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Pinning Dynamic Complex Networks by Time-Varying Controller-vertex Set

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***Abstract*—In this paper, we give a stability analysis of multi-agent system with a local pinning control algorithm for very general network topologies. These include determinately directed time varying topologies, the stochastically switching topologies. The pinned vertex set also varies with time, including deterministic and stochastic time-variations. We present sufficient conditions to guarantee the convergence of the pinning process: for the deterministic case, a time-varying pinned vertex set can stabilize the network of multi-agents with time-varying topologies if any vertex in the networks can be accessed by directed paths by at least one vertex in the pinned vertex set across all time intervals that are pre-defined; Similar results are also given for the stochastically switching case. As applications, numerical simulations based on the random waypoint model are given to verify our theoretical results.**

I. Introduction

A pinning control algorithm, which was first proposed to control the multi-mode laser systems [1], [2], is a viable strategy to drive networks of coupled oscillators onto some desired common reference trajectory. The general idea behind pinning control is to apply some local feedback controllers only to a fraction of network vertexes and the rest of vertexes can be propagated through the coupling among vertexes.

More close to the interest of this paper, in [4]-[12], the authors have stabilized a complex network to a homogeneous trajectory of the uncoupled system state. In these papers, authors considered two different pinning strategies: randomly pinning and selective pinning. [4], [5] concluded that according to the detailed complex network topology, selective pinning based on the connectivity degrees has better performance than totally randomly pinning. [9] indicated that the bounds of the network stabilizability are affected by not only the degrees of the pinned vertices but also the distance between the pinned vertex set and unpinned vertex set. More recently, the Lyapunov stability theory has been proposed to analyze the pinning controllability and the network stability (controllability) is converted to measuring the negative definiteness of one simple matrix characterizing the network topology. Hence, another important aspect of constructing pinning strategy is connected to the topology of the network.

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All aforementioned papers concern the complex dynamical system coupled via a time-invariant communication network. However, networks with time-varying topologies are considered to be more momentous. In [13]-[15], the pinning control strategies for networks with time varying topologies were addressed. The time-varying interaction network model based on mobile agents was investigated in [13], in which two pinning strategies: random selection that is static with time and stochastic selection that changes with time, are compared and concluded that the latter strategy performs better in some scenarios. [15] studied the pinning synchronization of a directed network with Markovian jump and nonlinear perturbations, and detailed pinning schemes were given based on the Lyapunov theory and the structure of the network.

A general stability analysis of pinning control a complex network with directed weighted time-varying graph topology is given in this paper. The selection of the pinning vertexes changes with time, too. By analyzing the structure of the network, the pinned set is strategically chosen to guarantee convergence of the network.

In addition, [16] studied the consensus problem in networks of multiagents with switching topologies modeled as adapted stochastic processes and proved that if there exist $T > 0$ and $\delta > 0$ such that the conditional expectation of the union of the δ-graph topologies across each T-length time interval has spanning trees, then the system reaches consensus. Furthermore, in [17], authors extended the results in [16] to networks with delays. These works inspire us to study the stability of pinning strategies in networks with general stochastically switching topologies, namely, induced by adapted stochastic processes, which include independent switching, Markovian jumping and hidden Markov switching as special cases.

This paper is organized as follows. In Section II, we present some definitions of graphs and give some notations required in this paper. In Section III, we investigate the pinning strategies of time varying complex networks and present the stability criteria. The dynamics of the pinning strategies of networks with switching topologies, which are modeled as adapted processes, are studied in section IV. An application is given in Section V to verify the theoretical results. We conclude this paper in Section VI.

II. Preliminary

For a matrix L, l_{ij} stands for the element of L on the ith row and jth column. $L^\top$ denotes the transpose of L and $L^s =$

$(L+L^\top)/2$ denotes the symmetry part of a square matrix L. Let E and O denote the identity matrix and zero matrix with proper dimensions, $\mathbf{1}$ and $\mathbf{0}$ denote the column vector with each element being 1 and 0 respectively. $Diag(L)$ is a diagonal matrix that has the same entry as L on the diagonal. For two matrices $A=[a_{ij}]$ and $B=[b_{ij}]$, $A \geq (>) B$ if for all i,j, $a_{ij} \geq (>) b_{ij}$.

