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The Stability and Bifurcation Analysis in High Dimensional Neural Networks with Discrete and Distributed Delays

Wenying Xu
Research Center for Complex Systems
and Network Sciences
and Department of Mathematics
Southeast University
Nanjing 210096, P. R. China
Email: wenyingxuwinnie@gmail.com

Jinde Cao
Research Center for Complex Systems
and Network Sciences
and Department of Mathematics
Southeast University
Nanjing 210096, P. R. China
Email: jdcao@seu.edu.cn

Min Xiao
College of Automation
Nanjing University of
Posts and Telecommunications
Nanjing, 210003, P. R. China
Email: candymanxm2003@aliyun.com

***Abstract*—This paper studies the stability and Hopf bifurcation in a high-dimension neural network involving the discrete and distributed delays. Such model extends the existing models of neural networks from low-dimension to high-dimension. Therefore, our model is much close to large real neural networks. Here, the delay is chosen as the bifurcation parameter and we obtain the sufficient conditions for the system keeping stable and undergoing the Hopf bifurcation. Moreover, the software package DDE-BIFTOOL is introduced to better display the properties of the system and the effect of gain parameters of the system and delay kernel on the onset of the bifurcation. The simulation results further justify the validity of our theoretical analysis.**

I. Introduction

Neural networks have been attracted many scholars' research interests since a simplified neural network model was proposed in [1] due to their wide applications in numerous areas such as image processing, optimizations, signal processing and so on. Such important applications motivate scholars to investigate the dynamical behaviors of neural networks and there have been some well-known theoretical and practical results.

The stability has been considered as a kind of representative dynamical behavior of neural networks and a great deal of existing work have been carried out on such a hot topic. Moreover, other dynamics of neural networks such as bifurcation and chaos also arouse much concern. For example, the dynamics of a neural network model involving two neurons was considered in [2]. Song et al. studied the Hopf bifurcation on a simplified BAM neural network with three neurons [3]. Then, a BAM neural network with four neurons with distributed delays was investigated and sufficient conditions for the stability and Hopf bifurcation were obtained [4], [5]. However, to the best knowledge of the authors, most existing works on the Hopf bifurcation focus on neuron networks with one, two, three or four neurons. Although the research on the simplified low-dimension neural network models can reflect the nature of neural networks to some extent, it may neglect some complex properties of large and real neural network systems. So it is necessary to discuss the dynamical behaviors and properties of high-dimension neural networks.

In addition, reference [6] proposed that time delays always are inevitable in the signal transmission and then put forward a neural network model with time delay. Subsequently, the neural network in [3] involving one delay was considered. Then, there also have been some literatures studying the stability and bifurcation of neural works with two, three or four time delays [7], [8], [9]. These above mentioned delays all are discrete delays. It should be pointed out that neural networks often have a spatial nature due to the presence of a great deal of parallel pathways with a variety of axon sizes and lengths, so a distribution of propagation delays over a period of time exists [10], [11]. There have been also a few papers concentrating on the Hopf bifurcation in neural network models with distributed delays [4], [12], [13].

Although both discrete and distributed delays in modeling neural networks are of importance, the stability and Hopf bifurcation for general neural networks with discrete and distributed delays have not received much research attention. However, in this paper, we investigate a problem of the stability and Hopf bifurcation in a high-dimension simplified BAM neural network model involving discrete and distributed delays. By analysing the characteristic equation of the linearized system, the sufficient conditions for the neural network remaining stable and undergoing the Hopf bifurcation are obtained. Besides, we discuss the effect of the number of neurons on the properties of the bifurcation. Furthermore, the software package DDE-BIFTOOL, which provides a tool for numerical bifurcation analysis of systems of delay differential equations, is introduced to deal with the Hopf bifurcation of neural networks and better display the effect of parameters on the onset of the bifurcation. Moreover, we further consider the influence of the parameter of the delay kernel on the stability of the network.

The rest of this paper is organized as follows: Section II discusses the conditions for the stability and Hopf bifurcation in a high-dimension neural network with discrete and distributed delays. Some simulation results are given in Section III to better justify our theoretical analysis. Finally, the paper is concluded in Section V.

