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Stability of Hopfield Neural Networks with Event-triggered Feedbacks

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***Abstract*— This paper investigates the convergence of Hopfield neural networks with an event-triggered rule to reduce the frequency of the neuron output feedbacks. The output feedback of each neuron is based on the outputs of its neighbours at its latest triggering time and the next triggering time of this neuron is determined by a criterion based on its neighborhood information as well. It is proved that the Hopfield neural networks are completely stable under this event-triggered rule. The main technique of proof is to prove the finiteness of trajectory length by the Łojasiewicz inequality. The realization of this event-triggered rule is verified by the exclusion of Zeno behaviors. Numerical examples are provided to illustrate the theoretical results and present the goal-seeking capability of the networks. Our result can be easily extended to a large class of neural networks.**

I. Introduction

IN [1] and [2], the author presented the famous Hopfield neural network model and its applications, where the continuous-time Hopfield network is described by

$$\begin{cases} C_i\dot{x}_i(t) = -\frac{x_i(t)}{R_i} + \sum_{j=1}^m \omega_{ij}y_j(t) + \theta_i \\ y_i(t) = g_i(\lambda_i x_i(t)),\ i = 1, \cdots, m \end{cases} \tag{1}$$

with each activation function $g_i(\cdot)$ being a sigmoidal function and symmetric weight condition ($\omega_{ij} = \omega_{ji}$ for all $i, j = 1, \cdots, m$). This model has a great variety of applications: for example, it can be used to search for the local minimum points and values of the quadratic objective function $\frac{-1}{2}\sum_{i,j=1}^m \omega_{ij}y_iy_j$ over the discrete set $\{0,1\}^m$ [3]-[5]. Particulary, this model can be applied for solving the traveling-sales problem [6]. From the analysis in [5], we know that as λ_i goes to infinity and θ_i is sufficiently small, the limit of the trajectory of (1) is located sufficiently near one of the local minimum points. Thus the study of the trajectory convergence (or stability) of the Hopfield neural network is a fundamental issue.

Over the past several decades, many researchers investigated stability of all kinds of (Hopfield) neural networks (see [7]-[11] and references therein). In several papers, the linearization technique and the classical LaSalle approach were used to prove the stability. However, these approaches would be invalid when the system had non-isolated equilibrium points (e.g., a manifold of equilibria) [12]. A new concept "absolute stability" was proposed in [4], [12], [14] to show that each trajectory of the neural network is convergent with any parameters and activation functions satisfying certain conditions. In these papers, the authors proved the absolute stability by proving the finiteness of the trajectory length. This idea was proposed in an earlier important paper [8]. The significant contributions of these works lie in that the stability analysis does not need any information of equilibria or even uniqueness/countableness of the equilibria. The key step of this proof is employing the celebrated Łojasiewicz inequality [15]-[16].

However, in the model, simultaneous outputs are used as feedbacks all the time, which is cost in practice for networks with a large number of neurons. In recent years, with the development of sensing, communications, and computing equipment, the concept of event-triggered control [17]-[23] and self-triggered control [24]-[27] have been proposed. The remarkable advantage of these two kinds of control techniques is that the frequency of feedbacks information exchange is significantly reduced.

In this paper, motivated by this idea, we investigate stability of Hopfield neural networks with event-triggered feedbacks. The stability we considered here is completely stable (see **Definition 1**). We present an event-triggered rule to reduce the frequency of the neuron output feedbacks. At each neuron, the output feedback is based on the outputs of its neighbours at its latest triggering time and the next triggering time of this neuron is determined by a criterion based on its neighborhood information as well. We prove that the Hopfield neural networks are completely stable under this event-triggered rule. The main technique used is using Łojasiewicz inequality to prove the finiteness of trajectory length. Then we prove that the event-triggered rule is realizable by the exclusion of Zeno behaviors. The event-triggered rule is distributed, i.e., each neuron only needs the information of its neighbors and itself, and asynchronous, i.e., all the neurons are not required to be triggered in a synchronous way, and independent to each other, i.e. triggering of an neuron will not affect or be affected by triggering of other neurons. It should be emphasized that our result can be easily extended to a large class of neural networks. For example, the standard cellular networks [28]-[29].