A directed graph $\mathcal{G}$ consists of a vertex set $\mathcal{V}(\mathcal{G}) = \{v_1,\cdots,v_n\}$ and a directed edge set $\mathcal{E}(\mathcal{G}) \subseteq \mathcal{V}(\mathcal{G}) \times \mathcal{V}(\mathcal{G})$, i.e., an edge is an ordered pair of vertices in $\mathcal{V}(\mathcal{G})$. $\mathcal{N}_i$ denotes the neighborhood of the vertex v_i, i.e. $\mathcal{N}_i = \{v_j : (v_i,v_j) \in \mathcal{E}(\mathcal{G})\}$. A (directed) *path* of length l from vertex v_i to v_j, denoted by $(v_{r_1}, v_{r_2}, \cdots, v_{r_{l+1}})$, is a sequence of $l+1$ distinct vertices $v_{r_1},\cdots,v_{r_{l+1}}$ with $v_{r_1}=v_i$ and $v_{r_{l+1}}=v_j$ such that $(v_{r_k}, v_{r_{k+1}}) \in \mathcal{E}(\mathcal{G})$. The graph $\mathcal{G}$ contains *a spanning tree* if there exist a vertex v_i and paths from v_i to any other vertices, and v_i is called the root. If for any two vertices v_i, v_j, there exists a path from v_i to v_j, then the graph is called *strongly connected*. A graph $\mathcal{G}_1$ is called a *component* of graph $\mathcal{G}$, if $\mathcal{V}(\mathcal{G}_1) \subseteq \mathcal{V}(\mathcal{G})$, $\mathcal{E}(\mathcal{G}_1) \subseteq \mathcal{E}(\mathcal{G})$.

An $n\times n$ matrix L is called *a Metzler matrix* with zero row sum if $l_{ij} \geq 0$ holds for all $i \neq j$ and $\sum_{j=1}^n l_{ij} = 0$ holds for all $i=1,\cdots,n$. An $n\times n$ matrix L can be associated with a directed graph $\mathcal{G}(L) = \{\mathcal{V},\mathcal{E}\}$ in such a way that $(v_j,v_i) \in \mathcal{E}(\mathcal{G}(L))$ if and only if $l_{ij} > 0$. With this correspondence, we also say L contains a spanning tree if $\mathcal{G}(L)$ contains a spanning tree. Furthermore, for a given $\delta > 0$, the δ-matrix of L, denoted by L^δ, is defined as

$$[L^\delta]_{ij} = \begin{cases} 1, & l_{ij} \geq \delta; \\ 0, & l_{ij} < \delta. \end{cases}$$

Definition 1: We say $L(t)$ has a δ-spanning tree across $[t_1,t_2)$, if $[\int_{t_1}^{t_2} L(s)ds]^\delta$ has a spanning tree.

The linearly coupled network with time-varying topology can be described as:

$$\dot{x}_i(t) = \sum_{j=1}^n l_{ij}(t)[x_j(t) - x_i(t)], \qquad i=1,\cdots,n \tag{1}$$

where $x_i(t) = [x_{i1},\cdots,x_{in}] \in \mathbb{R}^n$ are the states variables of agent i, $l_{ij}(t) \geq 0$ is the connection weight from agent j to i, and $l_{ii}(t) = -\sum_{j=1}^n l_{ij}(t)$.

In this paper, we study the pinning control problem of a directed network described by (1) to a homogeneous stationary state s.

For this purpose, we apply the pinning control strategy with local feedback controllers to a vertex subset $\mathcal{D}(t) \subseteq \mathcal{V}$, the pinned vertex set that can be time-varying, too. Consider the following model:

$$\dot{x}_i(t) = \sum_{j=1}^n l_{ij}(t)[x_j(t) - x_i(t)] + d_i(t)[s - x_i(t)], \;\; i=1,\cdots,n \tag{2}$$

where the nonnegative scale $d_i(t) \geq 0$ is the feedback control gain at the vertex i. The pinned vertex set $\mathcal{D}(t)$ is composed of all vertices with $d_i(t) > 0$.