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II. Main Results

This paper considers the stability and Hopf bifurcation in a $(n+1)$–dimension neural network with discrete and distribute delays as follows

$$\begin{cases} \dot{x}_1(t) &= -a_1x_1(t) + f_1(\int_{-\infty}^{t} F(t-s)x_{n+1}(s)ds) \\ \dot{x}_2(t) &= -a_2x_2(t) + f_2(\int_{-\infty}^{t} F(t-s)x_{n+1}(s)ds) \\ \vdots & \\ \dot{x}_n(t) &= -a_nx_n(t) + f_n(\int_{-\infty}^{t} F(t-s)x_{n+1}(s)ds) \\ \dot{x}_{n+1}(t) &= -a_{n+1}x_{n+1}(t) + g_1(x_1(t-\tau)) \\ & \quad +g_2(x_2(t-\tau)) + \cdots + g_n(x_n(t-\tau)) \end{cases} \tag{1}$$

where x_i, $(i = 1, 2, \cdots, n+1)$ denotes the state of the neuron i, the active functions f_j and g_j satisfy $f_j(0) = g_j(0) = 0$ and f_j, $g_j \in C^1(j = 1, 2, \cdots, n)$, a_i represents the stability of internal neurons processing, and the delay kernel F describes the influence of the neuron from the past to the current time t.

Remark 1: If $F(t-s) = \delta(t-s-\tau)$, where is the delta function , it is obvious that $F(s)$ satisfy $\int_0^{+\infty} F(s)ds = 1$, then system (1) becomes one only involving the discrete delay which has been studied in [14]. Here, we extend the stability and Hopf bifurcation of high dimensional neural networks to that of systems with the discrete and distributed delays.

Remark 2: At present, the analytical methods of Hopf bifurcation mainly depend on the distribution of the characteristic roots of the corresponding linearized system [3], [5], [7], [14]. This paper uses this method as well since making an advance in the analytical method is quite difficult and needs considerable work. However, our contribution is that we tackle the stability and Hopf bifurcation analysis problems for a class of general neural networks with discrete and distributed time-delays, and at the same time the software package DDE-BIFTOOL is introduced to better display the distribution of the characteristic roots and the effect of parameters on the onset of the bifurcation.

In this paper, we choose a weak delay kernel as follows

$$F(t) = \sigma e^{-\sigma t}, \quad \sigma > 0, \tag{2}$$

and let

$$x_{n+2}(t) = \int_{-\infty}^{t} F(t-s)x_{n+1}(s)ds, \tag{3}$$

then, Eq. (1) turns to be

$$\begin{cases} \dot{x}_1(t) &= -a_1x_1(t) + f_1(x_{n+2}(t)) \\ \dot{x}_2(t) &= -a_2x_2(t) + f_2(x_{n+2}(t)) \\ \vdots & \\ \dot{x}_n(t) &= -a_nx_n(t) + f_n(x_{n+2}(t)) \\ \dot{x}_{n+1}(t) &= -a_{n+1}x_{n+1}(t) + g_1(x_1(t-\tau)) \\ & \quad +g_2(x_2(t-\tau)) + \cdots \\ & \quad +g_n(x_n(t-\tau)) \\ \dot{x}_{n+2}(t) &= \sigma x_{n+1}(t) - \sigma x_{n+2}(t). \end{cases} \tag{4}$$

Its corresponding linearized system is

$$\begin{cases} \dot{x}_1(t) &= -a_1x_1(t) + f_1'(0)x_{n+2}(t) \\ \dot{x}_2(t) &= -a_2x_2(t) + f_2'(0)x_{n+2}(t) \\ \vdots & \\ \dot{x}_n(t) &= -a_nx_n(t) + f_n'(0)x_{n+2}(t) \\ \dot{x}_{n+1}(t) &= -a_{n+1}x_{n+1}(t) + g_1'(0)x_1(t-\tau) \\ & \quad +g_2'(0)x_2(t-\tau) + \cdots \\ & \quad +g_n'(0)x_n(t-\tau) \\ \dot{x}_{n+2}(t) &= \sigma x_{n+1}(t) - \sigma x_{n+2}(t). \end{cases} \tag{5}$$

