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A Review on Evolution of Lyapunov-Krasovskii Function in Stability Analysis of Recurrent Neural Networks with Single Time-varying Delay

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Abstract— **In the stability analysis of recurrent neural networks, one of the tasks is to reduce the conservativeness of the stability criterion. Along this routine, there are two ways to be considered. One is how to construct the Lyapunov-Krasovskii functional (LKF), and the other is how to use mathematical skills to estimate the derivatives of the LKF. The purpose of this paper is to present a brief review on the evolution on the construction of LKF for recurrent neural networks with single time-varying delay. By summarizing the observation, one can find the core elements in the construction of LKF. Moreover, one can find the evolution history on the delay-partitioning and its applications in the construction of LKF.**

I. Introduction

In the past few decades, neural networks (NNs) have received increasing interest owing to their applications in a variety of areas, such as signal processing, pattern recognition, static image processing, associative memory, combinatorial optimization and scientific areas. Up to now, stability of NNs with a time delay has been received attention since time delay is frequently encountered in NNs, and it is often a source of instability and oscillations in a system. Therefore, the stability of delayed neural networks has been investigated with considerable interest, and some stability criteria have been reported in the literature [1]–[24].

The stability criteria obtained can be generally classified into delay-independent and delay-dependent. Since the time delays encountered in neural networks are usually not very big, delay-dependent criteria, which include the information of time delays, are less conservative. As most of such criteria are derived via the Lyapunov theory, they all have a certain degree of conservatism. Reducing the conservatism has been the topic of much research [2], [3], [5], [6], [7].

With the Lyapunov stability theory, the reduction can be achieved mainly from two phases: choosing the Lyapunov-Krasovskii functional (LKF) and estimating its derivative [5]. The choice of the LKF is crucial for deriving less conservative criteria. Various types of the LKF have been constructed to discuss the delayed neural networks and they are reviewed as follows. 1) Discretized LKF: it reduces the conservatism of a simple type LKF, which uses one matrix in the whole delay interval, by partitioning the whole interval of the delay existing in the delayed neural networks into several subintervals and using different matrices in different subintervals. 2) Augmented LKF: it is constructed by augmenting the terms of the simple type LKF using state vectors and delayed state vectors of the delayed neural networks, the integral and the derivative of the state vectors, and so on. This technique introduces slack matrices (free-weighting matrices) into the LKF and thus it can reduce the conservatism by optimizing the combination of those matrices. 3) Delay-partitioning LKF: after rewriting the original delayed neural networks by the system with several additive delays, such LKF improves the results by applying the ideas of the discretized and augmented LKF together, i.e., partitioning the whole interval of the time delay and augmenting the terms of the simple LKF by the state vectors with subinterval delays. 4) LKF based on other ideas, such as the LKF including the slope of activation functions of the delayed neural networks] and the LKF including triple integral terms. Those functionals lead to less conservative results because of adding more model information into the LKF. For the derivative of the LKF, it is necessary to estimate the derivative for obtaining the criteria in terms of linear matrix inequalities (LMIs). Three main techniques, including the free-weighting matrix (FWM) approach, Jensen inequality and convex optimization approach, and their combinations, have been applied. The enlargement of the derivative of the LKF is commonly needed to deal with the integral terms of the derivative. The time delay introduced in the neural networks is mainly time-varying and bounded, for which the enlargement is usually caused by replacing the time-varying delay and/or its distance.

Motivated by above discussions, this paper will present a brief review on the construction of LKF for a kind of recurrent neural networks with single time-varying delay. By observing the evolution of the LKF, one may find some useful information in the construction of the LKF, which will help the readers to choose the suitable functional in the stability analysis of the delayed neural networks. These observations may also be helpful to deal with the stability and stabilization problems for nonlinear systems with delays.