Denote $(\Omega,\mathcal{F},P)$ a probability space with the state space Ω, σ-algebra $\mathcal{F}$, and probability measure $P(\cdot)$, and ω denotes an element of Ω. For a set $S \subseteq \Omega$, S^c denote the complement of S, i.e. $S^c = \Omega \backslash S$. $\mathbf{1}_S$ denotes the indicator function of S, i.e.,

$$\mathbf{1}_S = \begin{cases} 1, & \omega \in S \\ 0, & \omega \notin S \end{cases}$$

$E_P(\cdot)$ is the expectation with respect to P (sometimes abbreviated as $E(\cdot)$ if P is unambiguous). For any σ-algebra $\mathcal{G} \subseteq \mathcal{F}$, $E(\cdot|\mathcal{G})(P(\cdot|\mathcal{G}))$ is the conditional expectation (conditional probability) with respect to the sub-σ algebra $\mathcal{G}$.

Definition 2: (Adapted process) Let X_k be a stochastic process defined on the basic probability space $(\Omega,\mathcal{F},P)$, and let $\{\mathcal{F}^k\}$ be a filtration, i.e. a sequence of nondecreasing sub-σ-algebra of $\mathcal{F}$. If X_k is measurable with respect to $\mathcal{F}^k$, then the sequence $\{X_k,\mathcal{F}^k\}$ is called an adapted process.

III. Stability of networks with deterministic time-varying topologies and pinned vertex set

Let $l_{ii}(t) = -\sum_{j=1}^n l_{ij}(t)$, $L(t) = [l_{ij}(t)]_{i,j=1}^n \in \mathbb{R}^{n,n}$ and $D(t) = [d_1(t),\cdots,d_n(t)]^\top$. It can be seen that $L(t)$ is a Metzler matrix with zero row sum for any time t. In this paper, we assume the time-varying coupling matrix $L(t)$ satisfies:

Assumption $\mathcal{A}_1$: For any $t > t_0$, the elements $l_{ij}(t) \geq 0$ are piecewise continuous in $[t_0,+\infty)$.

From this assumption, one can verify the existence and uniqueness of the Cauchy problem of (2).

Before giving the main result, we restate the result in [14].

Lemma 1: Consider the consensus algorithm

$$\dot{v}(t) = L(t)v(t) \tag{3}$$

where $L(t)$ satisfies assumption $\mathcal{A}_1$. If there exist a time interval sequence $\{[t_q,t_{q+1})\}_{q=0}^\infty$ and $M > 0$ such that $\int_{t_q}^{t_{q+1}} l_{jk}(s)ds < M$ hold for all $j,k=1,\cdots,n$ and $L(t)$ has a δ-spanning tree across $[t_q,t_{q+1})$ with $\delta > 0$, then the system (3) can reach consensus.

Now, we are in the position to give the first main result of this paper.

Theorem 1: Suppose the time-varying coupling matrix $L(t)$ satisfies assumption $\mathcal{A}_1$ and there exist a time interval sequence $\{[t_q,t_{q+1})\}_{q=0}^\infty$ and $M > 0$ such that $\int_{t_q}^{t_{q+1}} l_{jk}(s)ds < M$ and $\int_{t_q}^{t_{q+1}} d_k(s)ds < M$ hold for all $j,k=1,\cdots,n$. If for any interval $[t_q,t_{q+1})$ and any vertex v_i, one of the following conditions is satisfied:

1) $\int_{t_q}^{t_{q+1}} d_i(s)ds > \delta$ with some $\delta > 0$,
2) there exists v_j such that $\int_{t_q}^{t_{q+1}} d_j(s)ds > \delta$ and the graph of $[\int_{t_q}^{t_{q+1}} L(s)ds]^\delta$ has a path from v_j to v_i,

then the homogeneous stationary state s of network (2) is stable

Proof. By adding an extra virtual static agent to the original network, which has a few nodes to be linked towards but without any incoming links, the stability of algorithm (2) is reconducted to consensus problem of the extended network.

Denote $y_i(t) = x_i(t), i = 1, \cdots, n, y_{n+1}(t) = s, y(t) = [y_1(t), \cdots, y_{n+1}(t)]^\top$, then (2) turns to be

$$\begin{cases} \dot{y}_i(t) = \sum_j l_{ij}(t)[y_j(t) - y_i(t)] + d_i(t)[y_{n+1}(t) - y_i(t)] \\ \dot{y}_{n+1}(t) = 0 \end{cases}$$

or in compact form,

$$\dot{y}(t) = M(t)y(t), \tag{4}$$

where

$$M(t) = \begin{bmatrix} L(t) - diag(D(t)) & D(t) \\ O & 0 \end{bmatrix}.$$

Hence, the pinning control problem can be reformulated as a consensus problem of the coupled system (4).

