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Exponential Synchronization for a Class of Networked Linear Parabolic PDE Systems via Boundary Control

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Abstract—**This paper addresses the problem of exponential synchronization via boundary control for a class of networked linear spatiotemporal dynamical networks consisting of N identical nodes, in which the spatiotemporal behavior of the each node is described by parabolic partial differential equations (PDEs). The purpose of this paper is to design boundary controllers ensuring the exponential synchronization of the networked parabolic PDE system. To do this, Lyapunov's direct method, the vector-valued Wirtinger's inequality, and the technique of integration by parts are employed. A sufficient condition on the existence of the boundary controllers is developed in term of standard of linear matrix inequality (LMI). Finally, numerical simulation results on a numerical example are presented to illustrate the effectiveness of the proposed design method.**

I. INTRODUCTION

COMPLEX dynamical networks (CDNs) can be used to describe the most systems in real world, where the nodes and edges represent individuals in the system and the connections among them. Typical examples include ecosystems, electrical power systems, social systems, the Internet, the WWW (World Wide Web) and so on. Since their wide and important applications, the topology and dynamical behavior of various CDNs have been intensively studied over the past few decades [1]-[8].

One of the interesting and significant phenomena in CDNs is the synchronization of all dynamical nodes in the network. Synchronization is a typical collective behavior and basic motion in nature. Since the pioneering work of Pecora and Carroll [9], synchronization of CDNs has been received increasing attention [10]-[13] due to its potential applications in secure communications, chemical reactions, biological systems, etc. In the case when the whole network cannot synchronize by itself, some controllers may be designed and applied to guide the network to synchronize. Therefore, a large amount of work has been devoted to the investigation of the synchronization on CDNs through designing appropriate controllers [14]-[18]. It must be pointed out that these results [9]-[18] are developed for the CDNs whose the node dynamics only depends on time and is assumed to be described by ordinary differential equations (ODEs) or delay differential equations (DDEs). In practice, the node dynamics in some CDNs like biological systems [19] is spatiotemporal in nature so that its behavior must depend on time as well as spatial position and is described by partial differential equations (PDEs).

Some authors have paid attention to the investigation of control and synchronization of spatiotemporal dynamical networks [20]-[22]. For example, pinning control and global as well as local control were respectively reported in [20] and [21] for spatiotemporal chaos, where the spatiotemporal chaos dynamics is approximately described by ODE model. Motivated by the results in [20] and [21], a robust H_∞ controller has been more recently developed in [22] to achieve synchronization of the coupled PDE systems with spatial coupling delay. Due to the truncation before the controller design, however, the results in [20]-[22] fail to take advantage of natural property of the systems. Moreover, the order of the model truncation is a tradeoff between model accuracy and real time computation.

Considering the fact that the aforementioned drawbacks resulting from the truncation before control design, some synchronization approaches have been developed based on the original PDE model to overcome the aforementioned drawbacks [23]-[27]. For example, based on the original PDE model, sufficient conditions on stability and passivity were presented in [23] and [24] by employing the Lyapunov's direct method for a class of reaction-diffusion neural networks. An approach via the original PDE model was proposed in [25] to examine the synchronization via an active controller of the coupled semi-linear parabolic PDE systems. Hu *et al.* [26] provided some sufficient conditions dependent on the diffusion coefficients guaranteeing the global exponential stability and synchronization for the reaction-diffusion delayed neural networks under the impulsive controllers. Wang and Wu [27] employed the Lyapunov's direct method and the inequality techniques to develop simple design methods for adaptive control laws ensuring synchronization and H_∞ synchronization of coupled reaction-diffusion neural networks with hybrid coupling. Notice that the controllers used in [25]-[27] are difficult to be implemented since their implementation needs arrays of actuators and sensors distributed over the entire spatial domain. Different from the controllers considered in [25]-[27], the implementation of boundary controllers only requires only few actuators located at the boundary of the spatial domain and is thus relatively easy. However, to the best authors' knowledge, few results are available for the exponential synchronization of CDNs through boundary control, which motivates this study.

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This paper considers the problem of exponential synchronization via boundary control for a class of networked linear parabolic PDE systems consisting of N identical nodes in one spatial dimension, where actuators of each node are only located at the one end of the spatial domain. By using the Lyapunov's direct method and the vector-valued Wirtinger's inequality, a simple design for the boundary controllers is developed for the networked linear parabolic PDE system and presented in term of standard linear matrix inequality (LMI). The suggested controllers can ensure the exponential synchronization of the networked parabolic PDE systems and are easily implemented since a finite number of actuators located at the boundary of the one dimensional spatial domain are required. Moreover, the design method can be directly implemented via the polynomial-time interior-point method [28] and [29]. Finally, a numerical example is given to illustrate the effectiveness of the proposed design method.

The remainder of this paper is organized as follows. Section II gives preliminaries and problem formulation. The sufficient condition on exponential synchronization of the networked parabolic PDE system is provided in Section III. Section IV presents an example to illustrate the effectiveness of the proposed method. Finally, Section V offers some concluding remarks.