The paper is organized as follows: in Section II, the problem formulation and preliminaries are given; in Section

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III, the stability of Hopfield neural networks with event-triggered feedback is discussed; in Section IV, examples with numerical simulation are provided to show the effectiveness of the theoretical results and illustrate its application; the paper is concluded in Section V.

Notions: $\mathbb{R}^m$ denotes m-dimensional real space. The notation $\|\cdot\|$ represents the Euclidean norm for vectors or the induced 2-norm for matrices, and $\|x\|_\infty = max_i|x_i|$ for $x \in \mathbb{R}^m$. $B_r(x_0) = \{x \in \mathbb{R}^m : \ \|x - x_0\| < r\}$ stands for the ball with center $x_0 \in \mathbb{R}^m$ and radius $r > 0$. For a function $F(x) : \ \mathbb{R}^m \to \mathbb{R}$, $\triangledown F(x)$ means the gradient of $F(x)$. For a set $Q \subseteq \mathbb{R}^m$ and a point $x_0 \in \mathbb{R}^m$, $dist(x_0, Q) = \inf_{y\in Q} \|x_0 - y\|$ indicates the distance from x_0 to Q.

II. Problem formulation and preliminaries

In this section, the neural network model considered in this paper is introduced, and some related definitions, concepts and lemmas are given.

A. Problem formulation

Consider a continuous-time Hopfield neural network with discontinuous output feedbacks as follows

$$\begin{cases} C_i\dot{x}_i(t) = -\frac{x_i(t)}{R_i} + \sum_{j=1}^m \omega_{ij} y_j(t^i_{k_i(t)}) + \theta_i \\ y_i(t) = g_i(\lambda_i x_i(t)), \ i = 1, \cdots, m \end{cases} \tag{2}$$

where $x_i(t)$ represents the state of the neuron i at time $t \geq 0$; $W = [\omega_{ij}]$ is the synaptic symmetric interconnection matrix, i.e., $\omega_{ij} = \omega_{ji}$ for all $i, j = 1, \cdots, m$; $\theta = [\theta_1, \cdots, \theta_m]^\top$ is the biasing input vector; $C_i > 0$ and $R_i > 0$ are neuron self-inhibition coefficients; λ_i is scale factor; each input-output activation function $g_i(\cdot)$ is a sigmoid function: $g_i(s) = 1/(1 + exp(-s))$. The strict increasing triggering event time sequence $\{t^j_k\}^\infty_{k=1}$ (to be defined) are neuron-wise and $t^j_1 = 0$, for all $j \in \mathcal{I}$ with $\mathcal{I} = \{1, \cdots, m\}$. At each t, each agent j pulls its neighbours with respect to an identical time point $t^j_{k_j(t)}$ with $k_j(t) = arg\max_{k'}\{t^j_{k'} \leq t\}$.

In order to design the appropriate triggering times, we define the state measurement error as:

$$e_i(t) = \sum_{j=1}^m \omega_{ij} y_j(t^i_k) - \sum_{j=1}^m \omega_{ij} y_j(t) \tag{3}$$

for $t \in [t^i_k, t^i_{k+1})$, $k = 0, 1, 2, ...$, and $i = 1, \cdots, m$. Let $C = diag[C_1, \cdots, C_m]$, $R = diag[R_1, \cdots, R_m]$, $\Lambda = diag[\lambda_1, \cdots, \lambda_m]$, $g(x) = [g_1(x_1), \cdots, g_m(x_m)]^\top$, $Dg(x) = diag[g'_1(x_1), \cdots, g'_m(x_m)]$ with $x \in \mathbb{R}^m$, $e(t) = [e_1(t), \cdots, e_m(t)]^\top$, $x(t) = [x_1(t), \cdots, x_m(t)]^\top$.

Denote

$$f_i(x(t)) = -\frac{x_i(t)}{R_i} + \sum_{j=1}^m \omega_{ij} y_j(t^i_{k_i(t)}) + \theta_i.$$

Let $f(x(t)) = [f_1(x(t)), \cdots, f_m(x(t))]^\top$. For $x_0 \in \mathbb{R}^m$, we regard x_0 as a constant function, thus $f_i(x_0)$ and $f(x_0)$ are well defined.