II. Problem Description and Preliminaries

Throughout this paper, we will consider the following classical Hopfield-type recurrent neural networks with single

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time-varying delay,

$$
\begin{aligned}
\dot{x}_i(t) =& -a_i x(t) + \sum_{j=1}^{n} w_{0ij} f_j(x_j(t)) \\
&+ \sum_{j=1}^{n} w_{1ij} f_j(x_j(t-\tau(t))),
\end{aligned} \tag{1}
$$

or in a compact matrix-vector form,

$$
\dot{x}(t) = -Ax(t) + W_0 f(x(t)) + W_1 f(x(t-\tau(t))), \tag{2}
$$

where $A = \mathrm{diag}(a_1, \cdots, a_n)$, $a_i > 0$, $W_0 = (w_{0ij})_{n\times n}$, $W_1 = (w_{1ij})_{n\times n}$, neuronal state $x(t) = (x_1(t), \cdots, x_n(t))^T$, activation function $f(x(t)) = (f_1(x_1(t)), \cdots, f_n(x_n(t)))^T$. $\tau(t)$ is a time varying delay. w_{0ij} denotes the constant connection weight of the i-th neuron on the j-th neuron at time t, w_{1ij} denotes the constant connection weight of the i-th neuron on the j-th neuron at time $t-\tau(t)$, $a_i > 0$ represents the rate with which the i-th neuron will reset its potential to the resting state in isolation when disconnected from the network and external inputs. $\tau(t)$ is nonnegative, bounded, and differentiable. The initial condition of (2) is supplemented by $x(\theta) = \varphi(\theta), \theta \in [-h_2, 0]$ with φ being an initial function.

In model (2), self-feedback matrix A, interconnection matrix W_0 and W_1 are usually known, which can be directly used in the construction of LKF and estimation of derivative of LKF. Therefore, there is no much space to be exploited in the interconnection matrices. However, different kind of activation function $f(x(t))$ and different kind of requirement on the time-varying delay may have significant influence on the stability analysis of the model (2). Therefore, in the following, we will give some assumption on the activation function and time varying delay.

Assumption D0: $\tau(t)$ is nonnegative, bounded, and differentiable with $0 \le \tau(t) \le h$ and $\dot{\tau}(t) \le \mu$, $h > 0, \mu \ge 0$ are positive known constants.

Assumption D1: $\tau(t)$ is nonnegative, bounded, and differentiable with $0 \le h_1 \le \tau(t) \le h_2$, $\dot{\tau}(t) \le \mu$, $h > 0, \mu \ge 0$ are positive known constants.

Assumption A0: Activation function is a global Lipschitz function, i.e.

$$
|f_i(s_1) - f_i(s_2)| \le l_i |s_1 - s_2|, \tag{3}
$$

where $l_i > 0$ is a known constant.

Assumption A1: Activation function satisfies

$$
0 \le \frac{f_i(s_1) - f_i(s_2)}{s_1 - s_2} \le l_i, \tag{4}
$$

where $l_i > 0$ is a known constant.

Assumption A2: Activation function satisfies

$$
l_i^- \le \frac{f_i(s_1) - f_i(s_2)}{s_1 - s_2} \le l_i^+, \tag{5}
$$

where l_i^- and $l_i^+ (i = 1, 2, \cdots, n)$ are known real scalars and they may be positive, negative, or zero, which means that the resulting activation functions may be non-monotonic [8].

Based on above assumptions, we will analyze and compare the different construction of LKF in the literature.

III. Different Construction of LKF Based on Different Assumption

In the following, 4 cases will be discussed for the construction of LKF for model (2).

Case 1) Assumption D0 and Assumption A0 hold.

This kind of combination is the classical assumption in the early study on the stability of recurrent neural networks with delays. In this Case 1, some M-matrix based stability results are established in [1], in which the following functional were established respectively,

$$
V_{c1}(t) = \sum_{i=1}^{n} r_i \Big(|x_i(t)| + \sum_{j=1}^{n} |w_{1ij}| l_j \int_{t-\tau(t)}^{t} x_j(s) ds \Big), \tag{6}
$$

with $\dot{\tau}(t) \le 0$, and

$$
V_{c2}(t) = \sum_{i=1}^{n} r_i \Big(|x_i(t)| + \sum_{j=1}^{n} |w_{1ij}| l_j \int_{t}^{\sigma(t)} x_j(s-\tau(s)) ds \Big), \tag{7}
$$

with $\tau(t)$ is differential function with $\tau(0) = 0$, $0 \le \dot{\tau}(t) < 1$, and $\sigma(t)$ is a differential inverse function of $t - \tau(t)$.