II. Preliminaries and Problem Formulation

Notations: $\Re$, $\Re^n$ and $\Re^{m\times n}$ denote the set of all real numbers, n-dimensional Euclidean space and the set of all $m\times n$ matrices, respectively. $\|\cdot\|$ and $\langle\cdot,\cdot\rangle_{\Re^n}$ denote the standard Euclidean norm and inner product for vectors, respectively. $\boldsymbol{A}\otimes\boldsymbol{B}$ means the Kronecker product of two matrices $\boldsymbol{A}$ and $\boldsymbol{B}$. Identity matrix of $n\times n$ dimension will be denoted by $\boldsymbol{I}_n$. For a symmetric matrix $\boldsymbol{M}$, $\boldsymbol{M}>(<,\le)0$ means that it is positive definite (negative definite, semi-negative definite, respectively). $\lambda_{\min}(\cdot)$ and $\lambda_{\max}(\cdot)$ stand for the minimum and maximum eigenvalues of a square matrix, respectively. $\mathcal{L}_2([0,L];\Re^n)$ is a Hilbert space of n-dimensional square integrable vector functions $\boldsymbol{\omega}(x)\in\Re^n$, $x\in[0,L]\subset\Re$ with the inner product and norm:

$$\langle\boldsymbol{\omega}_1,\boldsymbol{\omega}_2\rangle=\int_0^L\langle\boldsymbol{\omega}_1(x),\boldsymbol{\omega}_2(x)\rangle_{\Re^n}\,dx \text{ and } \|\boldsymbol{\omega}_1\|_2=\langle\boldsymbol{\omega}_1,\boldsymbol{\omega}_1\rangle^{1/2},$$

where $\boldsymbol{\omega}_1$, $\boldsymbol{\omega}_2\in\mathcal{L}_2([0,L];\Re^n)$. $\mathcal{W}^{l,2}([0,L];\Re^n)$ is a Sobolev space of absolutely continuous n-dimensional vector functions $\boldsymbol{\omega}(x):[0,L]\to\Re^n$ with square integrable derivatives $\frac{d^l\boldsymbol{\omega}(x)}{dx^l}$ of the order $l\ge1$ and with the norm $\|\boldsymbol{\omega}(\cdot)\|^2_{\mathcal{W}^{l,2}}=\int_0^L\sum_{i=0}^{l}\left(\frac{d^i\boldsymbol{\omega}(x)}{dx^i}\right)^T\left(\frac{d^i\boldsymbol{\omega}(x)}{dx^i}\right)dx$. The superscript '$T$' is used for the transpose of a vector or a matrix. The symbol '$*$' is used as an ellipsis in matrix expressions that are induced by symmetry, e.g.,

$$\begin{bmatrix}\boldsymbol{S}+[\boldsymbol{M}+\boldsymbol{N}+*] & \boldsymbol{X}\\ * & \boldsymbol{Y}\end{bmatrix}\triangleq\begin{bmatrix}\boldsymbol{S}+[\boldsymbol{M}+\boldsymbol{N}+\boldsymbol{M}^T+\boldsymbol{N}^T] & \boldsymbol{X}\\ \boldsymbol{X}^T & \boldsymbol{Y}\end{bmatrix}.$$

Consider a networked linear parabolic PDE system consisting of N identical nodes in one spatial dimension, in which the spatiotemporal dynamics of the i-th node is described by the following state-space model:

$$\begin{cases}\boldsymbol{y}_{i,t}(x,t)=\Theta\boldsymbol{y}_{i,xx}(x,t)+\boldsymbol{A}\boldsymbol{y}_i(x,t)+\sum_{j=1}^{N}g_{ij}\boldsymbol{y}_j(x,t)+\boldsymbol{J}(x,t)\\ \boldsymbol{y}_{i,x}(x,t)\big|_{x=0}=\boldsymbol{B}\boldsymbol{u}_i(t),\ \boldsymbol{y}_{i,x}(x,t)\big|_{x=L}=0\\ \boldsymbol{y}_i(x,0)=\boldsymbol{y}_{i,0}(x),\quad i\in\mathcal{N}\triangleq\{1,2,\cdots,N\}\end{cases}\tag{1}$$

where $\boldsymbol{y}_i(x,t)\triangleq[y_{i1}(x,t)\ \ \cdots\ \ y_{in}(x,t)]^T\in\Re^n$ is the state of the i-th node, the subscripts x and t stand for the partial derivatives with respect to x, t, respectively, $x\in[0,L]\subset\Re$ and $t\in[0,\infty)$ are the spatial position and time, respectively, and $\boldsymbol{u}_i(t)\in\Re^m$ is the boundary control input of the i-th node. $\Theta\in\Re^{n\times n}$, $\boldsymbol{A}\in\Re^{n\times n}$, and $\boldsymbol{B}\in\Re^{n\times m}$ represent the dispersal rate, connection matrices, and the control input matrix, respectively, $\boldsymbol{J}(x,t)\triangleq[J_1(x,t)\ \ \cdots\ \ J_n(x,t)]^T\in\Re^n$ is the external input, $\boldsymbol{G}\triangleq(g_{ij})_{N\times N}$ is the constant matrix describing the topological structure of network and the coupling strength between nodes for configuration. The parameter g_{ij} is defined as follows: if there exists a connection from node i to node j $(i\ne j)$, then $g_{ij}\ne0$; otherwise $g_{ij}=0$ $(i\ne j)$ and $g_{ii}=-\sum_{j=1,j\ne i}^{N}g_{ij}$, $i\in\mathcal{N}$. The coupling matrix $\boldsymbol{G}$ is not required to be symmetric or irreducible.