B. preliminaries

In this paper, we consider the completely stable (or convergence) system (2). Denote the set of equilibrium points for (2) as

$$\mathcal{S} = \{x \in \mathbb{R}^m : \ f(x) = 0\}.$$

Definition 1: [30] System (2) is said to be completely stable (or convergence) if for any trajectory $x(t)$ of (2), there exists $x_0 \in \mathcal{S}$ such that

$$\lim_{t\to\infty} x(t) = x_0.$$

Next we will show that all solutions for (2) are bounded, and there exists at least one equilibrium point.

Property 1: For any given triggering event time sequence $\{t^i_k, \ i \in \mathcal{I}, \ k = 1, 2, \cdots\}$, any initial condition in $\mathbb{R}^m$, there exists a unique solution for the piece-wise Cauchy problem (2), and the solution is bounded for all $t \geq 0$.

Proof: (a) For any given $\{t^i_k\}$ ordered as $0 = t_1 < t_2 < \cdots < t_k < \cdots$ (same items in $\{t^i_k\}$ treat as one). It is clear that in $[t_1, t_2]$, there exists a unique solution of (2) for any initial condition $x(t_1) \in \mathbb{R}^m$ (existence and uniqueness theorem [31]). Thus, we can regard $x(t_2)$ as the initial condition for next interval $[t_2, t_3]$. By induction, we can conclude that there exists a unique solution for the piece-wise Cauchy problem (2).

(b) [1]Since $0 < g_i(s) < 1$ for all $s \in \mathbb{R}$, then there exists $M_1 > 0$ such that

$$-\frac{x_i(t)}{R_i} - M_1 \leq f_i(x(t)) \leq -\frac{x_i(t)}{R_i} + M_1.$$

Thus for any given ε_0, there exists $r_0 > 0$ such that

$$\begin{cases} f_i(x(t)) < -\varepsilon_0, \ \forall x_i(t) \geq r_0 \\ f_i(x(t)) > \varepsilon_0, \ \forall x_i(t) \leq -r_0, \end{cases}$$

for all $i \in \mathcal{I}$. Thus

$$\mathcal{C} = \{x \in \mathbb{R}^m : \ \|x\|_1 \leq r_0\}$$

is positively invariant. Moreover, if $x(0) \notin \mathcal{C}$, $x(t)$ will enters $\mathcal{C}$ in finite time. This implies that all solutions for (2) are eventually confined in $\mathcal{C}$, hence they are bounded on $t \in [0, +\infty)$.

Obviously, $\mathcal{S}$ also is the equilibrium points set for the following continuous-time Hopfield neural network with continuous inputs

$$\begin{cases} C_i\dot{x}_i(t) = -\frac{x_i(t)}{R_i} + \sum_{j=1}^m \omega_{ij} y_j(t) + \theta_i \\ y_i(t) = g_i(\lambda_i x_i(t)), \ i = 1, \cdots, m \end{cases}$$

Thus from [12], we have

Property 2: For the equilibrium points set $\mathcal{S}$, the followings statements hold:

1) $\mathcal{S}$ is not empty;
2) there exists $r_1 > 0$ such that $\mathcal{S} \bigcap (\mathbb{R}^m \setminus B_{r_1}(\mathbf{0})) = \emptyset$.

[1]The proof of this part comes from [12] with minor modifications.

We consider the following candidate Lyapunov (or energy) function from [2]:

$$L(x) = \sum_{i=1}^{m}\left[\frac{1}{\lambda_i R_i}\int_0^{g_i(\lambda_i x_i)} g_i^{-1}(s)ds - \theta_i g_i(\lambda_i x_i)\right.$$
$$\left. - \frac{1}{2}\sum_{j=1}^{m}\omega_{ij} g_i(\lambda_i x_i) g_j(\lambda_j x_j)\right]. \tag{4}$$

A function $L:\ \mathbb{R}^m \to \mathbb{R}$ is said to be a strict Lyapunov function for (2), if $L \in C^1(\mathbb{R}^m)$, and the derivative of L along trajectories of (2), $\dot{L}(x(t))$, satisfies $\dot{L}(x) \le 0$ for $x \in \mathbb{R}^m$ and $\dot{L}(x) < 0$ for $x \notin \mathcal{S}$ (see [12]).

The following inequality, named Łojasiewicz inequality [15], will play an essential role in the proof our main result.