and the associated characteristic equation is

$$\begin{aligned} &\lambda^{n+2} + P_1\lambda^{n+1} + P_2\lambda^n + \cdots + P_{n+1}\lambda + P_{n+2} \\ &\quad -\sigma e^{-\lambda\tau}[Q_1\lambda^{n-1} + Q_2\lambda^{n-2} + \cdots + Q_n] = 0 \end{aligned} \tag{6}$$

where

$$\begin{aligned} P_k &= \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n+2} a_{i1}a_{i2}\cdots a_{ik}; \\ Q_1 &= \sum_{j=1}^{n} b_j; \\ Q_m &= \sum_{j=1}^{n} \left[b_j \sum_{\substack{1 \le i_1 < \cdots < i_{m-1} \le n; \\ i_1, i_2, \cdots i_{m-1} \ne j}} a_{i_1}a_{i_2}\cdots a_{i_{m-1}} \right]; \end{aligned}$$

with $k = 1, 2, \cdots n+2$, $m = 2, 3, \cdots, n$, $a_{n+2} = \sigma$ and $b_j = f_j'(0)g_j'(0)$.

Thus, $i\omega$ $(\omega > 0)$ is the root of Eq. (6) if and only if

$$\begin{aligned} &(i\omega)^{n+2} + P_1(i\omega)^{n+1} + P_2(i\omega)^n + \cdots + P_{n+1}(i\omega) + P_{n+2} \\ &\quad -\sigma e^{-i\omega\tau}\left[Q_1(i\omega)^{n-1} + Q_2(i\omega)^{n-2} + \cdots + Q_n\right] = 0. \end{aligned}$$

For convenience, in the following, we only consider the case of $n = 4k$. When $n = 4k+1$, $4k+2$ and $4k+3$, the analysis is similar, so it is omitted here.

When $n = 4k$, then the above equation can be rewritten as

$$\begin{aligned} &-\omega^{2n+4} + iP_1\omega^{2n+2} + P_2\omega^{2n} + \cdots + iP_{n+1}\omega^2 + P_{n+2} \\ &\quad -\sigma(\cos(\omega\tau) - i\sin(\omega\tau))[-iQ_1\omega^{2n-2} - Q_2\omega^{2n-4} \\ &\quad + \cdots + Q_n] = 0, \end{aligned}$$

that is

$$\begin{cases} \sigma[-Q_2\omega^{n-2} + Q_4\omega^{n-4} + \cdots + Q_n]\cos(\omega\tau) \\ +\sigma[-Q_1\omega^{n-1} + Q_3\omega^{n-3} + \cdots + Q_{n-1}\omega]\sin(\omega\tau) \\ = -\omega^{n+2} + P_2\omega^n + \cdots + P_{n+2} \\ \sigma[-Q_1\omega^{n-1} + Q_3\omega^{n-3} + \cdots + Q_{n-1}\omega]\cos(\omega\tau) \\ -\sigma[-Q_2\omega^{n-2} + Q_4\omega^{n-4} + \cdots + Q_n]\sin(\omega\tau) \\ = P_1\omega^{n+1} - P_3\omega^{n-1} + \cdots + P_{n+1}\omega \end{cases} \tag{7}$$

so we can obtain

$$\cos(\omega\tau) = \frac{M}{N} \tag{8}$$

with $M = (P_1\omega^{n+1} - P_3\omega^{n-1} + \cdots + P_{n+1}\omega)(-Q_1\omega^{n-1} + Q_3\omega^{n-3} + \cdots + Q_{n-1}\omega) + (-\omega^{n+2} + P_2\omega^n + \cdots + P_{n+2})(-Q_2\omega^{n-2} + Q_4\omega^{n-4} + \cdots + Q_n)$, $N = (-Q_2\omega^{n-2} + Q_4\omega^{n-4} + \cdots + Q_n)^2 + (-Q_1\omega^{n-1} + Q_3\omega^{n-3} + \cdots + Q_{n-1}\omega)^2$.