Notice that a solution $\boldsymbol{s}(x,t)\triangleq[s_1(x,t)\ \ \cdots\ \ s_n(x,t)]^T\in\Re^n$ of an isolated node satisfies the following parabolic PDE:

$$\begin{cases}\boldsymbol{s}_t(x,t)=\Theta\boldsymbol{s}_{xx}(x,t)+\boldsymbol{A}\boldsymbol{s}(x,t)+\boldsymbol{J}(x,t)\\ \boldsymbol{s}_x(x,t)\big|_{x=0}=\boldsymbol{s}_x(x,t)\big|_{x=L}=0\\ \boldsymbol{s}(x,0)=\boldsymbol{s}_0(x).\end{cases}\tag{2}$$

Here, $\boldsymbol{s}(x,t)$ may be an equilibrium point, a periodic orbit, or even a chaotic orbit in the phase space.

Define the error $\boldsymbol{e}_i(x,t)\triangleq\boldsymbol{y}_i(x,t)-\boldsymbol{s}(x,t)$, $i\in\mathcal{N}$. From (1) and (2), we have the following error system of the i-th node:

$$\begin{cases}\boldsymbol{e}_{i,t}(x,t)=\Theta\boldsymbol{e}_{i,xx}(x,t)+\boldsymbol{A}\boldsymbol{e}_i(x,t)+\sum_{j=1}^{N}g_{ij}\boldsymbol{e}_j(x,t)\\ \boldsymbol{e}_{i,x}(x,t)\big|_{x=0}=\boldsymbol{B}\boldsymbol{u}_i(t),\boldsymbol{e}_{i,x}(x,t)\big|_{x=L}=0\\ \boldsymbol{e}_i(x,0)=\boldsymbol{e}_{i,0}(x),\end{cases}\tag{3}$$

where $\boldsymbol{e}_{i,0}(x)\triangleq\boldsymbol{y}_{i,0}(x)-\boldsymbol{s}_0(x)$.

This paper considers the following identical state feedback controller for the i-th node of the networked PDE system (1):

$$\boldsymbol{u}_i(t) = \int_0^L \boldsymbol{K}\boldsymbol{e}_i(x,t)dx\,, \tag{4}$$

where $\boldsymbol{K} \in \Re^{m\times n}$ is the control gain matrix to be determined. Substituting (4) into (3) gives the closed-loop system of the i-th node

$$\begin{cases} \boldsymbol{e}_{i,t}(x,t) = \Theta\boldsymbol{e}_{i,xx}(x,t) + \boldsymbol{A}\boldsymbol{e}_i(x,t) + \sum_{j=1}^N g_{ij}\boldsymbol{e}_j(x,t) \\ \boldsymbol{e}_{i,x}(x,t)\big|_{x=0} = \boldsymbol{B}\int_0^L \boldsymbol{K}\boldsymbol{e}_i(x,t)dx, \boldsymbol{e}_{i,x}(x,t)\big|_{x=L} = 0 \\ \boldsymbol{e}_i(x,0) = \boldsymbol{e}_{i,0}(x), \end{cases} \tag{5}$$

Obviously, the origin is an equilibrium point of the system (5).

The objective of this study is to find a state feedback controller of the form (4) such that the networked parabolic PDE system (1) exponentially synchronizes the trajectory (2), i.e., the closed-loop networked parabolic PDE system (5) is exponentially stable. To this end, the following definition and lemmas are useful for the development in the sequel:

Definition 1. For some given boundary control inputs $\boldsymbol{u}_i(t)$, $i \in \mathcal{N}$, the networked parabolic PDE system (1) is said to be *exponential synchronization*, if there exists constants $\rho > 0$ and $\sigma > 0$ such that the following inequality holds for any $i \in \mathcal{N}$:

$$\|\boldsymbol{e}_i(\cdot,t)\|_2^2 \le \sigma\exp(-\rho t)\|\boldsymbol{e}_{i,0}(\cdot)\|_2^2\,,\ \forall\, t \ge 0\,.$$

Lemma 1. [30] Given matrices $\boldsymbol{A},\boldsymbol{B},\boldsymbol{C},\boldsymbol{D}$ with appropriate dimensions, one has

(1) $(\boldsymbol{A}\otimes\boldsymbol{B})^T = \boldsymbol{A}^T\otimes\boldsymbol{B}^T$,
(2) $(\boldsymbol{A}\otimes\boldsymbol{B})(\boldsymbol{C}\otimes\boldsymbol{D}) = (\boldsymbol{A}\boldsymbol{C})\otimes(\boldsymbol{B}\boldsymbol{D})$.

Lemma 2. [31] Let $\boldsymbol{z} \in \mathcal{W}^{1,2}([0,L];\Re^n)$ be a vector function with $\boldsymbol{z}(0) = 0$ or $\boldsymbol{z}(L) = 0$. Then, for a matrix $\boldsymbol{S} > 0$, one has the following integral inequality:

$$\int_0^L \boldsymbol{z}^T(s)\boldsymbol{S}\boldsymbol{z}(s)ds \le 4L^2\pi^{-2}\int_0^L \left(d\boldsymbol{z}(s)/ds\right)^T \boldsymbol{S}\left(d\boldsymbol{z}(s)/ds\right)ds\,. \tag{6}$$