Lemma 1: Consider a function $F(x)\ :\ \mathcal{B} \subseteq \mathbb{R}^m \to \mathbb{R}$, which is analytic in the open set $\mathcal{B}$. Suppose that the following set

$$\mathcal{S}_\nabla = \{x \in \mathcal{B}:\ \nabla F(x) = \mathbf{0}\}$$

is not empty. Then, for any $x_0 \in \mathcal{S}_\nabla$, there exist constants $r(x_0) > 0$ and $0 < v(x_0) < 1$, such that

$$\|\nabla F(x)\|_2 \ge |F(x) - F(x_0)|^{v(x_0)}$$

for $x \in B_{r(x_0)}(x_0)$.

Let $x(t),\ t \in [0, +\infty)$, be some trajectory of (2). For any $t > 0$, the length of $x(t)$ on $[0, t)$ is given by

$$l_{[0,t)} = \int_0^t \|\dot{x}(s)\| ds.$$

Significant contributions of [8] lies in that one can prove the existence of equilibrium and its stability (convergence of the trajectory) simultaneously by proving the finiteness of the length. This idea was also used in [12] to discuss global stability of analytic neural networks.

III. Continuous monitoring

We consider the case of symmetric case, i.e., $\omega_{ij} = \omega_{ji}$ for all $i, j \in \mathcal{I}$. The main result on complete stability for the neural network model (2) is presented in the following theorem.

Theorem 1: Set t_{k+1}^i as the time point such that for any fixed $\gamma \in (0,1)$

$$t_{k+1}^i = \max\left\{\tau \ge t_k^i:\ \left|\sum_{j=1}^{m}\omega_{ij} y_j(t_k^i) - \sum_{j=1}^{m}\omega_{ij} y_j(t)\right|\right.$$
$$\left. \le \gamma\left| - \frac{x_i(t)}{R_i} + \sum_{j=1}^{m}\omega_{ij} y_j(t_k^i) + \theta_i\right|,\ \forall t \in [t_k^i, \tau]\right\}. \tag{5}$$

Then, system (2) is completely stable.

Proof: The main techniques of the proof come from [12] with some modifications.

Firstly, we will prove following claim.

Claim 1: The $L(x)$ in (4) is a strict Lyapunov function for (2).

Note, for any point $x \in \mathbb{R}^m$

$$\begin{aligned}
&\frac{\partial}{\partial x_i} L(x) \\
&= \frac{1}{\lambda_i R_i}\lambda_i x_i \lambda_i g'(\lambda_i x_i) - \theta_i \lambda_i g'(\lambda_i x_i) \\
&\quad - \sum_{j=1}^{m}\omega_{ij} g_j(\lambda_j x_j)\lambda_i g'(\lambda_i x_i) \\
&= -\lambda_i g'(\lambda_i x_i)\left[-\frac{1}{R_i}x_i + \theta_i + \sum_{j=1}^{m}\omega_{ij} g_j(\lambda_j x_j)\right] \\
&= -\lambda_i g'(\lambda_i x_i) f_i(x).
\end{aligned} \tag{6}$$

Thus, for any trajectories $x(t)$ of (2)[2]

$$\begin{aligned}
&\frac{\partial}{\partial x_i} L(x(t)) \\
&= -\lambda_i g'(\lambda_i x_i(t))\left[-\frac{1}{R_i}x_i(t) + \theta_i + \sum_{j=1}^{m}\omega_{ij} y_j(t)\right] \\
&= -\lambda_i g'(\lambda_i x_i(t))\Big[f_i(x(t)) - e_i(t)\Big].
\end{aligned} \tag{7}$$

Then, the time derivative of $L(x(t))$ along (2) is

$$\begin{aligned}
\dot{L}(x(t)) &= \sum_{i=1}^{m}\frac{\partial}{\partial x_i} L(x(t))\frac{dx_i(t)}{dt} \\
&= -\sum_{i=1}^{m}\frac{\lambda_i}{C_i} g'(\lambda_i x_i(t))\Big[f_i(x(t)) - e_i(t)\Big] f_i(x(t))
\end{aligned} \tag{8}$$

From (5) and (8), and noting $g_i(s)$ is strict monotone increasing, we have

$$\begin{aligned}
&\dot{L}(x(t)) \\
&\le -\sum_{i=1}^{m}\frac{\lambda_i}{C_i} g'(\lambda_i x_i(t)) f_i(x(t)) f_i(x(t)) \\
&\quad + \gamma\sum_{i=1}^{m}\frac{\lambda_i}{C_i} g'(\lambda_i x_i(t)) f_i(x(t)) f_i(x(t)) \\
&= -(1-\gamma)\sum_{i=1}^{m}\frac{\lambda_i}{C_i} g'(\lambda_i x_i(t)) f_i(x(t)) f_i(x(t)) \\
&\le 0,
\end{aligned} \tag{9}$$

and $\forall x \notin \mathcal{S}$, there exits $i_0 \in \mathcal{I}$ such that $f_{i_0}(x) \ne 0$. Thus $\dot{L}(x) < 0$. This completes the proof of of Claim 1.