It should be pointed out that the assumption of $0 \le \dot{\tau}(t) < 1$ stems from the need to bound the growth of variations in the delay factor as a function of time in the viewpoint of mathematics. Moreover, because the M-matrix based stability results have no more variants in the expressions, therefore, it has been replaced by the LMI-based stability results in recent years.

In general case, if the activation function is in the form as that in (3), the corresponding stability results are usually in the M-matrix form or algebraic inequality form.

Observing the LKF (6) and (7), one can see that the second term in (6) and (7) is used to compensate the effect of delayed term in the estimation of the derivative of LKF (6) and (7). Coefficients r_i are used to satisfy the requirements of M-matrix, and no more slack variables can be inserted in the M-matrix based stability results.

Case 2 Assumption D0 and Assumption A1 hold.

This kind of combination is also the classical assumption in the early study on the stability of recurrent neural networks with delays. For example, using the Lyapunov diagonal stability method and conventional Lyapunov stability theory, Assumption A1 is always required.

In this Case 2, for the neural network (2), reference [2] gave the following LKF,

$$
\begin{aligned}
V_{HDD}(t) =& V_{h1}(t) + V_{h2}(t) + V_{h3}(t), \\
V_{h1}(t) =& x^T(t) P x(t) + 2 \sum_{i=1}^{n} \rho_i \int_{0}^{x_i(t)} f_i(s) ds
\end{aligned}
$$

$$V_{h2}(t) = \int_{t-\tau(t)}^{t} [x^T(s)Q_1x(s) + f^T(x(s))Q_2f(x(s))]ds + \int_{t-h_2}^{t} x^T(s)Q_3x(s)ds,$$

$$V_{h3}(t) = \int_{-h_2}^{0}\int_{t+\theta}^{t} \dot{x}^T(s)Z_1\dot{x}(s)dsd\theta, \quad (8)$$

and a novel global asymptotic stability criterion was derived. In the estimation of the derivative of LKF (8), the main contribution of [2] is to compute the term $\int_{-h_2}^{0}\int_{t+\theta}^{t}\dot{x}^T(s)Z_1\dot{x}(s)dsd\theta$ purposely, in which its derivative is often estimated as $h_2\dot{x}^T(t)Z_1\dot{x}(t) - \int_{t-\tau(t)}^{t} x^T(s)Q_3x(s)ds$, and the term $-\int_{t-h_2}^{t-\tau(t)} x^T(s)Q_3x(s)ds$ is ignored. This treatment may lead to considerable conservativeness.

Observing LKF (8), one can see that the time-varying delay $\tau(t)$ and its upper bound h_2, and the activation function form are simultaneously used, while the maximum upper bound of change rate of activation function l_i is not involved in the LKF (8).

Reference [3] proposed the following LKF to study the stability of (2) in the case,

$$\begin{aligned} V_{ZDD}(t) =& V_{z1}(t) + V_{z2}(t) + V_{z3}(t), \\ V_{z1}(t) =& x^T(t)Px(t) + 2\sum_{i=1}^{n}\rho_i\int_0^{x_i(t)} f_i(s)ds \\ &+ 2\sum_{i=1}^{n}\rho_i\int_0^{x_i(t)}(l_is - f_i(s))ds \\ V_{z2}(t) =& \int_{t-\tau(t)}^{t}[x^T(s)Q_1x(s) + f^T(x(s))Q_2f(x(s))]ds \\ &+ \int_{t-h_2}^{t} x^T(s)Q_4x(s)ds \end{aligned}$$

$$V_{z3}(t) = \int_{-h_2}^{0}\int_{t+\theta}^{t}\dot{x}^T(s)Z_1\dot{x}(s)dsd\theta, \quad (9)$$

By using the Jensens integral inequality and the Projection lemma, some global asymptotic stability criteria were established. Obviously, the delay information and the activation function information are simultaneously used, especially the maximum upper bound of change rate of activation function l_i is involved in the LKF (9).

