepsilon_1 - 4\varepsilon_1 < 0$. This implied that the solutions of system (14) and consequently the solutions of (13) are bounded since $|\beta_1|(N_3+N_5+N_7)$ is a constant. In other words, the solutions $u_1$ and $u_2$ are bounded in system (12). Similarly, we get other eigenvalues of system (12) as follows:

$$\lambda_{i1,i2} = \frac{-(k_i+\gamma_i) \pm \sqrt{(k_i+\gamma_i)^2 - 4k_i\varepsilon_i - 4\varepsilon_i}}{2} \quad (i = 2,3,4) \tag{16}$$

Based on the assumptions that $k_i > 0, 0 < \varepsilon_i \ll 1, b_i > 0$ $(i = 2,3,4)$, this leads $-(k_i+\gamma_i) < 0$ and $\lambda_{i1,i2} < 0$ if $(k_i+\gamma_i)^2 - 4k_1\varepsilon_1 - 4\varepsilon_i > 0$, or $\lambda_{i1,i2}$ are complex numbers with $Re\lambda_{i1,i2} < 0$ if $(k_i+\gamma_i)^2 - 4k_1\varepsilon_1 - 4\varepsilon_i < 0$ $(i = 2, 3, 4)$. Therefore, all the solutions of system (12) and consequently the solutions of system (6) are bounded.

*Lemma 2* Suppose that the matrix $A(= P + Q)$ is nonsingular, then system (9) has a unique equilibrium point.

*Proof* An equilibrium point $U^* = [u_1^*, u_2^*, ..., u_8^*]^T$ is the solution of the following algebraic equation

$$PU^* + QU^* = (P + Q)U^* = AU^* = 0 \tag{17}$$

Assuming that $U^*$ and $V^*$ are two equilibrium points of system (9), then we have

$$(P + Q)(U^* - V^*) = A(U^* - V^*) = 0 \tag{18}$$

Since $A$ is a nonsingular matrix, this implies that $U^* - V^* = 0$, and consequently $U^* = V^*$. Thus system (9) has a unique equilibrium point. Obviously, this equilibrium point is exactly the zero vector. Noting that the activation functions are continuous monotone bounded functions, satisfying that $f_j(0) = 0$ $(j = 1, 3, 5, 7)$. Therefore, system (9) has a unique equilibrium point implies that system (6) also has a unique equilibrium point. By means of Chafee's criterion of limit cycle, this paper discusses the oscillatory behavior of the solutions for a four coupled FHN neurons model where the synaptic strength of the self-connection, the neighborhood-interaction for each neuron and the time delays are different. In the following we provide some restrictive conditions to ensure that the unique equilibrium point of system (6) is unstable.

## III. Oscillation analysis

In this paper, we adopt the following norms of vectors, matrices and measure of a matrix [11]: $\|u(t)\| = \sum_{i=1}^{8} |u_i(t)|, \|P\| = max_j \sum_{i=1}^{8} |p_{ij}|, \|Q\| = max_j \sum_{i=1}^{8} |q_{ij}|, \mu(P) = lim_{\theta \to 0} \frac{\|E+\theta P\|}{\theta}$, which for the chosen norms reduce to $\mu(P) = max_{1 \le j \le 8}[p_{jj} + \sum_{i=1, i \ne j}^{8} |p_{ij}|]$. For more details of the measure of a matrix we refer to [11] and [12].

*Theorem 1* Suppose that $\beta_i > 0, b_i > 0, 0 < \varepsilon_i \ll 1, d_i < 0, 3(c_i + \alpha_i)^2 + 4d_i < 0 (i = 1, 2, 3, 4)$, the activation functions are continuous monotone increasing functions and $A = P + Q$ is a nonsingular matrix. Assume that $\mu(P)$ satisfies

$$|\mu(P)| < \|Q\| \tag{19}$$

then the trivial solution of system (9) is unstable, implying that the trivial solution of system (6) is also unstable. There exists an oscillatory solution for system (6).