Secondly, similar to [12], we have following result.

Claim 2: (Property 4 in [12]) There exist a finite number $m_0 \ge 1$ of different energy levels $L_j,\ j = 1, 2, \cdots, m_0$, such that each set of equilibrium points

$$\mathcal{S}_j = \{x \in \mathcal{S}:\ L(x) = L_j\}$$

$j = 1, 2, \cdots, m_0$, is not empty.

Without loss of generality, assume that the energy levels $L_j,\ j = 1, 2, \cdots, m_0$ are ordered as $L_1 > L_2 > \cdots > L_{m_0}$.

[2]Equation (6) represents the normal partial derivative of the function $L:\ \mathbb{R}^m \to \mathbb{R}$ in (4) and in order to avoid ambiguity, we point out that $f_i(x) = -\frac{x_i}{R_i} + \sum_{j=1}^{m}\omega_{ij} g_i(\lambda_i x_i) + \theta_i$, for all $i \in \mathcal{I}$. Equation (7) represents the partial derivative of the function L along trajectories of (2).

Thus there exists $\gamma > 0$ such that $L_j > L_{j+1} + 2\gamma$, for any $j = 1, 2, \cdots, m_0 - 1$. For any given $\epsilon > 0$, define

$$\Gamma_j = \{x \in \mathbb{R}^m : dist(x, \mathcal{S}_j) \le \epsilon\}$$

and

$$\mathcal{K}_j = \Gamma_j \bigcap \Big\{ x \in \mathbb{R}^m : \ L(x) \in [L_j - \gamma, L_j + \gamma] \Big\}.$$

Claim 3: (Property 5 in [12]) For $j = 1, 2, \cdots, m_0 - 1$, $\mathcal{K}_j$ is a compact set and $\mathcal{K}_j \bigcap \mathcal{S} = \mathcal{S}_j$.

Thirdly, we will prove following claim.

Claim 4: For any trajectory $x(t)$ of (2) and any given $t_0 \ge 0$, suppose $x(t_0) \in \mathcal{K}_j \setminus \mathcal{S}$ for some $j \in \{1, 2, \cdots, m_0\}$, then there exist a constant $c_j > 0$ and $v_j \in (0, 1)$ such that

$$\frac{|\dot{L}(x(t_0))|}{\|f(x(t_0))\|} \ge c_j |L(x(t_0)) - L_j|^{v_j}. \tag{10}$$

In order to avoid ambiguity, we point out that $f_i(x(t_0)) = -\frac{x_i(t^i_{k_i(t_0)})}{R_i} + \sum_{j=1}^m \omega_{ij} y_j(t^i_{k_i(t_0)}) + \theta_i$, for all $i \in \mathcal{I}$.

From (7) and (5), we have

$$\begin{aligned}\Big|\frac{\partial}{\partial x_i} L(x(t_0))\Big| &= \Big|\lambda_i g'(\lambda_i x_i(t_0)) \Big[f_i(x(t_0)) - e_i(t_0)\Big]\Big| \\ &\le \Big|(1+\gamma)\lambda_i g'(\lambda_i x_i(t_0)) f_i(x(t_0))\Big|. \end{aligned} \tag{11}$$

Let

$$h_j = \min_{i \in \mathcal{I}} [\min_{x \in \mathcal{K}_j} \frac{1}{(1+\gamma)\lambda_i g'(\lambda_i x_i)}] > 0.$$

Thus,

$$\|f(x(t_0))\| \ge h_j \|\nabla L(x(t_0))\|.$$

Additionally, letting

$$\xi_j = \min_{i \in \mathcal{I}} [\min_{x \in \mathcal{K}_j} (1-\gamma) \frac{\lambda_i}{C_i} g'(\lambda_i x_i)] > 0,$$

from (5) and (9), we have

$$|\dot{L}(x(t_0))| \ge \xi_j \|f(x(t_0))\|^2.$$

By the same arguments as in the proof of Lemma 2 in [12], we can complete the proof of Claim 4.