Reference [4] proposed the following LKF to study the stability of (2 in the case 2) with $\dot{\tau}(t) = 0$,

$$\begin{aligned} V_{SADD}(t) =& x^T(t)Px(t) + 2\alpha\sum_{i=1}^{n}\rho_i\int_0^{x_i(t)} f_i(s)ds \\ &+ 2(\alpha\gamma + \beta)\sum_{i=1}^{n}\rho_i\int_{t-\tau}^{t} f_i^2(x_i(s))ds \end{aligned} \quad (10)$$

By using some inequalities to estimate the derivative of $V_{SADD}(t)$, some less conservative global asymptotical stability results were established.

It is clear that LKF (10) is a special case of LKF (8) and LKF (9), which means that LKF (10) is the core element in the construction of effective LKF.

For the neural network (2), reference [5] gave the following LKF,

$$\begin{aligned} V_{ZDD}(t)|_k :=& V(x(t))|_{\tau(t)\in[\tau_{k-1},\tau_k]}, \\ V_{ZDD}(t)|_k =& V_{Z1}(t) + V_{Z2}(t) + V_{Z3}(t) + V_{Z4}(t), \end{aligned}$$

$$\begin{aligned} V_{Z1}(t) =& x^T(t)Px(t) + 2\sum_{i=1}^{n}\rho_i\int_0^{x_i(t)} f_i(s)ds, \\ V_{Z2}(t) =& \sum_{j=1}^{m}\int_{t-\tau_j}^{t-\tau_{j-1}}[x^T(s), f^T(x(s))]R_j[x^T(s), f^T(x(s))]^Tds, \\ V_{Z3}(t) =& \sum_{j=1}^{m}\delta_j\int_{-\tau_j}^{-\tau_{j-1}}\int_{t+\theta}^{t}\dot{x}^T(s)Z_j\dot{x}(s)dsd\theta, \\ V_{Z4}(t) =& \sum_{j=1}^{k-1}\int_{t-\tau_j}^{t-\tau_{j-1}}[x^T(s), f^T(x(s))]Q_j[x^T(s), f^T(x(s))]^Tds \\ &+ \int_{t-\tau(t)}^{t-\tau_{k-1}}[x^T(s), f^T(x(s))]Q_j[x^T(s), f^T(x(s))]^Tds, \end{aligned} \quad (11)$$

where $0 = \tau_0 < \tau_1 < \tau_2 < \cdots < \tau_m = h_2$, then the delay interval $[0, h_2]$ is decomposed into m segments, $\delta_j = \tau_j - \tau_{j-1}$. For any $t \geq 0$, there should exist an integer $k \in \{1, 2, \cdots, m\}$ such that $\tau(t) \in [\tau_{k-1}, \tau_k]$.

It is noted that a novel term $V_{Z4}(t)$ that is continuous at $\tau(t) = \tau_k$ is included in the Lyapunov-Krasovskii functional (11), which plays an important role in reducing conservativeness of the derived result. In the Lyapunov-Krasovskii functional (11), the different matrix pairs (Q_j, R_j, Z_j) can be chosen on the different delay intervals $[\tau_{j-1}, \tau_j]$.

In previous works, considerable attention has been paid to the case that the derivative of the time-varying delay $\tau(t)$ satisfies $\dot{\tau}(t) \leq \mu$. In fact, $\dot{\tau}(t)$ may have different upper bounds in various delay intervals, $\dot{\tau}(t) \leq \mu_k, \tau(t) \in [\tau_{k-1}, \tau_k], k = 1, 2, \cdots, m$. In this case, the traditional treatment in the estimation of the derivative of time varying delay is $\dot{\tau}(t) \leq \mu = \max\{\mu_1, \cdots, \mu_m\}$, which may lead to conservativeness inevitably. However, the case above can be taken fully into account by employing the Lyapunov-Krasovskii functional (11).

Observing the LKF (11), one can see that the LKF is a sum of a set of sub-LKF on the piece of time delay or fractal delay. That is, the global LKF is divided into many fractal local LKF, which may make good use of the information in different segment of time delay. This treatment is very similar with the sampling control in the control community. Correspondingly, many slack variables can be inserted into the solution space.