*Proof* From the assumptions, we know that the restrictive conditions of Lemma 1 and Lemma 2 are satisfied, each solution of system (6) is bounded and system (9) has a unique equilibrium point. We shall prove that this unique equilibrium point is unstable. Since $\beta_i > 0 (i = 1, 2, 3, 4)$, and the activation functions are monotone increasing functions, so $l_j > 0 (j = 1, 3, 5, 7)$. In other words, each entry $q_{ij}$ of the matrix $Q$ is positive or zero. The proof is accomplished by means of a Lyapunov functional $V(t, u) = (V_1(t, u), V_2(t, u), ..., V_8(t, u))^T$, where

$$V_i(t, u) = |u_i(t)| + \sum_{j=1}^{8} q_{ij} \int_{t-\tau_j}^{t} |u_j(s)ds|, \quad (i = 1, 2, ..., 8) \tag{20}$$

when $u_i(t) > 0$ we have $|u_i(t)| = u_i(t)$ and $V_i(t, u) = u_i(t) + \sum_{j=1}^{8} q_{ij} \int_{t-\tau_j}^{t} u_j(s)ds$. Calculating the derivative of $V_i(t, u)$ through system (9) as $u_i(t) > 0$, we get

$$\begin{aligned} & V_1(t, u)|_{(9)} \\ = \; & u_1'(t) + \sum_{j=1}^{8} q_{1j}(u_j(t) - u_j(t - \tau_j)) \\ = \; & d_1u_1 - u_2 + \beta_1[l_3(u_3(t - \tau_3)) + l_5(u_5(t - \tau_5)) \\ & + l_7(u_7(t - \tau_7))] + \beta_1(l_3u_3 + l_5u_5 + l_7u_7) \\ & - \beta_1[l_3(u_3(t - \tau_3)) - l_5(u_5(t - \tau_5)) \\ & - l_7(u_7(t - \tau_7))] \\ = \; & d_1u_1 - u_2 + \beta_1(l_3u_3 + l_5u_5 + l_7u_7) \end{aligned} \tag{21}$$

$$V_2(t, u)|_{(9)} = u_2'(t) = \varepsilon_1u_1 - \gamma_1u_2 \tag{22}$$

Similarly, we have

$$V_3(t, u)|_{(9)} = d_3u_3 - u_4 + \beta_2(l_1u_1 + l_5u_5 + l_7u_7) \tag{23}$$

$$V_4(t, u)|_{(9)} = \varepsilon_2u_3 - \gamma_2u_4. \tag{24}$$

$$V_5(t, u)|_{(9)} = d_5u_5 - u_6 + \beta_3(l_1u_1 + l_3u_3 + l_7u_7) \tag{25}$$

$$V_6(t, u)|_{(9)} = \varepsilon_3u_5 - \gamma_3u_6. \tag{26}$$

$$V_7(t, u)|_{(9)} = d_7u_7 - u_8 + \beta_4(l_1u_1 + l_3u_3 + l_5u_5) \tag{27}$$

$$V_8(t, u)|_{(9)} = \varepsilon_4u_7 - \gamma_4u_8. \tag{28}$$

So, the derivative of $V(t, u)$ through system (9) as $u_i(t) > 0 (i = 1, 2, ..., 8)$ can be represented as a matrix form

$$\begin{aligned} & V(t, u)|_{(9)} \\ = \; & (V_1(t, u)|_{(9)}, V_2(t, u)|_{(9)}, ..., V_8(t, u)_{(9)})^T \\ = \; & PU(t) + QU(t) \end{aligned} \tag{29}$$

While as $u_i(t) < 0$ then $|u_i(t)| = -u_i(t)$ and $V_i(t, u) = -u_i(t) + \sum_{j=1}^{8} q_{ij} \int_{t-\tau_j}^{t} (-u_j(s))ds = -[u_i(t) + \sum_{j=1}^{8} q_{ij} \int_{t-\tau_j}^{t} u_j(s)ds]$, and

$$\begin{aligned} & V_1(t, u)|_{(9)} \\ = \; & -[u_1'(t) + \sum_{j=1}^{8} q_{1j}[u_j(t) - (u_j(t - \tau_j))] \\ = \; & -[d_1u_1 - u_2 + \beta_1(l_3u_3 + l_5u_5 + l_7u_7)] \\ = \; & d_1(-u_1) - (-u_2) + \beta_1(l_3(-u_3) \\ & + l_5(-u_5) + l_7(-u_7)) \end{aligned} \tag{30}$$

