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Stability of a neutral delay neuron system in the critical case

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***Abstract*—In this paper, the asymptotic stability properties of neutral-type neuron system are studied mainly in the critical case when the exponential stability is not possible. In the case of a critical value of the coefficient in neutral-type neuron system, the difficulty for our investigation is the fact that the spectrum of the linear operator is asymptotically approximated to the imaginary axis. Hence, based on the energy method, the asymptotic stability results for neutral-type neuron system are derived, and a complete analysis of the stability diagram is presented.**

Keywords—neuron system; asymptotic stability; neutral-tyep; critical case;

I. INTRODUCTION

Systems with neutral differential equations arise when modeling biological, physical, neurophysiological, etc. processes whose rate of change of state at any moment of time t is determined not only by the present states, but also by past states[1-9]. In the research field of neural networks, some particular neutral differential systems have been proven to be highly useful when the problem has capability to memorize dynamic information. Usually, the transmission of a signal through the brain and neural chains requires some time, and is more complicated than other transmission, such as temperature transmission, etc., although it is possible to propose an evolution neural model without delay also in this case. However, the use of the delay equation may be advantageous, since there is no need to analyze details of the transmission, and it allows us to add a term to the delay τ to make up for the time of signal transmission [5].

How can the states of a neuron at previous moments of time influence the present state of evolution of the neuron? It is well known that at each moment the neuron 'knows' only the situation at that moment, and hence can react only to this. Even if we take into consideration the memory of human beings, we should be able to consider only the memory state of the current moment.

A natural issue when dealing neuron system of neutral-type is to prove existence, uniqueness and global exponential stability of the equilibrium based on various techniques, for example, the linearization technique [1], Lyapunov approach [2], linear matrix inequality and neutral transformation [3,4], Lyapunov-Krasovskii functional and the descriptor system approach[11], and so on. However, the mentioned-above papers [1-4, 11] usually address this question by considering neutral-type neuron system with $|p|<1$, i.e., guarantee the exponential stability of the difference operator. But, the critical value $|p|=1$ raises technical difficulties and seldom involve in yet at present. This is because the spectrum of the linearized operator is asymptotically approximated to the left of the imaginary axis, and furthermore, the exponential stability is lost.

A lot of studies on delay equations start from the local stability analysis of some special solutions. For this purpose, the standard method is to analyze its variational equation around that special solution. If the special solution is a constant, the variational equation takes the form of linear scalar delay equation. The stability of the trivial solution (i.e., the zero solution) of the variational equation depends on the location of the roots of an associated characteristic equation. Although this is an extensively studied topic, most of the results derived so far apply only for retarded equations. Systematic results for the more general neutral delay equations are still very much in demand. Furthermore, an old contribution to the critical $|p|=1$ is discussed in [10]. In the linear case, algebraic rate of convergence is proven. In nonlinear case with $p=-1$ algebraic convergence is also proven in that paper, but assuming small C^1 data. In both cases, the main tools involved are asymptotic expansions of characteristic roots, Laplace transforms and function series.

The main work of our paper is two aspects. First, we generalize a single neuron system of neutral-type in [11] to a more general neutral-type neuron system. Then, we push forwards the analysis of this neutral-type neuron system in its critical case, especially to remove (when possible) the assumption of small solutions, and also to consider less regular initial data. A strategy based on energy analysis is followed.

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This paper is organized as follows. Section II treats the full nonlinear NDDE. The stability of the zero solution in the homogenous case is proven. Generalization to the non-homogeneous case with constant external input is made in Section III. Finally, some conclusions are drawn in Section IV, and some future lines of research are given.

II. STABILITY IN THE HOMOGENEOUS CASE

In this section, we first give some definitions and lemmas used in this paper.

Definition 1. Lebesgue spaces are function spaces associated to measure spaces (X, M, μ), where X is a set, M is a $\sigma-$algebra of subsets of X, and μ is a countably additive measure on M. Let $L^2(X,\mu)$ be the space of those complex-valued measurable functions on X for which the Lebesgue integral of the square of the absolute value of the function is finite, *i.e.*, for a function f in $L^2(X,\mu)$, $\int_X |f|^2 d\mu < \infty$, and where functions are identified if and only if they differ only on a set of measure zero.