III. EXPONENTIAL SYNCHRONIZATION

The aim of this section is to present an LMI-based sufficient condition on the exponential synchronization of the networked parabolic PDE system (1). We consider the following Lyapunov functional for the closed-loop system (5):

$$V(t) = \sum_{i=1}^N \int_0^L \boldsymbol{e}_i^T(x,t)\boldsymbol{P}\boldsymbol{e}_i(x,t)dx \tag{7}$$

where $0 < \boldsymbol{P} \in \Re^{n\times n}$ is a real matrix to be determined. The time derivative of $V(t)$ given by (7) along the solution of the i-th node of the closed-loop networked parabolic PDE system (5) is given by

$$\begin{aligned} \dot{V}(t) &= 2\sum_{i=1}^N \int_0^L \boldsymbol{e}_i^T(x,t)\boldsymbol{P}\boldsymbol{e}_{i,t}(x,t)dx \\ &= 2\int_0^L \sum_{i=1}^N \boldsymbol{e}_i^T(x,t)\boldsymbol{P}\Theta\boldsymbol{e}_{i,xx}(x,t)dx \\ &\quad + \int_0^L \sum_{i=1}^N \boldsymbol{e}_i^T(x,t)[\boldsymbol{P}\boldsymbol{A} + *]\boldsymbol{e}_i(x,t)dx \\ &\quad + 2\int_0^L \sum_{i=1}^N \boldsymbol{e}_i^T(x,t)\boldsymbol{P}\sum_{j=1}^N g_{ij}\boldsymbol{e}_j(x,t)dx\,. \end{aligned} \tag{8}$$

By integrating by parts and taking into account the boundary condition of (5), we can find that for any $i \in \mathcal{N}$,

$$\begin{aligned} &\int_0^L \boldsymbol{e}_i^T(x,t)\boldsymbol{P}\Theta\boldsymbol{e}_{i,xx}(x,t)dx \\ &= \boldsymbol{e}_i^T(x,t)\boldsymbol{P}\Theta\boldsymbol{e}_{i,x}(x,t)\Big|_{x=0}^{x=L} - \int_0^L \boldsymbol{e}_{i,x}^T(x,t)\boldsymbol{P}\Theta\boldsymbol{e}_{i,x}(x,t)dx \\ &= -\boldsymbol{e}_i^T(0,t)\boldsymbol{P}\Theta\boldsymbol{B}\int_0^L \boldsymbol{K}\boldsymbol{e}_i(x,t)dx - \int_0^L \boldsymbol{e}_{i,x}^T(x,t)\boldsymbol{P}\Theta\boldsymbol{e}_{i,x}(x,t)dx \end{aligned}$$

which implies

$$\begin{aligned} 2\int_0^L \boldsymbol{e}_i^T(x,t)\boldsymbol{P}\Theta\boldsymbol{e}_{i,xx}(x,t)dx &= -2\boldsymbol{e}_i^T(0,t)\boldsymbol{P}\Theta\boldsymbol{B}\int_0^L \boldsymbol{K}\boldsymbol{e}_i(x,t)dx \\ &\quad - \int_0^L \boldsymbol{e}_{i,x}^T(x,t)[\boldsymbol{P}\Theta + *]\boldsymbol{e}_{i,x}(x,t)dx\,. \end{aligned} \tag{9}$$

Then, we have the following theorem:

Theorem 1. For the networked parabolic PDE system (1), if there exist a $n\times n$ matrix $\boldsymbol{Q} > 0$ and a $m\times n$ matrix $\boldsymbol{Z}$ satisfying the following LMI:

$$\Psi \triangleq \begin{bmatrix} -0.25L^{-2}\pi^2[\boldsymbol{I}_N\otimes\Theta\boldsymbol{Q} + *] & \boldsymbol{I}_N\otimes\Theta\boldsymbol{B}\boldsymbol{Z} \\ * & \Psi_{22} \end{bmatrix} < 0 \tag{10}$$

where

$$\Psi_{22} \triangleq [\boldsymbol{I}_N\otimes(\boldsymbol{A}\boldsymbol{Q} - \Theta\boldsymbol{B}\boldsymbol{Z}) + \boldsymbol{G}\otimes\boldsymbol{Q} + *]\,,$$

then there exists a state feedback controller of the form (4) such that the networked parabolic PDE system (1) exponentially synchronizes the trajectory (2), i.e., the closed-loop networked parabolic PDE system (5) is exponentially stable. In this case, the gain matrix $\boldsymbol{K}$ of the controller (4) can be given as

$$\boldsymbol{K} = \boldsymbol{Z}\boldsymbol{Q}^{-1} \tag{11}$$

Proof. Assume that LMI (10) is satisfied for matrices $\boldsymbol{Q} > 0$ and $\boldsymbol{Z}$. Set

$$\boldsymbol{Q} = \boldsymbol{P}^{-1},\ \boldsymbol{Z} = \boldsymbol{K}\boldsymbol{Q}\,. \tag{12}$$

By employing Lemma 1, pre- and post-multiplying both sides of the matrix Ψ by the block diagonal matrix $\mathrm{diag}\{\boldsymbol{I}_N \otimes \boldsymbol{Q}^{-1}, \boldsymbol{I}_N \otimes \boldsymbol{Q}^{-1}\}$, respectively and using (12), we can get

$$\begin{aligned}\bar{\Psi} &\triangleq \mathrm{diag}\{\boldsymbol{I}_N \otimes \boldsymbol{Q}^{-1}, \boldsymbol{I}_N \otimes \boldsymbol{Q}^{-1}\}\Psi\mathrm{diag}\{\boldsymbol{I}_N \otimes \boldsymbol{Q}^{-1}, \boldsymbol{I}_N \otimes \boldsymbol{Q}^{-1}\}\\ &= \begin{bmatrix} -0.25L^{-2}\pi^2\boldsymbol{I}_N \otimes [\boldsymbol{P}\Theta + *] & \boldsymbol{I}_N \otimes \boldsymbol{P}\Theta\boldsymbol{B}\boldsymbol{K} \\ * & \bar{\Psi}_{22}\end{bmatrix},\end{aligned} \tag{13}$$

where

$$\bar{\Psi}_{22} \triangleq [\boldsymbol{I}_N \otimes (\boldsymbol{P}\boldsymbol{A} - \boldsymbol{P}\Theta\boldsymbol{B}\boldsymbol{K}) + \boldsymbol{P} \otimes \boldsymbol{G} + *]\,.$$