The assumptions imply that $\int_{t_q}^{t_{q+1}} M(s)ds$ has a δ-spanning tree. Then from Lemma 1, we can conclude that system (4) can reach consensus. This completes the proof.

From the Perron-Frobenius theorem [21], we have the following lemma.

Lemma 2: For any Metzler matrix $A \in \mathbb{R}^n$ with zero row sum and $a_{ii} < 0, i = 1, \cdots, n$, there exists a permutation matrix P such that

$$P^\top AP = \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} & \cdots & \hat{A}_{1K} \\ \mathbf{0} & \hat{A}_{22} & \cdots & \hat{A}_{2K} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \hat{A}_{KK} \end{bmatrix}$$

where $\hat{A}_{ii}$ is irreducible, $i = 1, \cdots, K$. The graph corresponding to $\hat{A}_{ii}$ is named *strongly connected component* of the graph $\mathcal{G}(A)$.

Since every $\mathcal{G}(\hat{A}_{ii})$ is strongly connected, $i = 1, \cdots, K$, then every vertex has paths to the rest vertices in $\mathcal{G}(\hat{A}_{ii})$. By Theorem 1, we can obtain the following theorem.

Theorem 2: Suppose there exists an infinite time interval sequence $\{[t_q, t_{q+1})\}_{q=0}^{\infty}$ such that $\int_{t_q}^{t_{q+1}} l_{jk}(s)ds < M, \int_{t_q}^{t_{q+1}} d_k(s)ds < M, j, k = 1, \cdots, n$ with some $M > 0$. Define $R_q = \{k | \int_{t_q}^{t_{q+1}} d_k(s)ds > \delta\}$. If there exists $\delta > 0$ such that R_q contains at least one vertex in each strongly connected component of the graph of $[\int_{t_q}^{t_{q+1}} L(s)(s)ds]^\delta$, then the pinning controlled directed network (2) is stable at the homogeneous stationary state s.

IV. Pinning networks of multiagents with switching topologies modeled as adapted stochastic processes

Consider the system

$$\dot{x}(t) = L_k x(t) + D_k[s - x(t)], \ t \in [t_{k-1}, t_k), \tag{5}$$

where $t_0 < t_1 < \cdots$, L_k is a random Metzler matrix with zero row sum, D_k is a random diagonal matrix with nonnegative elements for each k. We assume the switching time sequence is also stochastic and independent of the stochastic process of the graph topology. Denote $\Delta t_k = t_k - t_{k-1}$, $x_k = x(t_k)$.

Assumption $\mathcal{A}_2$. For any $i \neq j$, $[L_k]_{ij} < M_1$ and $[D_k]_{ii} < M_1$ with some $M_1 > 0$.

Assumption $\mathcal{A}_3$. There exist two adapted processes $\{\Delta t_k, \mathcal{F}_k^1\}$, $\{[L_k, D_k], \mathcal{F}_k^2\}$ on two probability spaces $\{\Omega_1, \mathcal{F}^1, P_1\}, \{\Omega_2, \mathcal{F}^2, P_2\}$. $(\Delta t_k, \mathcal{F}_k^1)$ is independent with $([L_k, D_k], \mathcal{F}_k^2)$.

Assumption $\mathcal{A}_4$. There exist $T_2 > T_1 > 0, T_4 > T_3 > 0$ such that

$$T_1 < E(\Delta t_{k+1} | \mathcal{F}_k^1) < T_2, \quad T_3 < E((\Delta t_{k+1})^2 | \mathcal{F}_k^1) < T_4, \tag{6}$$

Definition 3: For any $p > 0$, we call the system (5) L_p stable if for any given initial distribution of $x(t_0)$ with $E(\|x(t_0)\|^p) < \infty$,

$$\lim_{k \to \infty} E(\|x_k - s\|^p) = 0.$$

Before giving the main results, we restate Theorem 3.3 in [16].