According to Eq. (7), we have

$$\omega^{2n+4} + A_1\omega^{2n+2} + A_2\omega^{2n} + \cdots + A_{n+1}\omega + A_{n+2} = 0 \quad (9)$$

in which

$$\begin{aligned}
A_1 &= P_1^2 - 2P_2; \\
A_2 &= P_2^2 + 2P_4 - 2P_1P_3; \\
A_3 &= P_3^2 - 2P_6 - 2P_2P_4 + 2P_1P_5 - \sigma^2 Q_1^2; \\
&\vdots \\
A_m &= P_m^2 + 2[\sum_{\substack{s_1+s_2=2m \\ 0\le s_1<s_2\le n+2}} (-1)^{m+s_1} P_{s_1}P_{s_2}] \\
&\quad -\sigma^2[Q_{m-2}^2 + 2(\sum_{\substack{t_1+t_2=2m-4 \\ 1\le t_1<t_2\le n}} (-1)^{m+t_1} Q_{t_1}Q_{t_2})] \\
&\vdots \\
A_{n+2} &= P_{n+2}^2 - Q_n^2
\end{aligned}$$

in which $P_0 = 1$.

Let $z = \omega^2$, then

$$h(z) = z^{n+2} + A_1 z^{n+1} + A_2 z^n + \cdots + A_{n+1}\omega + A_{n+2}. \quad (10)$$

1) If $A_{n+2} > 0$ and $h'(z) > 0$, then Eq. (10) has no roots. That means Eq. (6) doesn't have a pair of pure imaginary roots.

2) If $A_{n+2} < 0$, since $\lim_{z\to+\infty} h(z) = +\infty$, then Eq. (10) at least has a positive root, which implies Eq. (6) has a pair of imaginary roots. Without loss of generality, we assume that Eq. (10) has $n+2$ positive roots labeled as z_i, $(1 \le i \le n+2)$, then $\omega_i = \sqrt{z_i}$, and $\tau_i^{(j)} = (\arccos(M/N) + 2j\pi)/\omega_i$, $(j = 1, 2, \cdots)$. Furthermore, we definite that

$$\tau_0 = \tau_{i_0}^{(0)} = \min\{\tau_i^{(0)} | i = 1, 2, \cdots, n+2\},$$

and $\omega_0 = \omega_{i_0}$.

When $\tau = 0$, then Eq. (6) becomes

$$\lambda^{n+2} + P_1\lambda^{n+1} + P_2\lambda^n + (P_3 - \sigma Q_1)\lambda^{n-1} + \cdots + (P_{n+1} - \sigma Q_{n-1})\lambda + (P_{n+2} - \sigma Q_n) = 0. \quad (11)$$

All roots of the above equation (11) have positive real parts if and only if

(H1)

$D_1 = P_1 > 0;$

$$D_2 = \begin{vmatrix} P_1 & P_3 - \sigma Q_1 \\ 1 & P_2 \end{vmatrix} > 0;$$

$$D_3 = \begin{vmatrix} P_1 & P_3 - \sigma Q_1 & P_5 - \sigma Q_3 \\ 1 & P_2 & P_4 - \sigma Q_2 \\ 0 & P_1 & P_3 - \sigma Q_1 \end{vmatrix} > 0;$$

$\vdots$

$$D_{n+2} = \begin{vmatrix} P_1 & P_3 - \sigma Q_1 & P_5 - \sigma Q_3 & \cdots & 0 \\ 1 & P_2 & P_4 - \sigma Q_2 & \cdots & 0 \\ \vdots & \vdots & \cdots & \cdots & \vdots \\ 0 & 0 & \cdots & \cdots & P_{n+2} - \sigma Q_n \end{vmatrix} > 0.$$

Lemma 1: Consider the exponential polynomial

$$\begin{aligned}
&P(\lambda, e^{-\lambda R_1}, \ldots, e^{\lambda R_m}) \\
&= \lambda^n + p_1^{(0)}\lambda^{n-1} + \cdots + p_{n-1}^{(0)}\lambda + p_n^{(0)} + [p_1^{(1)}\lambda^{n-1} + \cdots \\
&+ p_{n-1}^{(1)}\lambda + p_n^{(1)}]e^{-\lambda R_1} + \cdots + [p_1^{(m)}\lambda^{n-1} + \cdots + p_{n-1}^{(m)}\lambda \\
&+ p_n^{(m)}]e^{-\lambda R_m}
\end{aligned}$$

where $R_i \ge 0$ $(i = 1, 2, \ldots, m)$ and p_j^i $(i = 0, 1, \ldots, m;$ $j = 1, 2, \ldots, n)$ are constants. As $(R_1, R_2, \ldots, R_m)$ vary, the sum of the order of zeros of $P(\lambda, e^{-\lambda R_1}, \ldots, e^{-\lambda R_m})$ on the open right half plane can change only if a zero appears on or crosses the imaginary axis.