Definition 2. The space $L^\infty(S,\mu)$ is defined as follows. We start with the set of all measure functions from S to C (or R) which are essentially bounded, *i.e.*, bounded up to a set of measure zero. Again two such functions are identified if they are equal almost everywhere.

Definition 3. Sobolev spaces, denoted by H^s or $W^{s,2}$, are Hilbert spaces. These are a special kind of function space in which differentiation may be performed, but that (unlike other Banach spaces such as the Hölder spaces), support the structure of an inner product.

Definition 4. The Hölder spaces $C^{k,\alpha}(\Omega)$, where Ω is an open subset of some Euclidean space and $k \geq 0$ an integer, consists of those functions on Ω having continuous derivatives up to order k and such that the kth partial derivatives are Hölder continuous with exponent α, where $0 < \alpha \leq 1$.

In this paper, we consider the following neural model which is modeled by nonlinear neutral delay differential equation (NDDE) and search for the solution $x \in H^1_{loc}\left((-1,+\infty),\Re\right)$:

$$\begin{cases} \dfrac{d}{dt}[x(t)-px(t-\tau)] = -f(x(t)) + g(x(t-\tau)) + I(t), t \geq 0. \\ x(t) = x_0(t) \in H^1\left((-\tau,0),\Re\right), \quad -\tau < t < 0 \end{cases} \quad (1)$$

If $t-\tau \to \infty$ as $t \to \infty$, then Eq. (1) is said to have the property of completely forgetting the past. This means that the values of the solution x on any finite interval do not influence the right-hand side of the equation for sufficiently large t. In other words, the rate of change of the neuron at any moment is determined by the states of the neuron at preceding moments which are not too remote. Sometimes such equations are said to have fading memory.

In order to study the stability of (1), we need make the following assumptions

$$\begin{cases} p = \pm 1, \\ f \in C^1(\Re,\Re), \quad f' > 0,\ f(0) = 0 \\ |g| \leq \gamma |f|, \text{ with } 0 \leq \gamma < 1 \end{cases} \quad (2)$$

In some cases, we also require the following additional assumptions

$$\begin{cases} |g'| \leq \gamma |f'|, \text{ with } 0 \leq \gamma < 1, \\ \lim\limits_{y\to\pm\infty} f(y) = \pm\infty. \end{cases} \quad (3)$$

Let $s = \tau t$, then Eq. (1) becomes

$$\begin{cases} \dfrac{d}{dt}[y(t)-py(t-1)] = -\tau f(y(t)) + \tau g(y(t-1)) + \tau I(t)], \\ y(t) = y_0(t) \in H^1\left((-1,0),\Re\right), \quad -1 < t < 0. \end{cases} \quad (4)$$

First, we introduce basic functions and inequalities in order to state a main result:

$$F(y) = \tau\int_0^y f(z)dz, \quad G(y) = \tau\int_0^y g(z)dz, \quad H(y) = 2(F(y) - pG(y)), \quad (5)$$

with f, g, γ and p defined in (1)-(2).

Lemma 1. For all y, F and H satisfy the following inequality

$$0 \leq 2(1-\gamma)F(y) \leq H(y) \leq 2(1+\gamma)F(y)$$

Proof. Assume $y>0$. By (2) and (5), it follows

$$|G(y)| \leq \tau\int_0^y |g(z)|dz \leq \tau\gamma\int_0^y |f(z)|dz = \gamma F(y)$$

Similarly, if $y<z<0$, then $|f(z)| = -f(z)$ and

$$|G(y)| \leq \tau\int_y^0 |g(z)|dz \leq \tau\gamma\int_y^0 |f(z)|dz = \gamma F(y)$$

As a result, we get $0 \leq |G| \leq \gamma F$ for all y. Using (5) and $p = \pm 1$ concludes the proof. ∎