So as $u_i(t) < 0$ we get

$$V(t, u)|_{(9)} = P(-U(t)) + Q(-U(t)) \tag{31}$$

Thus, we always have

$$V(t, u)|_{(9)} = P|U(t)| + Q|U(t)| \tag{32}$$

From (32), since $|\mu(P)| < \|Q\|$, therefor for any $u_i(t) \ne 0$, we have $V(t, u)|_{(9)} > 0$, and the trivial solution of system (9) is unstable. Similar to [10, Theorem 4.1.1], one can prove that the trivial solution or the unique equilibrium point of

system (6) is also unstable. Since all solutions of system (6) are bounded, according to Chafee's criterion, there exists an oscillatory solution of system (6).
In the following, we consider the case that some $\beta_i(i = 1, 2, 3, 4)$ are not larger than zero.

*Theorem 2* Suppose that $b_i > 0, 0 < \varepsilon_i \ll 1, d_i < 0, 3(c_i+\alpha_i)^2+4d_i < 0 (i = 1, 2, 3, 4)$, the activation functions are continuous monotone functions and $A$ is a nonsingular matrix. Let $\rho_k = \rho_{k1} + i\rho_{k2}(\rho_{k2}$ may equal to zero) and $\sigma_k = \sigma_{k1} + i\sigma_{k2}(\sigma_{k2}$ may equal to zero) $(k = 1, 2, ..., 8)$ denote the eigenvalues of matrices $P$ and $Q$ respectively. If for some $k, |\rho_{k1}| < \sigma_{k1}$, then the trivial solution of system (9) is unstable, implying that system (6) has an oscillatory solution.

*Proof* The assumptions guarantee that system (9) has a unique equilibrium point. Setting $\tau^* = \min\{\tau_1, \tau_3, \tau_5, \tau_7\}$, and considering first $\tau_i = \tau(\tau \leq \tau^*, i = 1, 3, 5, 7)$ in system (9). Since $\rho_k$ and $\sigma_k(\sigma_{k2}\ (k = 1, 2, ..., 8)$ are eigenvalues of matrices $P$ and $Q$ respectively, then the characteristic equation of system (9) is the following:

$$\prod_{k=1}^{8}(\lambda - \rho_k - \sigma_k e^{-\lambda\tau}) = 0 \tag{33}$$

Without loss of generality, we assume that the $\rho_k$ satisfying the conditions of Theorem 2 is $\rho_1$. Therefore, we consider the following equation:

$$\lambda - \rho_1 - \sigma_1 e^{-\lambda\tau} = 0 \tag{34}$$

Let $\lambda = \lambda_1 + i\lambda_2$. Separating the real and imaginary parts from (34) yields the following two equations:

$$\lambda_1 - \rho_{11} - \sigma_{11}e^{-\lambda_1\tau}\cos(\lambda_2\tau) = 0 \tag{35}$$

$$\lambda_2 - \rho_{12} + \sigma_{12}e^{-\lambda_1\tau}\sin(\lambda_2\tau) = 0 \tag{36}$$