Lemma 2: If $h(z_i) \neq 0$, then the following inequality holds:

$$\text{Re}\left(\frac{d\lambda}{d\tau}\right)\Bigg|_{\tau=\tau_i^{(j)}} \neq 0. \quad (12)$$

Proof: Differentiating both sides of Eq. 6 with respect to τ, we obtain

$$\begin{aligned}
&[(n+2)\lambda^{n+1} + (n+1)P_1\lambda^n + \cdots + P_{n+1}\lambda \\
&+ P_{n+2}]\frac{d\lambda}{d\tau} + \sigma[\tau e^{-\lambda\tau}\frac{d\lambda}{d\tau} + \lambda e^{-\lambda\tau}][Q_1\lambda^{n-1} \\
&+ Q_2\lambda^{n-2} + \cdots + Q_n] - \sigma e^{-\lambda\tau}[(n-1)\lambda^{n-2}Q_1 \\
&+ (n-2)\lambda^{n-3}Q_2 + \cdots + Q_{n-1}]\frac{d\lambda}{d\tau} = 0.
\end{aligned}$$

Here, we also only consider the case of $n = 4k$, in fact, other three cases can be dealt with by the same method. Then

$$\begin{aligned}
&\left(\frac{d\lambda}{d\tau_i^{(j)}}\right)^{-1}\Bigg|_{\tau_i^{(j)}} \\
&= \frac{\sigma[(n-1)Q_1\omega_i^{n-2} + i(n-2)Q_2\omega_i^{n-3} + \cdots + Q_{n-1}]}{\sigma(i\omega_i)(-iQ_1\omega_i^{n-1} - Q_2\omega_i^{n-2} + \cdots + Q_n)} \\
&\quad + \frac{[i(n+2)\omega_i^{n+1} + (n+1)P_1\omega_i^n + \cdots + P_{n+1}])}{\sigma(i\omega_i)(-iQ_1\omega_i^{n-1} - Q_2\omega_i^{n-2} + \cdots + Q_n)} \\
&\quad \cdot (\cos(\omega_i\tau_i^{(j)}) + i\sin(\omega_i\tau_i^{(j)}) - \frac{\tau_i^{(j)}}{i\omega_i}.
\end{aligned}$$

By using Eq. (7), we have

$$\text{Re}\left(\frac{d\lambda}{d\tau_i^{(j)}}\right)^{-1}\Bigg|_{\tau_i^{(j)}} = \frac{z_i h'(z)}{\Delta} \quad (13)$$

with $\Delta = \sigma^2[(Q_1\omega^n - Q_3\omega^{n-2} + \cdots + Q_{n-3}\omega^4 - Q_{n-1}\omega^2)^2 + (Q_2\omega^{n-1} - Q_4\omega^{n-3} + \cdots + Q_{n-1}\omega^3 - Q_n\omega)^2]$.

Obviously, Δ and z_i are positive.
Moreover,

$$\operatorname{sign}\left(\operatorname{Re}\left(\frac{d\lambda}{d\tau_i^{(j)}}\right)\Bigg|_{\tau_i^{(j)}}\right) = \operatorname{sign}\left(\operatorname{Re}\left(\frac{d\lambda}{d\tau_i^{(j)}}\right)^{-1}\Bigg|_{\tau_i^{(j)}}\right).$$

Thus, this lemma has been proved. ■

Theorem 1: For system (1), the following results hold:

(1) if $A_{n+2} > 0$, $h'(z) > 0$ and (H1) hold, then the equilibrium of system (1) is asymptotically stable for $\tau \geq 0$;

(2) if $A_{n+2} < 0$, and (H1) hold, then the equilibrium of system (1) is asymptotically stable for $\tau \in [0, \tau_0]$; meantime, system (1) undergoes the Hopf bifurcation when $\tau = \tau_i^{(j)}$.