In the following, we derive asymptotic stability of the zero solution for system (4). Hence, we have

Theorem 1. Assume $y(t)$ be the solution of (4) without external input I=0. For all t>0, we define

$$E(t) = \int_{t-1}^t \left\{\left(\frac{dy(\xi)}{d\xi} + \tau f(y(\xi))\right)\right\}^2 d\xi \quad (6)$$

Then we have

$$\sup_{t>0} E(t) + \sup_{t>0} F(y(t)) + \tau^2 \int_0^{+\infty} (f(y(t)))^2 dt < +\infty, \quad (7)$$

where F is defined in (5). It follows the asymptotic stability of the origin

$$\lim_{t\to+\infty} y(t) = 0. \quad (8)$$

Proof. If $I(t) = 0$, Eq. (4) can become

$$\frac{dy(t)}{dt} + \tau f(y(t)) = p\left(\frac{dy(t-1)}{dt} + p\tau g(y(t-1))\right). \quad (9)$$

For $p = \pm 1$ and taking the square, we obtain

$$\left\{\frac{dy(t)}{dt} + \tau f(y(t))\right\}^2 = \left\{\left(\frac{dy(t-1)}{dt} + p\tau g(y(t-1))\right)\right\}^2 \quad (10)$$

By (5), the right-hand side of (10) at *t*-1 is rewritten

$$\left\{\left(\frac{dy(t-1)}{dt}+p\tau g(y(t-1))\right)\right\}^2$$
$$=\{\left[\frac{dy(t-1)}{dt}+\tau f(y(t-1))\right]$$
$$-\left[\tau f(y(t-1))-p\tau g(y(t-1))\right]\}^2$$
$$=\left[\frac{dy(t-1)}{dt}+\tau f(y(t-1))\right]^2-\frac{dH(t-1)}{dt}$$
$$+\tau^2[f(y(t-1))-pg(y(t-1))]$$
$$[f(y(t-1))-p\tau g(y(t-1))-2f(y(t-1))]$$
$$=\left[\frac{dy(t-1)}{dt}+\tau f(y(t-1))\right]^2$$
$$-\tau^2[f^2(y(t-1))-g^2(y(t-1))]-\frac{dH(t-1)}{dt}. \quad (11)$$

The above equation is replaced into (10), which yields

$$\left\{\left(\frac{dy(t)}{dt}+\tau f(y(t))\right)\right\}^2$$
$$+\tau^2[f^2(y(t-1))-g^2(y(t-1))]+\frac{dH(t-1)}{dt} \quad (12)$$
$$=\left[\frac{dy(t-1)}{dt}+\tau f(y(t-1))\right]^2$$

Integrating (12) over [0, *t*], we get for all *t*>1

$$\int_0^t\left\{\left(\frac{dy(\xi)}{d\xi}+\tau f(y(\xi))\right)\right\}^2 d\xi$$
$$+\tau^2\int_{-1}^{t-1}[f^2(y(\xi))-g^2(y(\xi))]d\xi+H(y(t-1)) \quad (13)$$
$$-H(y(-1))=\int_{-1}^{t-1}\left\{\left(\frac{dy(\xi)}{d\xi}+\tau f(y(\xi))\right)\right\}^2 d\xi$$

Equation (13) is simplified into the energy equality

$$\int_{t-1}^t\left\{\left(\frac{dy(\xi)}{d\xi}+\tau f(y(\xi))\right)\right\}^2 d\xi$$
$$+\tau^2\int_0^{t-1}[f^2(y(\xi))-g^2(y(\xi))]d\xi+H(y(t-1))$$
$$=\int_{-1}^0\left\{\left(\frac{dy_0(\xi)}{d\xi}+\tau f(y_0(\xi))\right)\right\}^2 d\xi \quad (14)$$
$$+H(y_0(-1))-\tau^2\int_{-1}^0[f^2(y_0(\xi))-g^2(y_0(\xi))]d\xi$$