We shall show that $\lambda_1 > 0$, and there is an eigenvalue which has positive real part in system (34). Let $f(\lambda_1) = \lambda_1 - \rho_{11} - \sigma_{11}e^{-\lambda_1\tau}\cos(\lambda_2\tau)$, then $f(\lambda_1)$ is a continuous function of $\lambda_1$. If $\rho_{11} > 0$, then we can select a suitable delay $\tau$ such that $\sigma_{11}\cos(\lambda_2\tau) > -\rho_{11}$. Therefore, $f(0) = -\rho_{11} - \sigma_{11}\cos(\lambda_2\tau) < 0$. Noting that $e^{-\lambda_1\tau} \to 0$ as $\lambda_1 \to +\infty$, there exists a suitably large $\lambda_1(> 0)$ such that $f(\lambda_1) = \lambda_1 - \rho_{11} - \sigma_{11}e^{-\lambda_1\tau}\cos(\lambda_2\tau) > 0$. By the continuity of $f(\lambda_1)$, there exists a positive $\lambda_1^* \in (0, \lambda_1)$ such that $f(\lambda_1^*) = 0$. If $\rho_{11} < 0$, since $|\rho_{11}| < \sigma_{11}(\sigma_{11} \neq 0)$, then there exists a suitable delay $\tau_0$ and a positive $\lambda_0$ such that $\sigma_{11}\cos(\lambda_2\tau_0) < -\rho_{11}$, and $\lambda_0 - \rho_{11} - \sigma_{11}e^{-\lambda_0\tau_0}\cos(\lambda_2\tau_0) < 0$ both hold. Subsequently $f(0) = -\rho_{11} - \sigma_{11}\cos(\lambda_2\tau_0) > 0$, and $f(\lambda_0) = \lambda_0 - \rho_{11} - \sigma_{11}e^{-\lambda_0\tau_0}\cos(\lambda_2\tau_0) < 0$. Again from the continuity of $f(\lambda_1)$, there exists a positive $\lambda_1^{**} \in (0, \lambda_0)$ such that $f(\lambda_1^{**}) = 0$. Thus, there is an eigenvalue of system (34) that has a positive real part. This implies that the trivial solution of system (9) is unstable and consequently the trivial solution of system (6) is unstable. Since all solutions of system (6) are bounded, according to the Chafee's criterion, system (6) generates an oscillatory solution. Based on the theory of delayed differential equation [13, 14], the instability of the solution will be maintained as the time delay increases. Therefore, for any $\tau_i \geq \tau(i = 1, 3, 5, 7)$, the trivial solution of system (9) is still unstable, implying that the trivial solution of system (6) is also unstable. So, for any $\tau_i \geq \tau(i = 1, 3, 5, 7)$, system (6) generates an oscillatory solution. In these conditions since we select a suitable delay $\tau$ such that system (6) has an oscillatory solution, we can refer to this process as a delay induced oscillation.