Lemma 3: Considering system

$$\dot{x}(t) = L_k x(t), \quad t \in [t_{k-1}, t_k), \tag{7}$$

where L_k is a random Metzler matrix. $\Delta t_k = t_k - t_{k-1}$. If $\{\Delta t_k, \mathcal{F}_k^1\}, \{[L_k, D_k], \mathcal{F}_k^2\}$ satisfy assumptions $\mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4$ and there exist $m \in \mathbb{N}^+, \delta > 0$ such that $E(\sum_{q=1}^m L_{k+q} | \mathcal{F}_k^2)$ contains a δ-spanning tree for $k = 0, 1, 2, \cdots$, then system (7) achieves L_p consensus for any $p > 0$.

Now, we give the sufficient conditions for L_p stability of network (5).

Theorem 3: If $\{\Delta t_k, \mathcal{F}_k^1\}, \{[L_k, D_k], \mathcal{F}_k^2\}$ satisfy assumptions $\mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4$ and for any vertex v_i, one of the following conditions is satisfied:

1) $[E(\sum_{q=1}^m D_{k+q} | \mathcal{F}_k^2)]_i > \delta$ with some $\delta > 0$,
2) there exist a vertex v_j with $[E(\sum_{q=1}^m D_{k+q} | \mathcal{F}_k^2)]_j > \delta$ and a path of graph of $[E(\sum_{q=1}^m L_{k+q} | \mathcal{F}_k^2)]^\delta$ from v_j to v_i,

then for any $p > 0$, system (5) is L_p stable.

Proof. Similar to the proof of Theorem 1, we suppose an extra virtual static agent is imposed to the network. Denote $y_i(t) = x_i(t), i = 1, \cdots, n, y_{n+1}(t) = s, y(t) = [y_1(t), \cdots, y_{n+1}(t)]^\top$, then (5) turns to be

$$\dot{y}(t) = M_k y(t), \quad t \in [t_{k-1}, t_k) \tag{8}$$

where

$$M_k = \begin{bmatrix} L_k - diag(D_k) & D_k \\ O & 0 \end{bmatrix}.$$

The assumptions imply that $E(\sum_{q=1}^m M_{k+q} | \mathcal{F}_k^2)$ contains a δ-spanning tree. Then, from lemma 3, we can conclude that L_p consensus of (8) is reached. The proof is completed.

As applications, we apply Theorem 3 to independent stochastically switching and Markovian switching topologies.

Corollary 1: Suppose $\{[L_k, D_k]\}$ is an independent sequence and $\{[L_k, D_k]\}, \{\Delta t_k, \mathcal{F}_k^1\}$ satisfy assumptions $\mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4$. If for any vertex v_i, one of the following conditions is satisfied:

1) $[\sum_{q=1}^m E(D_{k+q})]_i > \delta$ with some $\delta > 0$,

2) there exists v_j with $[\sum_{q=1}^{m} E(D_{k+q})]_j > \delta$ and the graph of $[\sum_{q=1}^{m} E(L_{k+q})]^{\delta}$ has a path from v_j to v_i,

then system (5) is L_p stable for any $p > 0$.

Proof. Let $\mathcal{F}_k^2 = \sigma([L_1, D_1] \cdots, [L_k, D_k])$. Since $\{L_k\}$ and $\{D_k\}$ are independent sequences, we have

$$E(\sum_{q=1}^{m} L_{k+q}|\mathcal{F}_k^2) = \sum_{q=1}^{m} E(L_{k+q}|\mathcal{F}_k^2) = \sum_{q=1}^{m} E(L_{k+q})$$

and $E(\sum_{q=1}^{m} D_{k+q}|\mathcal{F}_k^2) = \sum_{q=1}^{m} E(D_{k+q})$. The conditions here coincide with conditions in Theorem 3, hence, conclusions then comes from Theorem 3.

Now, we consider the case where the graph topologies are induced by a homogeneous Markov chain with a stationary distribution π.

Corollary 2: Suppose $\{[L_k, D_k]\}, \{\Delta t_k, \mathcal{F}_k^1\}$ satisfy assumptions $\mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4$ and $\{[L_k, D_k]\}$ is a homogeneous Markov chain with a stationary distribution π. If there exists $\eta > 0$, such that for any v_i, either $[E_\pi(D_1)]_i > \eta$ or there exist a vertex v_j with $[E_\pi(D_1)]_j > \eta$ and a path of graph of $E_\pi(L_1)$ from v_j to v_i, then system (5) is L_p stable for any $p > 0$.