Note that, in the previous references before the end of 2007 [6], time delay $\tau(t)$ satisfies the interval $[0, h_2]$, and in the treatment of $[0, \tau(t)]$, it is usually enlarged as h_2. In fact, another term $h_2 - \tau(t)$ was also regarded as h_2. Therefore, the practical interval delay $\tau(t) \in [0, h_2]$ was enlarged as $[0, 2h_2]$

in the estimation of the derivative of LKF, which may lead to a conservative result. In fact, due to the continuous of time varying delay, $h_2 = \tau(t) + (h_2 - \tau(t))$ holds. Then in the estimation of the derivative of LKF, h_2 can be decomposed into the sum of $\tau(t)$ and $(h_2 - \tau(t))$, which may decrease the conservativeness. On the other hand, the range of time-varying delay for NNs considered in previous references is from 0 to an upper bound. In practice, a time-varying interval delay is often encountered, that is, the range of delay varies in an interval for which the lower bound is not restricted to 0. In this case, the stability criteria for NNs with time-varying delay in previous references are conservative because they do not take into account the information of the lower bound of delay [6]. This is the following Case 3 to be discussed.

Case 3 Assumption D1 and Assumption A1 hold.

In the real world, time-varying delay is a positive and bounded function. Therefore, a usual way to describe the delay is to restrict the delay in a interval. In theory, the delay may attenuate to zero, which can incorporate the case of no delay. In fact, system with or without delay is different in essence. Therefore, different problem has different way. Based on these understanding, it is reasonable to make the Assumption D1 hold.

In this case, reference [6] gave the following LKF for the neural network (2),

$$
\begin{aligned}
V_{HhDD}(t) &= V_{hh1}(t) + V_{hh2}(t) + V_{hh3}(t),\\
V_{hh1}(t) &= x^T(t)Px(t) + 2\sum_{i=1}^{n}\rho_i\int_0^{x_i(t)} f_i(s)ds\\
V_{hh2}(t) &= \int_{t-\tau(t)}^{t}[x^T(s)Q_1x(s) + f^T(x(s))Q_2f(x(s))]ds\\
&\quad + \int_{t-h_1}^{t} x^T(s)Q_3x(s)ds + \int_{t-h_2}^{t} x^T(s)Q_4x(s)ds,\\
V_{hh3}(t) &= \int_{-h_2}^{0}\int_{t+\theta}^{t}\dot{x}^T(s)Z_1\dot{x}(s)dsd\theta\\
&\quad + \int_{-h_2}^{-h_1}\int_{t+\theta}^{t}\dot{x}^T(s)Z_2\dot{x}(s)dsd\theta, \qquad (12)
\end{aligned}
$$

and a novel global asymptotic stability criterion was derived.

In this Case 3, reference [7] considered the neural network (2) with two additive time-varying delay components, i.e., $\tau(t) = \tau_1(t) + \tau_2(t)$, and constructed a general LKF. In fact, if the two additive time-varying delay $\tau_1(t) + \tau_2(t)$ is regarded as a single time varying delay $\tau(t)$, the corresponding LKF in [7] is just the same (12). for ,

Observing the LKF (12) [6] and in [7], the common feature is that the interval delay information is involved in the LKF (12), and the conventional term of activation function is also involved in (12). Obviously, the core in (12) is also the first functional $V_{hh1}(t)$.

Traditionally, the activation functions are assumed to be continuous, differentiable, monotonically increasing and bounded, such as the function of sigmoid-type. However, in many electronic circuits, the input-output functions of amplifiers may be neither monotonically increasing nor continuously differentiable, hence non-monotonic functions can be more appropriate to describe the neuron activation in designing and implementing an artificial neural network [8]. In the following, the general activation function will be discussed.

Case 4 Assumption D1 and Assumption A2 hold.

In Assumption (5), l_i^- and l_i^+ are allowed to be positive, negative or zero. Hence, the resulting activation functions could be non-monotonic, and more general than the usual sigmoid functions, and Assumption (4) is a special case of Assumption (5).

Since Assumption (5) may assume neither differentiability nor strict monotonicity, then Assumption (5) improves the Assumption (4). At the same time, Assumption (5) can be equivalent to the Assumption (3), in which both activation function may be non-monotonic. It is the Assumption (5) that may bridge the stability results of delayed neural networks in different kind of expression form.