By Lemma 1 and Eq. (14), one deduces the energy inequality for all $\xi>1$

$$\int_{t-1}^t\left\{\left(\frac{dy(\xi)}{d\xi}+\tau f(y(\xi))\right)\right\}^2 d\xi$$
$$+(1-\gamma^2)\tau^2\int_0^{t-1}f^2(y(\xi))d\xi+2(1-\gamma)F(y(t-1))$$
$$\le\int_{-1}^0\left\{\left(\frac{dy_0(\xi)}{d\xi}+\tau f(y_0(\xi))\right)\right\}^2 d\xi$$
$$+2(1+\gamma)F(y_0(-1))-(1-\gamma^2)\tau^2\int_{-1}^0 f^2(y_0(\xi))d\xi=C_0 \quad (15)$$

Since the three terms on the left side of (15) are positive, furthermore, for all *t*>1, we obtain the following three inequalities:

$$F(y(t))\le\frac{C_0}{2(1-\gamma)},\ \tau^2\int_0^{+\infty}f^2(y(t))dt\le\frac{C_0}{1-\gamma^2},\ E(t)\le C_0 \quad (16)$$

The first inequality in (16) implies $y\in L^\infty(0,+\infty)$. On any bounded set *K*, there exists *d*>0 such that $d|y|\le|f(y)|\le {|y|}/{d}$ on *K*. Using the second inequality in (16) and the boundedness of *y*, one obtains $y\in L^2(0,+\infty)$, one deduces that $\frac{dy(t)}{dt}$ is uniformly bounded in $L^2(T-1,T)$ and hence *y* is bounded in the Hölder space $C^{0,\frac{1}{2}}(0,+\infty)$. It proves that *y* tends towards 0. ■

The proof of the stability for system (1) is based on an energy method. Notice that the inequality (7) allows convergence towards zero with an algebraic rate. Actually, there is not exponential stability of the equilibrium *y*=0.

Remark 1. If *p*=0 and $\gamma=0$ in system (4) with (2), then the NDDE is a simple ODE. Our computation provides for all *T*>0:

$$2F(y(T))+\int_0^T\left((\frac{dy(t)}{dt})^2+\tau^2f^2(y(t))\right)dt=2F(y(0)),$$

which implies the global stability of *y=0*. Moreover, the stability of *y*=0 is exponential since $f'(0)>0$.

Remark 2. In the usual case $|p|<1$, then a stronger energy estimate can be obtained by our proof: $y\in H^1(0,+\infty)$, and the stability is exponential.

Remark 3. In the present case $|p|=1$, the $y\in L^2\cap L^\infty$ and *y* is uniformly bounded in $H^1(\tau-1,\tau)$ for all τ.

Remark 4. If $|p|=1$ and $\gamma=1$, then the stability is obtained, but the asymptotic stability can be lost.

III. STABILITY IN COPNSTANT EXTERNAL INPUT

In this section, we consider a non-homogeneous NDDE (4) with constant external input *I*:

$$\begin{cases}\frac{d}{dt}[y(t)-py(t-1)]=-\tau f(y(t))+\tau g(y(t-1))+\tau I,\\ y(t)=y_0(t)\in H^1(-1,0),\quad -1<t<0\end{cases} \quad (17)$$

with assumption (2). For the sake of clarity, we analyze successively *g*=0 and $g\neq 0$.

Theorem 2(Case *g*=0). Assume *y*(*t*) be the solution of Eq. (17). One of three cases may occur:

(i)If *I*=*f*(*e*), then *e* is the unique globally attractive solution;

(ii)If I>sup f (or I<inf f), then $\lim_{t\to+\infty} y(t) = +\infty$ (or $-\infty$) ;

(iii)If I=sup f (or I=inf f), then there is no convergence towards a constant solution.

Proof. We consider successively the three cases.

Case 1: I=$f(e)$:

This case occurs if $\lim_{y\to\pm\infty} f(y) = \pm\infty$. Replacing

$$z = y - e,\ f_e(z) = f(e+z) - f(e)\,,$$

into (17) yields the homogeneous NDDE

$$\frac{dz(t)}{dt} - p\frac{dz(t-1)}{dt} = -\tau f_e'(z(t)) + ...$$

Since $f_e(0) = 0$ and $f_e' > 0$, the assumptions of Theorem 1 are satisfied: $\lim_{t\to+\infty} z(t) = 0$ and thus y tends asymptotically towards e.