## IV. Simulation Results

In system (6), the parameter values are fixed as $\alpha_1 = -1.6, \alpha_2 = -1.3, \alpha_3 = -1.2, \alpha_4 = -1.1; b_1 = 0.16, b_2 = 0.25, b_3 = 0.12, b_4 = 0.15; c_1 = 1.3, c_2 = 1.302, c_3 = 1.305, c_4 = 1.308; d_1 = -0.695, d_2 = -0.698, d_3 = -0.704, d_4 = -0.708; \beta_1 = 1.5, \beta_2 = 0.5, \beta_3 = 0.45, \beta_4 = 0.18; \varepsilon_1 = 0.05, \varepsilon_2 = 0.025, \varepsilon_3 = 0.085, \varepsilon_4 = 0.035$, respectively. It is easily checked that the conditions of Lemma 1 and Lemma 2 hold. The activation function is selected as $\tanh(u)$. In this case, $\gamma_1 = \gamma_3 = \gamma_5 = \gamma_7 = 1$, and $|\mu(P)| = 0.994, \|Q\| = 2.35$. Therefore, $|\mu(P)| < \|Q\|$. Based on Theorem 1, there exists an oscillatory solution (see Fig.1). In order to compare the effect of the time delays, in Fig. 2 we changed the time delays from (1, 2, 3, 2) to (4, 7, 9, 6), the other parameters are kept the same as in Figure 1. We see that, in these conditions, the oscillatory frequency decreases when the delays are increased. Both in Fig. 3 and Fig. 4, apart from time delays, we take the following parameters: $\alpha_1 = -1.3, \alpha_2 = -1.6, \alpha_3 = -1.4, \alpha_4 = -1.5; b_1 = 0.25, b_2 = 0.16, b_3 = 0.24, b_4 = 0.18; c_1 = 1.268, c_2 = 1.582, c_3 = 1.894, c_4 = 1.685; d_1 = -0.5, d_2 = -0.48, d_3 = -0.65, d_4 = -0.68; \beta_1 = 1.18, \beta_2 = 0.6, \beta_3 = 0.75, \beta_4 = 0.42; \varepsilon_1 = 0.085, \varepsilon_2 = 0.095, \varepsilon_3 = 0.075, \varepsilon_4 = 0.098$, respectively. The activation function is selected as $\arctan(u)$. We still keep $\gamma_1 = \gamma_3 = \gamma_5 = \gamma_7 = 1$. The eigenvalues of matrices $P$ and $Q$ are $\rho_1 = -0.1763, \rho_2 = -0.4917, \rho_3 = -0.4586, \rho_4 = -0.2374, \rho_5 = -0.2606 + 0.1664i, \rho_6 = -0.2606 - 0.1664i, \rho_7 = -0.2476 + 0.2025i, \rho_8 = -0.2476 - 0.2025i$, and $\sigma_1 = 2.1341, \sigma_2 = -0.9937, \sigma_3 = -0.4721, \sigma_4 = -0.6638, \sigma_5 = 0, \sigma_6 = 0, \sigma_7 = 0, \sigma_8 = 0$, respectively. Since $|\rho_1| = 0.1763 < \sigma_1$, there is an oscillatory solution as predicted by Theorem 2. From the figures, we see that the frequency of oscillation also decreases as delays are increased. In order to show the effect of $\beta_i$, in Fig. 5, we only change the values of $\beta_2 = -0.6, \beta_4 = -0.42$, the values $\beta_1, \beta_3$ and the other parameters are all the same as in Fig. 3 and Fig. 4. The oscillation is still maintained, however, the oscillatory frequency is higher.

## V. Conclusion

This paper discusses a four coupled FHN neurons model where the synaptic strength of the self-connection, the neighborhood-interaction for each neuron and the time delays are different. By using Chafee's criterion of limit cycle, two theorems are provided to determine the oscillatory behavior of the solutions. The present theoretical developments provide a new set of conditions to investigate the nature of excitable cell models and develop a better understanding of the dynamic behavior of networks made up of spiking neurons.

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Fig. 1 Oscillatory behavior of the solutions, activation function: tanh(u), delays: (1, 2, 3, 2).

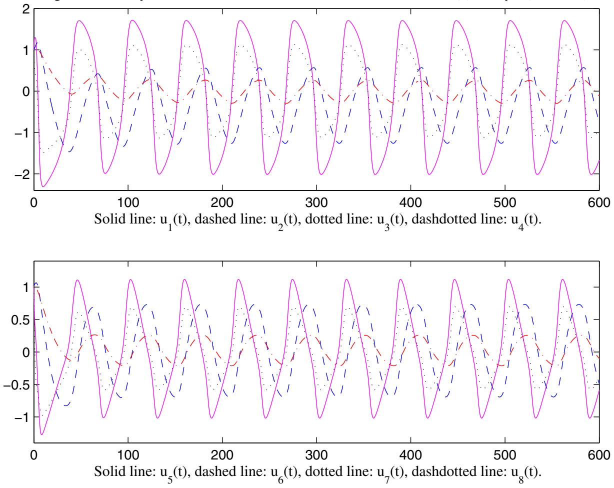



Fig. 2 Oscillatory behavior of the solutions, activation function: tanh(u), delays: (4, 7, 9, 6).