III. Simulation Results

In this section, we will consider such a model of the neural network (1) with 8 neurons in wich $a_1 = a_2 = \cdots = a_n = a = 1$, $a_9 = b = 2$, $f(x) = c\tanh(x) = \tanh(x)$, $g(x) = d\tanh(x) = -1.2\tanh(x)$ and $\sigma = 2$.

Firstly, we choose $\tau = 0.2$. From Fig. 1, it is obvious that the characteristic equation of system (1) has positive roots. Fig. 2 depicts the change of the real parts of system (1) versus τ. Moreover, we can easily know that it has a root with zero real part when $\tau \in [0, 0.2)$ and at the 22th point in Fig. 3. Then we use the software package DDE-BIFTOOL to compute the values of ω_0 and τ_0 which are 2.079 and 0.197 respectively. When $\tau = 0.197$, the characteristic equation indeed has a pair of pure imaginary roots (see Fig. 4).

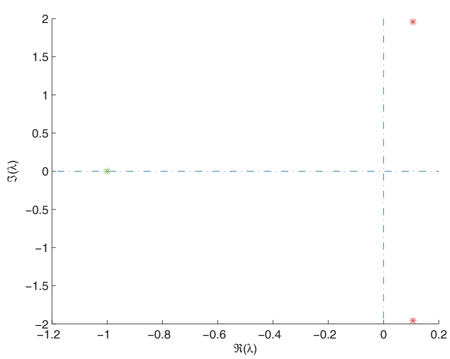

Fig. 1. The distribution of the characteristic roots of system (1) with $\tau = 0.2$ whose real parts are computed up to $\text{Re}(\lambda) \geq -1.5$.

Besides, it can be easily seen that the equilibrium of system (1) is asymptotically stable (see Fig. 5 and Fig. 6) when $\tau < \tau_0 = 0.2$. However, as the τ increases and exceeds the critical value τ_0, then the system undergoes the Hopf bifurcation (see Fig. 7 and Fig. 8).

Next, we investigate the effect of the number of neurons and parameters of system (1) on the critical value τ_0. To begin with, Table I reveals that the value of τ_0 decreases as we increase the number of neurons. It can be computed that the values of τ_0 are 4.094, 0.7262 and 0.197 respectively at $n = 2$, 4 and 8. This implies that the region of stability of the system will be smaller with the increase of n. It may be because higher dimensional systems have more complicated properties so that the change of time delay τ has a greater influence on the stability of the system.

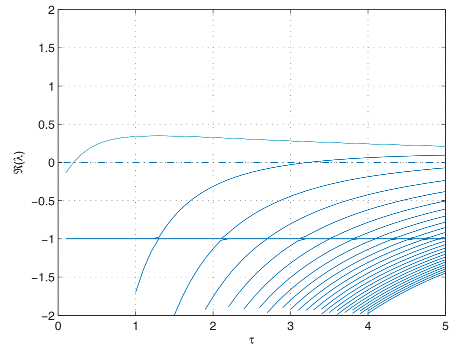

Fig. 2. Real parts of the characteristic equation versus τ.

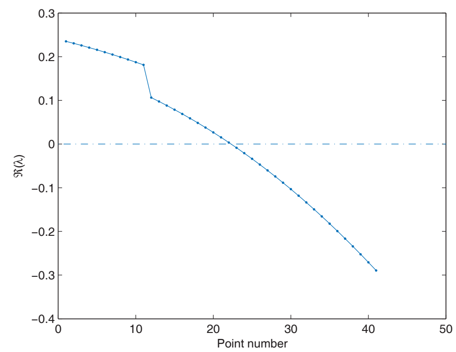

Fig. 3. Real parts of the characteristic equation versus the point number along the branch.

TABLE I. The effect of the number of neurons on the value of ω_0 and τ_0 with $a = 1$, $b = 2$, $c = 1$, $d = -1.2$ and $\sigma = 2$

n	ω_0	τ_0
2	0.5158	4.094
4	1.361	0.7262
8	2.079	0.197

What's more, the relationship between parameters of system (1) and the critical value τ_0 will be discussed. Fig. 9 and Fig. 10 show the similar trend, that is, the critical value τ_0 increases as the gain parameter a or b increases with other parameters remaining the same. Besides, we also find that τ_0 goes to zero when a tends to zero or b goes to 1 which means the system is unstable for all $\tau > 0$ with a sufficiently small positive parameter a or b being close to 1. There is a downward trend on the value of τ_0 as d increases (see Fig. 11) while the increase of d leads to τ_0 being increasing (see Fig. 12). Note that regardless of whether c going from the right part of coordinate axis to zero or d going from the left part of coordinate axis to zero, the value of τ_0 tends to infinite. That