Case 2: I>sup f (or I<inf f).

Assume $\sup f < I < +\infty$, and introduce $\delta = I - \sup f > 0$.

- If p=+1, then $\frac{dy(t)}{dt} - \frac{dy(t-1)}{dt} = \tau(I - f(y)) \geq \delta,$ so that $\frac{dy(n+\tau)}{d\tau} \geq y_0'(\tau) + n\delta,$ with $\tau \in [-1, 0]$. As a result, $\lim_{t\to+\infty} \frac{dy(t)}{dt} = +\infty,$ and hence $\lim_{t\to+\infty} y(t) = +\infty$.
- If $p = -1$, then $\frac{dy(t)}{dt} + \frac{dy(t-1)}{dt} \geq \tau\delta$. Integration on $[\tau - 1, \tau]$ yields $y(\tau) - y(\tau - 2) \geq \delta$, and once again $\lim_{t\to+\infty} y(t) = +\infty$.

Case 3: I=sup f (or I=inf f).

Let us assume $y(t) \to e$ when $t \to +\infty$. Let $y_n(s)$ be $y(n+s)$ for $s \in [0,1]$. We have $y_n(.) \to e$ in $L^\infty([0,1])$ and $y_n' \to 0$ in the sense of distribution. The NDDE can be rewritten as follows: $y_n' + py_{n-1}' + f(y_n) = I$ .Taking the weak limit, we get $f(e) = I$ which is impossible since f<I. As a result, y(.) cannot converge towards a constant.

Using the same argument, we can state that y(.) cannot converge towards a periodic continuous solution. More precisely, let us assume that $y_n(s) \to e(s)$ where e(.) is continuous.

Writing $y_n(0) = y(n) = y((n-1)+1) = y_{n-1}(1)$ yields e(0)=e(1) i.e., e is a 1-peridioc continuous function. Taking the weak limit in the NDDE yields a differential equation for $e(.) : (1-p)e'(t) + f(e(t)) = I$.

• If p=1, then $f(e(t))$=I, which is impossible.

• If $p = -1$, then $e \in C^1([0,1])$ by the equation. Let s_0 be a maximizer of e(.) on the compact set [0, 1], then $e'(s_0) = I$ and by the differential equation, $f(d(s_0)) = I$ which is again impossible.

As a consequence, y(.) cannot converge towards a periodic continuous solution. ■

Now, let us examine the NDDE (17) with all the terms.

Theorem 3 (Case $g \neq 0$). Under the assumption (3), the unique e such as $f(e) - g(e) = I$ is the unique globally attractive solution of (17).

Proof. Assumption (3) easily imply that there exists a unique e satisfying $f(e) + g(e) = I$. Replacing

$$z = y - e,\ f_e(z) = f(e+z) - f(e),\ g_e(z) = g(e+z) - g(e)\,,$$

into (17) yields the homogeneous NDDE

$$\frac{dz(t)}{dt} - p\frac{dz(t-1)}{dt} + f_e'(z(t)) - g_e'(z(t-1)) = 0\,.$$

Inequality (3) implies that $\gamma|f_e| \geq |g_e|$. Since $f_e(0) = 0$ and $f_e' > 0$, all the assumptions of Theorem 1 are satisfied, and hence $z \to 0$ asymptotically. ■

Without assumptions (3) in Theorem, one may encounter more complex situations, with 2 or more solutions.

IV. CONCLUSIONS

It is well-known that for the neutral-type system, the analysis of stability conditions is more complicated than a retarded-type system. We have shown that the condition of asymptotic stability for neutral-type neuron system in the critical case can be derived by means of the energy method. Existence and asymptotic stability of periodic solutions for system in the critical case and under external periodic stimulus are a important problem deserving of study.

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