Generally speaking, Assumption (3) is mainly suitable for the stability results in the M-matrix form and algebraic inequality dorm, while Assumption (4) and Assumption (5) are suitable for the stability results in the matrix inequality form, especially the linear matrix inequality form.

In the case of constant delay, reference [8] constructed the following LKF to study the global exponential stability of model (2),

$$
\begin{aligned}
V_{LDD}(t) =& e^{2kt}x^T(t)Px(t) + e^{2kt}\int_{t-\tau}^{t} f^T(x(s))Q_1f(x(s))ds\\
&+ e^{2kt}\int_0^{\tau}\int_{t-\theta}^{t} f^T(x(s))Z_1f(x(s))dsd\theta.
\end{aligned} \qquad (13)
$$

Obviously, the main elements in (13) are the same as those in LKF (8) and LKF (9) and LKF (10).

Reference [9] discussed the exponential stability for the neural network (2), and the following LKF was proposed,

$$
\begin{aligned}
&V_{WDD}(t) = V_{w1}(t) + V_{w2}(t) + V_{w3}(t),\\
&V_{w1}(t) = x^T(t)Px(t)\\
&V_{w2}(t)\\
&= \int_{t-\tau(t)}^{t} e^{\alpha(s-t)}(x^T(s)\ f^T(x(s)))Q_1(x^T(s)\ f^T(x(s)))^Tds\\
&\quad + \int_{t-\sigma}^{t} e^{\alpha(s-t)}(x^T(s)\ f^T(x(s)))Q_2(x^T(s)\ f^T(x(s)))^Tds\\
&\quad + \int_{t-h_2}^{t} e^{\alpha(s-t)}(x^T(s)\ f^T(x(s)))Q_3(x^T(s)\ f^T(x(s)))^Tds\\
&\quad + \int_{t-h_1}^{t} e^{\alpha(s-t)}(x^T(s)\ f^T(x(s)))Q_4(x^T(s)\ f^T(x(s)))^Tds\\
&V_{w3}(t) = \int_{-\sigma}^{0}\int_{t+\theta}^{t} e^{\alpha(s-t)}\dot{x}^T(s)Z_1\dot{x}(s)dsd\theta\\
&\qquad + \int_{-\sigma}^{-h_1}\int_{t+\theta}^{t} e^{\alpha(s-t)}\dot{x}^T(s)Z_2\dot{x}(s)dsd\theta, \quad (14)
\end{aligned}
$$

where $\sigma = (h_1 + h_2)/2$.

Comparing the LKF (14) with LKF (8)-LKF (13), the mean value of subinterval of delay was involved in the LKF (14), which meant that more delay information was used in the LKF.

Reference [10] studied the following recurrent neural networks with time delay,

$$\dot{x}(t) = -Ax(t) + W_0 f(W_2 x(t)) + W_1 f(W_2 x(t - \tau(t))), \tag{15}$$

which has combined the local field model and static model in a unified framework. In the delay-independent stability analysis, by introducing more information on the activation functions of the neurons into the chosen Lyapunov-Krasovskii functional (LKF), a new delay-independent stability criterion is obtained in terms of a simple linear matrix inequality (LMI). In the delay-dependent stability analysis, by employing an integral inequality and convex combination technique, some novel delay-dependent stability criteria are derived.

Specially, in reference [10], the following LKFs were proposed, respectively,

$$\begin{aligned} V_{HDI}(t) =& x^T(t)Px(t) + \int_{t-\tau(t)}^{t} x^T(s)Qx(s)ds \\ &+ 2\sum_{i=1}^{n} \rho_i \int_{0}^{W_{2i}x(t)} [f_i(s) - l_i^-]ds \\ &+ 2\sum_{i=1}^{n} \delta_i \int_{0}^{W_{2i}x(t)} [l_i^+ - f_i(s)]ds, \end{aligned} \tag{16}$$