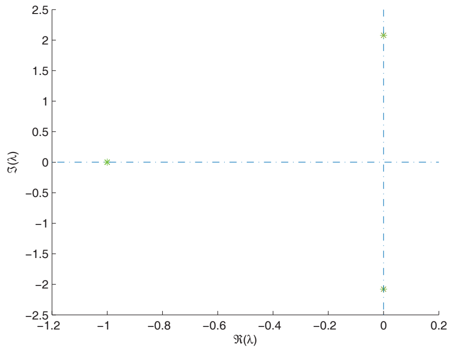

Fig. 4. The distribution of the corrected characteristic roots of system (1) with $\tau = 0.197$ whose real parts are computed up to $\mathrm{Re}(\lambda) \geq -1.5$: a pair of pure imaginary eigenvalues is clearly visible.

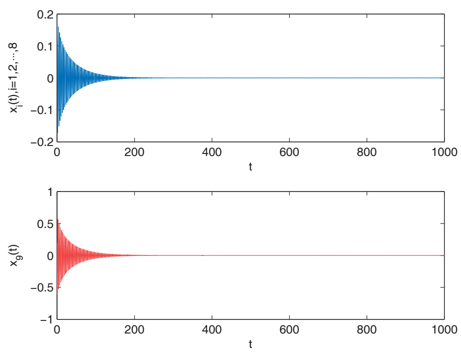

Fig. 5. The trajectories of system (1) with $\tau = 0.18 < \tau_0$: the equilibrium is asymptotically stable.

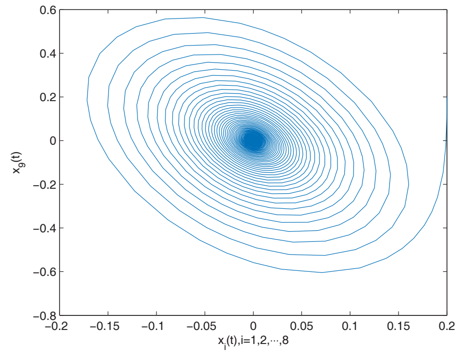

Fig. 6. The phase plot of system (1) with $\tau = 0.18 < \tau_0$: the equilibrium is asymptotically stable.

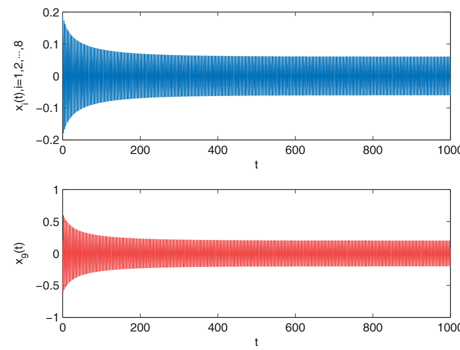

Fig. 7. The trajectories of system (1) with $\tau = 0.2 > \tau_0$: bifurcating periodic solutions occur.

implies that system (1) is asymptotically stable for all $\tau > 0$. Furthermore, we consider the effect of the gain parameter of the delay kernel on the stability of system (1). It is revealed in Fig. 13 that the critical value τ_0 becomes much smaller as σ is much bigger. Therefore, the stabilized zone can be modulated properly by choosing appropriate parameters of the system and the gain parameter of the delay kernel.

IV. Conclusions

The stability and Hopf bifurcation of a class of high dimensional neural networks have been investigated in this paper. Such models involve the discrete and distributed delays which make our model close to large and real neural networks with n neurons and mix delays. By analysing the characteristic equation of the corresponding linearized system, we have obtained the sufficient conditions for the system keeping stable and undergoing the bifurcation. Further, the software package DDE-BIFTOOL has been proposed to study the dynamics of such a class of high dimensional system and has better displayed the distribution of characteristic roots and the effect of gain parameters of the system and delay kernel on the onset of the bifurcation. Furthermore, the Hopf bifurcation analysis for neural networks is a well-studied area and thus making an advance in the problem will need considerable work.