Fourthly, like [12], we can prove that the length of $x(t)$ on $[0, +\infty)$ is finite.

Claim 5: (Theorem 3 in [12]) For any trajectories $x(t)$ of (2) has finite length on $[0, +\infty)$, i.e.,

$$l_{[0,+\infty)} = \int_0^{+\infty} \|\dot{x}(s)\| ds = \lim_{t \to +\infty} \int_0^t \|\dot{x}(s)\| ds < +\infty.$$

Finally, by the same argument as the proof of Theorem 1 in [12], we can prove that for any trajectories $x(t)$ of (2), there exists $\lim_{t \to +\infty} x(t) = x_1 = constant$, where x_1 is an equilibrium point of (2).

Next we will prove that the above event-triggered rule (5) is realizable, i.e., the inter-event times (the time length between two continuous triggering times), are lower bounded away from zero, which is also known as not exhibiting Zeno behavior [32]. This is proven in the following theorem:

Theorem 2: For the event-triggered rule in Theorem 1 and any initial condition of each neuron, at any time $t \ge 0$ there exists at least one neuron j_1 for which the next inter-event time is strictly positive.

Proof: The main techniques of the proof in this part come from [17] with some modifications. We will show that there exists at least one neuron such that its next inter-event interval is bounded from below by a certain time $\tau_D > 0$. Denoting

$$j_1 = arg \max_{i \in \mathcal{I}} \Big| -\frac{x_i(t)}{R_i} + \sum_{j=1}^m \omega_{ij} y_j(t^i_{k_i(t)}) + \theta_i \Big|.$$

Thus

$$|C_{j_1} \dot{x}_{j_1}(t)| \ge \frac{1}{\sqrt{m}} \|C\dot{x}(t)\|.$$

In addition,

$$\Big| \sum_{j=1}^m \omega_{ij} y_j(t^i_k) - \sum_{j=1}^m \omega_{ij} y_j(t) \Big| = |e_i(t)| \le \|e(t)\|.$$

So

$$\frac{|e_i(t)|}{C_{j_1} |\dot{x}_{j_1}(t)|} \le \frac{\sqrt{m}\|e(t)\|}{\|C\dot{x}(t)\|}.$$

We can calculate the time derivative of $\frac{\|e(t)\|}{\|C\dot{x}(t)\|}$:

$$\begin{aligned} &\frac{d}{dt}\frac{\|e(t)\|}{\|C\dot{x}(t)\|} \\ =&\frac{[e(t)]^\top[-\dot{y}(t)]}{\|e(t)\|\|C\dot{x}(t)\|} - \frac{\|e(t)\|[C\dot{x}(t)]^\top C\ddot{x}(t)}{\|C\dot{x}(t)\|^2\|C\dot{x}(t)\|} \\ =&\frac{[e(t)]^\top[-W\Lambda Dg(\Lambda x(t))\dot{x}(t)]}{\|e(t)\|\|C\dot{x}(t)\|} \\ &- \frac{\|e(t)\|[C\dot{x}(t)]^\top C[-C^{-1}R^{-1}\dot{x}(t)]}{\|C\dot{x}(t)\|^2\|C\dot{x}(t)\|} \\ \le& k_1 + k_2 \frac{\|e(t)\|}{\|C\dot{x}(t)\|} \end{aligned}$$

where $k_1 = max_{\tau \in [t, +\infty)} \|W\Lambda Dg(\Lambda x(\tau))\| \le \frac{1}{4}\|W\Lambda\|$, $k_2 = \|C^{-1}R^{-1}\|$.

Via comparison principle, we have $\frac{\|e(t)\|}{\|C\dot{x}(t)\|} \le \phi(t, \phi_0)$, where $\phi(t, \phi_0)$ is the solution of following differential equation

$$\begin{cases} \frac{d\phi}{dt} = k_2\phi + k_1 \\ \phi(t, \phi_0) = \phi_0 \end{cases}.$$

Hence the inter-event times of agent v_{j_1} are bounded from below by the time τ_D which satisfies $\phi(\tau_D, 0) = \frac{\gamma}{\sqrt{m}}$. We can calculate τ_D as follows.

$$\int_0^{\frac{\gamma}{\sqrt{m}}} \frac{d\phi}{k_2\phi + k_1} = \int_0^{\tau_D} dt,$$

which yields

$$\tau_D = \frac{1}{k_2} ln(1 + \frac{k_2}{k_1}\frac{\gamma}{\sqrt{m}}). \tag{12}$$

This completes the proof.