and

$$\begin{aligned} V_{HDD}(t) =& V_{h1}(t) + V_{h2}(t) + V_{h3}(t), \\ V_{h1}(t) =& x^T(t)Px(t) + \int_{t-\tau(t)}^{t-h_1} x^T(s)Qx(s)ds \\ &+ 2\sum_{i=1}^{n} \rho_i \int_{0}^{W_{2i}x(t)} [f_i(s) - l_i^-]ds \\ &+ 2\sum_{i=1}^{n} \delta_i \int_{0}^{W_{2i}x(t)} [l_i^+ - f_i(s)]ds, \\ V_{h2}(t) =& \int_{t-h_1}^{t} x^T(s)Q_1x(s)ds + \int_{t-h_2}^{t-h_1} x^T(s)Q_2x(s)ds, \\ V_{h3}(t) =& \int_{-h_1}^{0}\int_{t+\theta}^{t} \dot{x}^T(s)R_1\dot{x}(s)dsd\theta \\ &+ \int_{-h_2}^{-h_1}\int_{t+\theta}^{t} \dot{x}^T(s)R_2\dot{x}(s)dsd\theta, \end{aligned} \tag{17}$$

which are designed for delay-independent stability criterion and delay-dependent stability criterion, respectively.

The LKF (16) is different from the ones in [12] and [13]. Both l_i^+ and l_i^- are taken into account in (16), while only l_i^- is considered in [12] and [13]. Therefore, the LKF (16) is more general than the ones in [12] and [13].

The chosen LKF in (17) is different from that existing in [6] since there are some redundancies exist in the LKF in [6]. In fact, LKF in [6] includes such two terms $\mathcal{L}_1 = \int_{-h_2}^{0}\int_{t+\theta}^{t} \dot{z}^T(s)Z_1\dot{z}(s)dsd\theta$ and $\mathcal{L}_2 = \int_{-h_2}^{-h_1}\int_{t+\theta}^{t} \dot{z}^T(s)Z_2\dot{z}(s)dsd\theta$, where $Z_1 > 0, Z_2 > 0$. Clearly, $\mathcal{L}_1$ can be decomposed into $\mathcal{L}_1 = \mathcal{L}_{11} + \mathcal{L}_{12}$, where $\mathcal{L}_{11} = \int_{-h_2}^{-h_1}\int_{t+\theta}^{t} \dot{z}^T(s)Z_1\dot{z}(s)dsd\theta$ and $\mathcal{L}_{12} = \int_{-h_1}^{0}\int_{t+\theta}^{t} \dot{z}^T(s)Z_1\dot{z}(s)dsd\theta$. Obviously, $\mathcal{L}_{11}$ is redundant because it can be absorbed by $\mathcal{L}_2$ with Z_2. Nevertheless, such redundancies do not exist in the LKF (17).

Summarizing up above 4 cases, the construction of LKF is mainly concerned with the types of activation function and the restriction of time-varying delay. In general, more information about the recurrent neural network or the system model are involved in the LKF, the derived the stability results may be less conservative at the expense of the computational complexity and the complicated expression form. While if the core information of the system is involved in the LKF, the obtained stability results may also be effective with less computational complexity. Therefore, for different purpose of the design, one may choose different level of LKF. Pursuing the optimal performance of the quantitative stability, it seem no a limit, while the expense of the devotion may increase sharply. If we only want to find a satisfactory performance, it is better to construct a LKF involving the core elements of the system. No free lunch.

Another topic is to estimate the derivative of the LKF, despite the construction of the LKF is compact. This aspect includes many mathematical skills and mathematical tool, such as Jensens integral inequality, the projection lemma, free weight matrix method, integral inequality and convex combination technique. Altogether, a suitable LKF can significantly decrease the conservativeness of the stability criterion if some suitable mathematical estimation methods are used in the derivative of LKF.

IV. Conclusions

In this paper, a brief review on the construction of Lyapunov-Krasovskii functional in the stability proof of recurrent neural networks with single time-varying delay was presented. The emphasis is placed on the combination of the time delay information and the activation function information. Different combination of time delay and activation function, the stability result for the concerned delayed neural networks may be different. No a general method can cover all the case. Even for the case of constant delay, it is not reasonable to expect that the stability result of neural networks with constant delay is a special case of stability result of neural networks with time-varying delay, because different case may construct different LKF and adopt different estimation method to compute the derivative of LKF. This brief review will be helpful to the construction of LKF in the stability analysis of recurrent neural networks with delay, and good performance can be expected to obtain.

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