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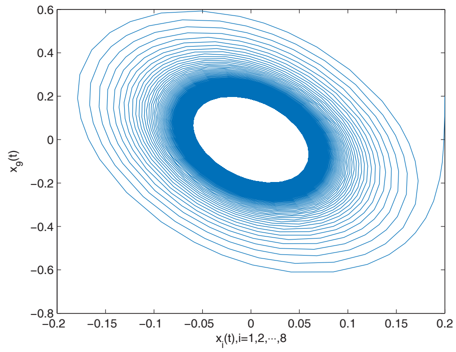

Fig. 8. The phase plot of system (1) with $\tau = 0.2 > \tau_0$: bifurcating periodic solutions occur.

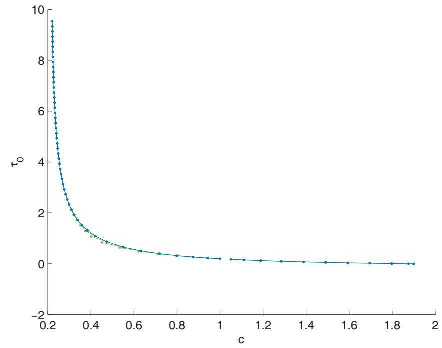

Fig. 11. The relationship between the gain parameter of system (1) c and τ_0 with $a = 1$, $b = 2$, $d = -1.2$ and $\sigma = 2$.

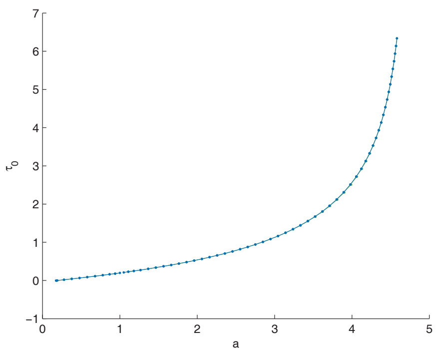

Fig. 9. The relationship between the gain parameter of system (1) a and τ_0 with $b = 2$, $c = 1$, $d = -1.2$ and $\sigma = 2$.

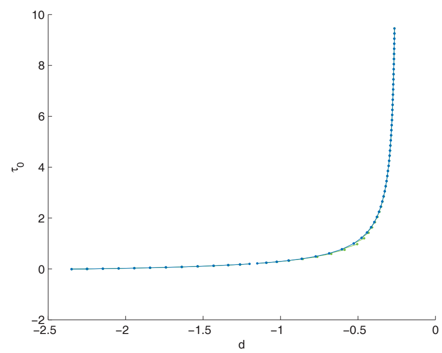

Fig. 12. The relationship between the gain parameter of system (1) d and τ_0 with $a = 1$, $b = 2$, $c = 1$ and $\sigma = 2$.

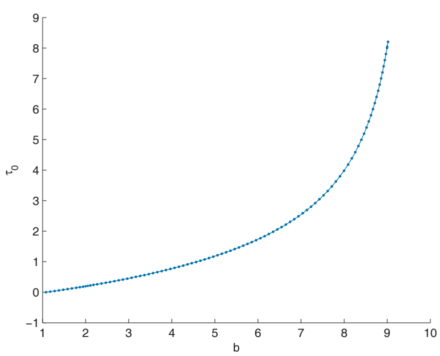

Fig. 10. The relationship between the gain parameter of system (1) b and τ_0 with $a = 1$, $c = 1$, $d = -1.2$ and $\sigma = 2$.

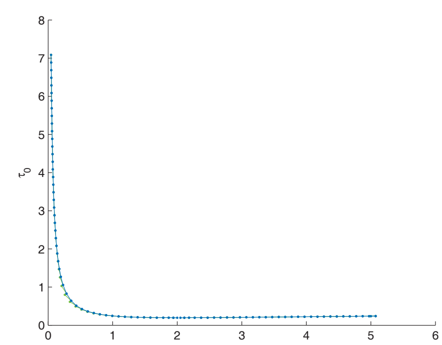

Fig. 13. The relationship between the gain parameter of the weak kernel σ and τ_0 with $a = 1$, $b = 2$, $c = 1$ and $d = -1.2$.

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