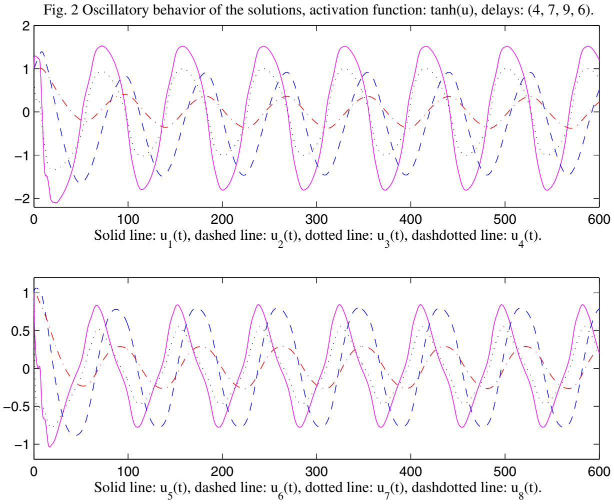



Fig. 3 Oscillatory behavior of the solutions, activation function: arctan (u), delays: (1, 2, 3, 2)

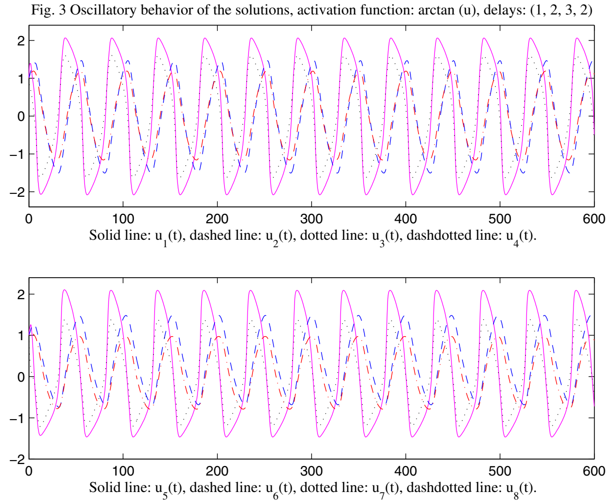



Fig. 4 Oscillatory behavior of the solutions, activation function: arctan (u), delays: (4, 8, 6, 9)

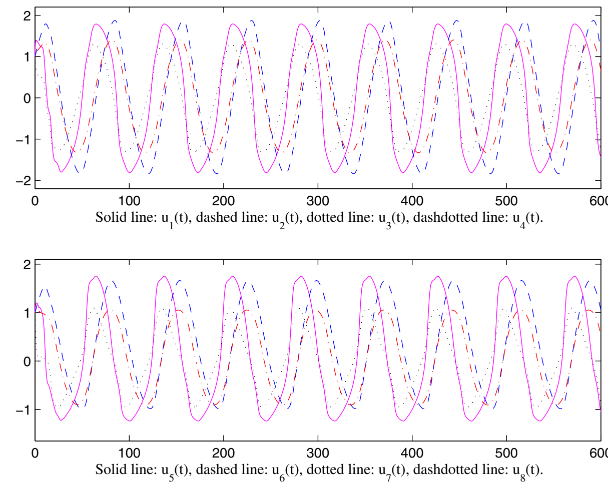
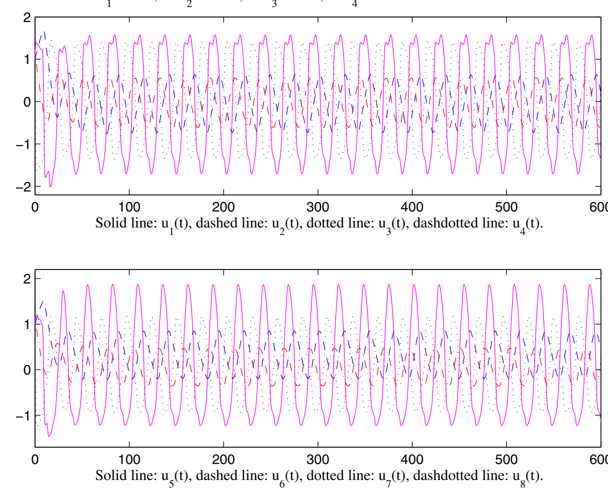



Fig. 5 Oscillatory behavior of the solutions, activation function: arctan(u), delays: (4, 8, 6, 9). $\beta_1 = 1.18$, $\beta_2 = -0.6$, $\beta_3 = 0.42$, $\beta_4 = -0.42$.

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