From (12) and (13), we can get from LMI (10) that $\bar{\Psi} < 0$ since $\boldsymbol{Q} > 0$. The inequality $\bar{\Psi} < 0$ implies $\boldsymbol{I}_N \otimes [\boldsymbol{P}\Theta + *] > 0$. We can thus further obtain

$$[\boldsymbol{P}\Theta + *] > 0\,. \tag{14}$$

Using Lemma 2 and considering (14), we can get for any $i \in \mathcal{N}$

$$\begin{aligned}&-\int_0^L \boldsymbol{e}_{i,x}^T(x,t)[\boldsymbol{P}\Theta + *]\boldsymbol{e}_{i,x}(x,t)dx\\ &\qquad \le -\frac{\pi^2}{4L^2}\int_0^L \bar{\boldsymbol{e}}_i^T(x,t)[\boldsymbol{P}\Theta + *]\bar{\boldsymbol{e}}_i(x,t)dx\,.\end{aligned} \tag{15}$$

where $\bar{\boldsymbol{e}}_i(x,t) \triangleq \boldsymbol{e}_i(x,t) - \boldsymbol{e}_i(0,t)$.

Substituting (15) into (9), gives

$$\begin{aligned}&2\int_0^L \boldsymbol{e}_i^T(x,t)\boldsymbol{P}\Theta\boldsymbol{e}_{i,xx}(x,t)dx\\ &\quad \le -2\boldsymbol{e}_i^T(0,t)\boldsymbol{P}\Theta\boldsymbol{B}\int_0^L \boldsymbol{K}_i\boldsymbol{e}_i(x,t)dx\\ &\quad -\frac{\pi^2}{4L^2}\int_0^L \bar{\boldsymbol{e}}_i^T(x,t)[\boldsymbol{P}\Theta + *]\bar{\boldsymbol{e}}_i(x,t)dx\,.\end{aligned} \tag{16}$$

Hence, from (16), the expression (8) can be rewritten as

$$\begin{aligned}\dot{V}(t) &\le -2\sum_{i=1}^N \boldsymbol{e}_i^T(0,t)\boldsymbol{P}\Theta\boldsymbol{B}\int_0^L \boldsymbol{K}\boldsymbol{e}_i(x,t)dx\\ &+\int_0^L \sum_{i=1}^N \boldsymbol{e}_i^T(x,t)[\boldsymbol{P}\boldsymbol{A} + *]\boldsymbol{e}_i(x,t)dx\\ &-\frac{\pi^2}{4L^2}\sum_{i=1}^N\int_0^L \bar{\boldsymbol{e}}_i^T(x,t)[\boldsymbol{P}\Theta + *]\bar{\boldsymbol{e}}_i(x,t)dx\\ &+2\int_0^L \sum_{i=1}^N \boldsymbol{e}_i^T(x,t)\boldsymbol{P}\sum_{j=1}^N g_{ij}\boldsymbol{e}_j(x,t)dx\\ &= 2\sum_{i=1}^N\int_0^L \bar{\boldsymbol{e}}_i^T(x,t)\boldsymbol{P}\Theta\boldsymbol{B}\boldsymbol{K}\boldsymbol{e}_i(x,t)dx\\ &+\int_0^L \sum_{i=1}^N \boldsymbol{e}_i^T(x,t)[\boldsymbol{P}\boldsymbol{A} - \boldsymbol{P}\Theta\boldsymbol{B}\boldsymbol{K} + *]\boldsymbol{e}_i(x,t)dx\\ &-\frac{\pi^2}{4L^2}\sum_{i=1}^N\int_0^L \bar{\boldsymbol{e}}_i^T(x,t)[\boldsymbol{P}\Theta + *]\bar{\boldsymbol{e}}_i(x,t)dx\\ &+2\int_0^L \sum_{i=1}^N \boldsymbol{e}_i^T(x,t)\boldsymbol{P}\sum_{j=1}^N g_{ij}\boldsymbol{e}_j(x,t)dx\\ &= \int_0^L \tilde{\boldsymbol{e}}^T(x,t)\bar{\Psi}\tilde{\boldsymbol{e}}(x,t)dx\,.\end{aligned} \tag{17}$$

where $\tilde{\boldsymbol{e}}(x,t) \triangleq [\bar{\boldsymbol{e}}^T(x,t) \quad \boldsymbol{e}^T(x,t)]^T$.

According to the matrix theory, one can find a scalar $\rho > 0$ such that the matrix inequality $\bar{\Psi} < 0$ can be written as

$$\bar{\Psi} + \rho\boldsymbol{I} \le 0\,. \tag{18}$$

Substitution of (18) into (17), derives

$$\dot{V}(t) \le -\rho\|\tilde{\boldsymbol{e}}(\cdot,t)\|_2^2 \le -\rho\|\boldsymbol{e}(\cdot,t)\|_2^2\,. \tag{19}$$