Remark 1: Here we should point out that although for some neurons the next inter-event time may infinite, from the proof process of Theorem 1, we still can conclude that the Hopfield neural networks are completely stable under the event-triggered rule (5).

Remark 2: As the discussion in [21], since larger τ_D implies less control updating times, a larger τ_D implies that less resources are needed for the equipment, like embedded microprocessors, to communicate between neurons. On the other hand, a smaller $\dot{L}(x(t))$ means a faster convergence speed. We know from (12) that τ_D is increasing with respect to γ. Then a larger γ leads to less control updating times for each neuron, while a smaller γ leads to bigger low bound of the system convergence rate according to (9). It should be emphasized that we can not say a smaller γ leads to faster convergence according to (9). Therefore, the protocol designer should choose a proper γ for a compromise between control actuation times and system convergence rate.

IV. Examples

In this section, three numerical examples are given to demonstrate the effectiveness of the presented results and their application.

Example 1, consider a 2-dimension Hopfield neural network with

$$C = R = \Lambda = I_2,\ \theta = \begin{bmatrix} 1 \\ -1 \end{bmatrix},\ W = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}.$$

The initial value of each neuron is randomly selected within the interval $[-1, 1]$ in our simulations. Figure 1 shows that the state $x(t)$ converges to $\nu = [2.7072, -1.6021]^\top$.

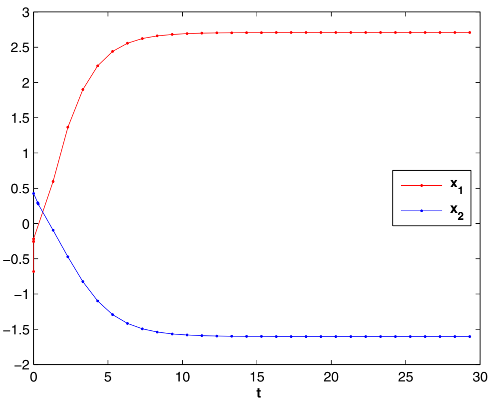

Fig. 1. Convergence of $x(t)$ with $x(0) = [-0.6802, 0.4267]^\top$ and $\gamma = 0.99$.

Then, the parameter γ takes different values while adopting the triggered principles provided in Theorem 1 and the corresponding simulation results are listed in Table I. The T_1 in the table denotes the first time when $\|x(t) - \nu\| \leq 0.0001$, which can be seen as an index representing the convergence speed of the event-triggered rule. All the data in this table are the averages over 50 overlaps. It can be seen that all the actual minimum inter-event times are greater than the corresponding τ_D calculated by (12). The minimum value of event interval and the actual number of event decreases with respect to γ, and T_1 increases with respect to γ, which are consistent with the theoretical analysis. The τ_m in the table represents the minimum value of the length of event interval. The N_m in the table stand for number of event times.

TABLE I
Simulation results with different γ under the triggered principle in Theorem 1.

γ	τ_D	τ_m	N_m	T_1
0.1	0.0901	0.6072	42.10	31.6612
0.2	0.1727	0.8920	26.90	32.5330
0.3	0.2491	1.0560	21.16	32.8354
0.4	0.3200	1.1643	17.92	33.0597
0.5	0.3862	1.2014	15.58	33.0533
0.6	0.4483	1.2020	14.48	33.2765
0.7	0.5068	1.2018	12.70	33.4677
0.8	0.5620	1.2028	13.22	33.4744
0.9	0.6144	1.2043	12.76	33.4854

Noting the definition of (4), if we choose $\|\theta\|$ sufficient small and let $\lambda_i \to +\infty$ for all $i \in \mathcal{I}$, then

$$L(x) \approx -\sum_{i=1}^{m} \frac{1}{2} \sum_{j=1}^{m} \omega_{ij} g_i(\lambda_i x_i) g_j(\lambda_j x_j).$$

Next, as an application of our result, we will give two simple examples of seeking local minimum point of quadratic polynomial $H(y) = \frac{-1}{2} y^\top W y$. As an illustration, we still consider 2-dimension problem. To minimize $H(y) = \frac{-1}{2}[\omega_{11} y_1^2 + \omega_{22} y_2^2 + \omega_{12} y_1 y_2]$ over $\{0, 1\}^2$. Denote $\bar{y}(\Lambda) = \lim_{t \to +\infty} g(\Lambda x(t))$, where $x(t)$ is the trajectory of (2). Thus $\bar{y}(\Lambda)$ is the local minimum point of $H(y)$ if $\|\theta\|$ sufficient small and $\lambda_i \to +\infty$ for all $i \in \mathcal{I}$.