Since $\boldsymbol{P} > 0$, it is easily observed that $V(t)$ given by (7) satisfies the following inequality:

$$p_1\|\boldsymbol{e}(\cdot,t)\|_2^2 \le V(t) \le p_2\|\boldsymbol{e}(\cdot,t)\|_2^2 \tag{20}$$

where $p_1 \triangleq \lambda_{\min}(\boldsymbol{P})$ and $p_2 \triangleq \lambda_{\max}(\boldsymbol{P})$ are two positive scalars. Using (20), the inequality (19) can be represented as

$$\begin{aligned}p_1\|\boldsymbol{e}(\cdot,t)\|_2^2 &\le V(t) \le V(0)\exp(-p_2^{-1}\rho t)\\ &\le p_2\|\boldsymbol{e}_0(\cdot)\|_2^2\exp(-p_2^{-1}\rho t)\end{aligned} \tag{21}$$

Therefore, we have

$$\|\boldsymbol{e}(\cdot,t)\|_2^2 \le p_2p_1^{-1}\|\boldsymbol{e}_0(\cdot)\|_2^2\exp(-p_2^{-1}\rho t),\ t \ge 0\,. \tag{22}$$

From (22) and Definition 1, we can conclude that the closed-loop networked parabolic PDE system (5) is exponentially stable, i.e., the control law (4) can guide the networked parabolic PDE system (1) to exponentially synchronize the trajectory (2). From (12), we have (11). The proof is complete.

Theorem 1 presents an LMI-based condition for the existence of an identical state feedback controller of the form (4) for the exponential synchronization of the networked parabolic PDE system (1). The corresponding control gain matrix $\boldsymbol{K}$ can be constructed as (11) via the feasible solutions to LMI (10), which can be solved efficiently using the existing LMI optimization techniques [28] and [29].

Remark 1. Notice that an LMI-based fuzzy boundary control design has more recently proposed in [32] for a class of semi-linear parabolic PDE systems. Different from the fuzzy boundary control design in [32], this paper provides an LMI-based sufficient condition on the exponential synchronization via boundary control of a linear networked parabolic PDE system with N identical nodes in one spatial dimension.

Remark 2. In comparison to the existing results on the synchronization of complex dynamical reaction-diffusion networks [26] and [27], the main difference of this paper lies in that this paper utilizes a boundary control law to achieve the exponential synchronization whereas the results [26] and [27] employ the distributed controllers (i.e., actuators and sensors are distributed over the entire spatial domain) to force the network synchronize. Moreover, although this paper only considers a linear networked parabolic PDE system with N identical nodes in one spatial dimension, the result in this paper can be extended to address more general networked PDE systems with N nodes, like networked PDE systems with N non-identical nodes in two/three spatial dimensions. This important issue will be addressed in future study.

Remark 3. Even though the result in Theorem 1 is developed only for the case the boundary condition of (1), the same design method can be directly obtained in a similar manner for the case of boundary conditions $\boldsymbol{y}_{i,x}(x,t)\big|_{x=0} = \boldsymbol{B}\boldsymbol{u}_i(t)$ and $\boldsymbol{y}_i(L,t) = 0$, $i \in \mathcal{N}$, the case of boundary conditions $\boldsymbol{y}_{i,x}(x,t)\big|_{x=0} = 0$ and $\boldsymbol{y}_{i,x}(x,t)\big|_{x=L} = \boldsymbol{B}\boldsymbol{u}_i(t)$, $i \in \mathcal{N}$, or the case of boundary conditions $\boldsymbol{y}_i(0,t) = 0$ and $\boldsymbol{y}_{i,x}(x,t)\big|_{x=L} = \boldsymbol{B}\boldsymbol{u}_i(t)$, $i \in \mathcal{N}$.

IV. Numerical Simulation

In this section, in order to illustrate the effectiveness of the proposed theoretical result, we consider the exponential synchronization problem of the following networked linear parabolic PDE system with five identical nodes in one spatial dimension:

$$\begin{cases} y_{i,t}(x,t) = y_{i,xx}(x,t) + y_i(x,t) + \sum_{j=1}^{5} g_{ij} y_j(x,t) \\ \qquad + \sin(\pi x)\,\mathrm{sgn}(\cos(t)) \\ y_{i,x}(x,t)\big|_{x=0} = u_i(t), y_{i,x}(x,t)\big|_{x=1} = 0 \\ y_i(x,0) = y_{i,0}(x) \end{cases} \tag{23}$$

where $y_i(x,t) \in \Re$, $i \in \{1,2,\cdots,5\}$. The coupling matrix $\boldsymbol{G}$ is chosen as

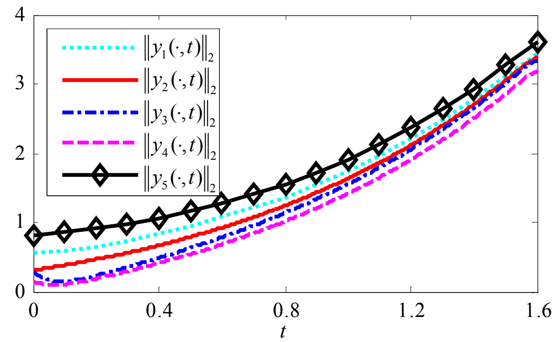

Fig. 1 Open-loop trajectories of $\|y_i(\cdot,t)\|_2$, $i \in \{1,2,\cdots,5\}$

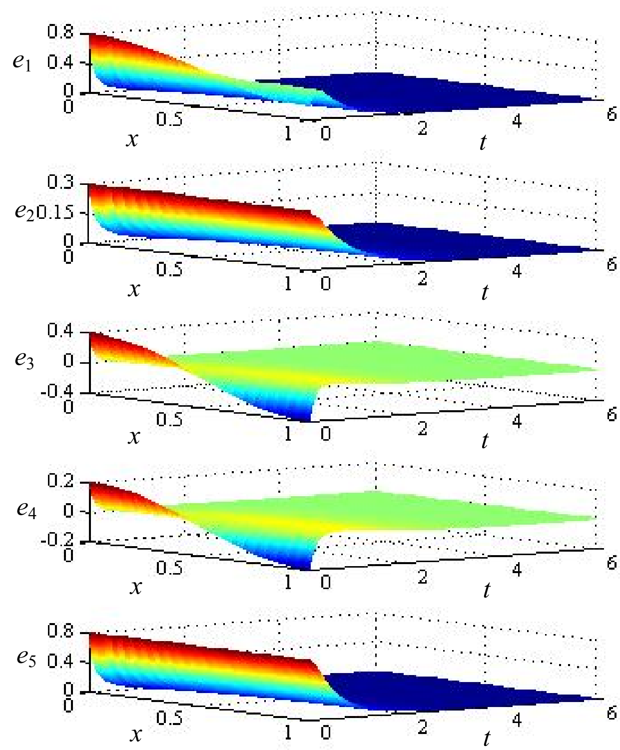

Fig. 2 Closed-loop error profiles of evolution of $e_i(x,t)$, $i \in \{1,2,\cdots,5\}$

$$\boldsymbol{G} \triangleq \begin{bmatrix} -2.4746 & 0.6557 & 0.0357 & 0.8491 & 0.9340 \\ 0.6787 & -2.5718 & 0.7577 & 0.7431 & 0.3922 \\ 0.6555 & 0.1712 & -1.5645 & 0.7060 & 0.0318 \\ 0.2769 & 0.0462 & 0.0971 & -1.2437 & 0.8235 \\ 0.6948 & 0.3171 & 0.9502 & 0.0344 & -1.9966 \end{bmatrix}.$$

Let the initial conditions be $y_{1,0}(x) = 0.5 + 0.3\cos(\pi x)$, $y_{2,0}(x) = 0.3$, $y_{3,0}(x) = 0.4\cos(\pi x)$, $y_{4,0}(x) = 0.2\cos(\pi x)$, and $y_{5,0}(x) = 0.8$, $x \in [0,1]$, respectively. When $u_i(t) = 0$, $i \in \{1,2,\cdots,5\}$, Fig. 1 gives the open-loop trajectories of

$\|y_i(\cdot,t)\|_2$, $i\in\{1,2,\cdots,5\}$ under these initial conditions. It is clear from Fig. 1 that the nodes of the networked PDE system (1) are not synchronized.

Set $L=1$, $\Theta=1$, $\boldsymbol{A}=1$, and $\boldsymbol{B}=1$. Solving LMIs (10) and using (11), the control gain parameter $\boldsymbol{K}$ is derived as $\boldsymbol{K}=4.2689$. Applying the controller (4) with this control parameter to the system (23), Fig. 2 shows the closed-loop error profiles of evolution of $y_i(x,t)$, $i\in\{1,2,\cdots,5\}$ under the same initial conditions. Obviously, the proposed control law (4) can guide the networked parabolic PDE system (23) to synchronize. Fig. 3 indicates the corresponding trajectories of control inputs $u_i(t)$, $i\in\{1,2,\cdots,5\}$.

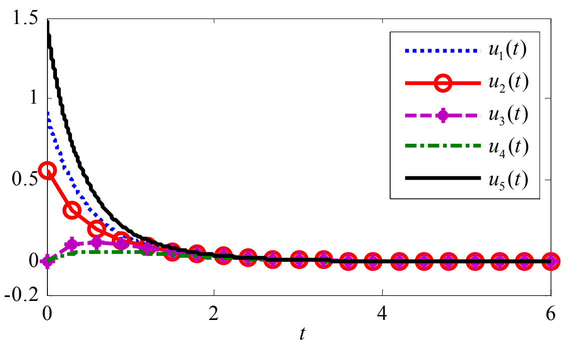

Fig.3 Trajectories of control inputs $u_i(t)$, $i\in\{1,2,\cdots,5\}$

V. Conclusions

In this paper, we have addressed the exponential synchronization problem of a networked linear parabolic PDE system with N identical nodes in one spatial dimension. It has been proved in detail that some boundary controllers can guide the spatiotemporal dynamical networks to synchronize if the network coefficients satisfy a given LMI condition. The control gain parameter is easily obtained via feasible solutions to the given LMI. Furthermore, suggested boundary controllers are easily implemented since only few actuators located at the boundary of the one-dimensional spatial domain are utilized. Finally, a numerical example is simulated, and the achieved simulation results verify well the theoretical result.

References

[1] M. Gilli, "Strange attractors in delayed cellular neural networks," *IEEE Trans. Circuits Syst. I*, vol. 40, no. 11, pp. 849–853, 1993.

[2] S. Wasserman and K. Faust, *Social Network Analysis*, New York: Cambridge University Press, 1994.

[3] S.-H. Strogatz, "Exploring complex networks," *Nature*, vol. 410, pp. 268-276, 2001.

[4] E. M. J. Newman, "The structure and function of complex networks," *SIAM Rev.*, vol. 45, no. 2, pp. 167-256, 2003.

[5] J. Lü and G. Chen, "A time-varying complex dynamical network model and its controlled synchronization criteria," *IEEE Trans. Automat. Control*, vol. 50, no. 6, pp. 841-846, 2005.

[6] S. Boccaletti, V. Latora, Y. Moreno, *et al.*, "Complex networks: Structure and dynamics," *Phys. Rep.*, vol. 424, nos. 4-5, pp. 175-308, 2006.