Example 2, let

$$C = R = I_2,\ W = \begin{bmatrix} 1.5 & -0.2 \\ -0.2 & 1.1 \end{bmatrix}.$$

Figure 2 shows that the terminal limit $\bar{y}(\Lambda)$ converge to a local minimum point $[1, 0]^\top$ as $\lambda_i \to +\infty$ for all $i \in \mathcal{I}$.

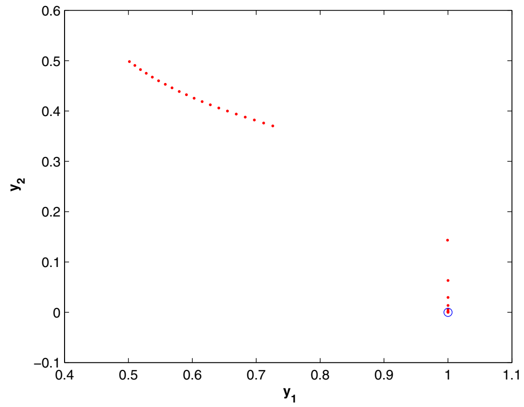

Fig. 2. The limits $\bar{y}(\Lambda)$ converge to a local minimum point $[1, 0]^\top$. We select $\theta = [0.001, 0.001]^\top$, $\gamma = 0.99$, and random initial data within the interval $[-5, 5]$. $\lambda_1 = \lambda_2$ are select from 0.01 to 100.

Example 3, let

$$C = R = I_2,\ W = \begin{bmatrix} 1.0965 & -1.0787 \\ -1.0787 & 1.2535 \end{bmatrix}.$$

Obviously, the objective function has two local minimum points: $[1, 0]^\top$ and $[0, 1]^\top$. Figure 3 shows that the terminal limit $\bar{y}(\Lambda)$ converge to such two local minimum points as $\lambda_i \to +\infty$ for all $i \in \mathcal{I}$.

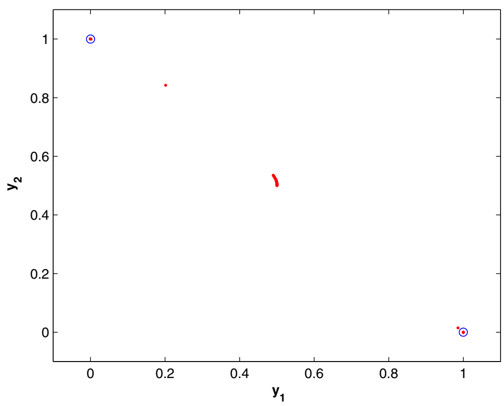

Fig. 3. The limits $\bar{y}(\Lambda)$ converge to two local minimum points $[1, 0]^\top$ and $[0, 1]^\top$. We select $\theta = [0.001, 0.001]^\top$, $\gamma = 0.99$, and random initial data $x(0)$ within the interval $[-5, 5]$. $\lambda_1 = \lambda_2$ are select from 0.01 to 100.

V. Conclusion

In this paper, an event-triggered output feedback rule for Hopfield neural network with symmetric synaptic neuron interconnection matrix is proposed. It is proved that the Hopfield neural network is completely stable under this event-triggered rule, and that Zeno behavior can be excluded. In addition, this event-triggered rule is not only distributed and asynchronous but also independent. As a result, the information exchange frequency among all neurons can be significantly reduced. It should be point out that the main technique to prove the absolute stability in main result is finite-length of trajectory, and the core of the proof of finite-length of trajectory is the Łojasiewicz inequality [12]. Three numerical examples are given to demonstrate the effectiveness of the presented results and their application. Our future work will include the self-triggered formulation and event-triggered stability of other more general systems.

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