[7] E. Bullmore and O. Sporns, "Complex brain networks: graph theoretical analysis of structural and functional systems," *Nat. Rev. Neurosci.*, vol. 10, no. 3, pp. 186-198, 2009.

[8] R. Cohen and S. Havlin, *Complex Networks: Structure, Robustness and Function*, New York: Cambridge University Press, 2010.

[9] L.-M. Pecora and T.-L. Carroll, "Synchronization in chaotic systems," *Phys. Rev. Lett.*, vol. 64, no. 8, pp. 821–824, 1990.

[10] S. Boccaletti, J. Kurths, G. Osipov, et al., "The synchronization of chaotic systems," *Phys. Rep.*, vol. 366, nos. 1-2, pp. 1-101, 2006.

[11] A. Arenas, A. Díaz-Guilera, J. Kurths, et al., "Synchronization in complex networks," *Phys. Rep.*, vol. 469, no. 3, pp. 93-153, 2008.

[12] M. Eisencraft, R.-D. Fanganiello, and L.-H.-A. Monteiro, "Chaotic synchronization in discrete-time systems connected by bandlimited channels," *IEEE Commun. Lett.*, vol. 15, no. 6, pp. 671-673, 2011.

[13] V. Nicosia, M. Valencia, M. Chavez, *et al.*, "Remote synchronization reveals network symmetries and functional modules," *Phys. Rev. Lett.*, vol. 110, no. 17, pp. 174102-1~174102-5, 2013.

[14] J. Zhou, J. Lu, and J. Lü, "Adaptive synchronization of an uncertain complex dynamical network," *IEEE Trans. Automat. Control*, vol. 51, no. 4, pp. 652-656, 2006.

[15] W. Yu and J. Cao, "Synchronization control of stochastic delayed neural networks," *Phys. A*, vol. 373, pp. 252–260, 2007.

[16] J. Yao, Z. Guan, D. J. Hill, "Passivity-based control and synchronization of general complex dynamical networks," *Automatica*, vol. 45, no. 9, pp. 2107-2113, 2009.

[17] Z.-G Wu, P. Shi, H. Su, et al., "Exponential synchronization of neural networks with discrete and distributed delays under time-varying sampling," *IEEE Trans. Neural Netw. Learn. Syst.*, vol. 23, no. 9, pp. 1368-1376, 2012.

[18] W. Yu, G. Chen, J. Lü, *et al.*, "Synchronization via pinning control on general complex networks," *SIAM J. Control Optim.*, vol. 51, no. 2, pp. 1395-1416, 2013.

[19] J.-D. Murray, *Mathematical Biology*, Berlin: Springer-Verlag, 1989.

[20] R.-O. Grigoriev, M.-C. Cross, and H.-G. Schuster, "Pinning control of spatiotemporal chaos," *Phys. Rev. Lett.*, vol. 79, no. 15, pp. 2795-2798, 1997.

[21] N. Parekh, S. Parthasarathy, and S. Sinha, "Global and local control of spatiotemporal chaos in coupled map lattices," *Phys. Rev. Lett.*, vol. 81, no. 7, pp. 1401-1404, 1998.

[22] K.-N. Wu, C.-X. Li, B.-S. Chen, *et al.*, "Robust H_∞ synchronization of coupled partial differential systems with spatial coupling delay," *IEEE Trans. Circuits-II*, vol. 60, no. 7, pp. 451-455, 2013.

[23] Z. Wang and H. Zhang, "Global asymptotic stability of reaction-diffusion Cohen-Grossberg neural networks with continuously distributed delays," *IEEE Trans. Neural Netw.*, vol. 21, no. 1, pp. 39-49, 2010.

[24] J.-L. Wang, H.-N. Wu, and L. Guo, "Passivity and stability analysis of reaction-diffusion neural networks with Dirichlet boundary conditions," *IEEE Trans. Neural Netw.*, vol. 22, no. 12, pp. 2105-2116, 2011.

[25] S. Codreanu, "Synchronization of spatiotemporal nonlinear dynamical systems by an active control," *Chaos Solition. Fract.*, vol. 15, no. 3, pp. 507-510, 2003.

[26] C. Hu, H. Jiang, and Z. Teng, "Impulsive control and synchronization for delayed neural networks with reaction-diffusion terms," *IEEE Trans. Neural Netw.*, vol. 21, no. 1, pp. 67-81, 2010.

[27] J.-L. Wang and H.-N. Wu, "Synchronization and adaptive control of an array of linearly coupled reaction-diffusion neural networks with hybrid coupling," *IEEE Trans. Cybern.*, Available online, doi: 10.1109/TCYB.2013.2283308, 2013.

[28] S. Boyd, L. EI Ghaoui, E. Feron., and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*, PA: SIAM, 1994.

[29] P. Gahinet, A. Nemirovskii, A. J. Laub, and M. Chilali, *LMI Control Toolbox for Use with Matlab*, Natick, MA: The Math.Works Inc., 1995.

[30] A.-N. Langville and W.-J. Stewart, "The Kronecker product and stochastic automata networks," *J. Comput. Appl. Math.*, vol. 167, no. 2, pp. 429–447, 2004.

[31] A. Seuret and F. Gouaisbaut, "Wirtinger-based integral inequality: Application to time-delay systems," *Automatica*, vol. 49, no. 9, pp. 2860-2866, 2013.

[32] H.-N. Wu, J.-W. Wang, and H.-X. Li, "Fuzzy boundary control design for a class of nonlinear parabolic distributed parameter systems," *IEEE Trans. Fuzzy Syst.*, Available online, doi:10.1109/TFUZZ.2013.269698, 2013.