Proof. Let $\mathcal{F}_k^2 = \sigma([L_1, D_1] \cdots, [L_k, D_k])$. From the Markov property, we have $E(L_{k+1}|\mathcal{F}_k^2) = E(L_{k+1}|L_k)$. Note that $\{L_k\}$ and $\{D_k\}$ are uniformly ergodic, hence,

$$\lim_{m\to\infty} E(\frac{1}{m}\sum_{q=1}^{m} L_{k+q}|L_k) = E_\pi(L_k) = E_\pi(L_1)$$

and

$$\lim_{m\to\infty} E(\frac{1}{m}\sum_{q=1}^{m} D_{k+q}|D_k) = E_\pi(D_1).$$

Therefore, we can find m' such that for any $m > m'$ and any v_i, either $[E(\frac{1}{m}\sum_{q=1}^{m} D_{k+q}|\mathcal{F}_k)]_i > \frac{\eta}{2}$ or there exist a vertex v_j with $[E(\frac{1}{m}\sum_{q=1}^{m} D_{k+q}|\mathcal{F}_k)]_j > \frac{\eta}{2}$ and a path of graph of $[E(\frac{1}{m}\sum_{q=1}^{m} L_{k+q}|\mathcal{F}_k)]^{\delta}$ from v_j to v_i with some constant $\delta > 0$. The conclusion follows from Theorem 3.

V. Numerical Simulations

In this section, we give some numerical examples to illustrate the theoretical results.

A. Deterministic case

In this example, we have $m = 5$ agents. The time-varying coupling matrix and pinned vertex set is switching among the following ones:

$$L_1 = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \; D_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix},$$

$$L_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \; D_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

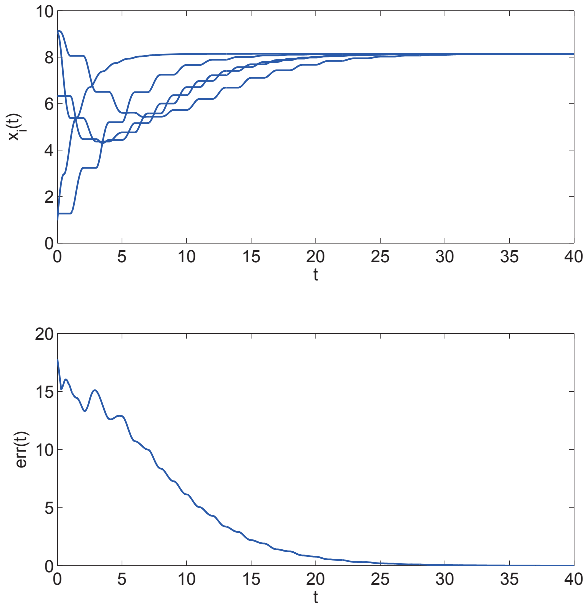

Fig. 1. The dynamical behaviors of $x_i(t)$ and $err(t)$ with respect to time.

Let $\{t_k\}$ be the time sequence:

$$0 = t_0 < t_1 < t_2 < \cdots < t_n < \cdots,$$

with $\Delta t_k = t_k - t_{k-1}$, $a(t) = |sin(\pi t)|$ be the coupling strength and $b(t) = |cos(\pi t)|$ be the pinning scale. We choose

$$L(t) = \begin{cases} a(t)L_1, & t \in [t_{2k}, t_{2k+1}) \\ a(t)L_2, & t \in [t_{2k+1}, t_{2(k+1)}) \end{cases}$$

and $\Delta t_k = \Delta = 1$, $D(t) = b(t)D_1 = b(t)D_2$. Hence, $\int_{t_{2k}}^{t_{2(k+1)}} [D(s)]_5 ds > \delta$ and $\int_{t_{2k}}^{t_{2(k+1)}} L(s)ds$ has a δ-spanning tree and v_5 is the root, here $\delta = \frac{2}{\pi}$. Therefore, all the conditions in Theorem 1 hold. We use the following quality to measure the stability of the pinning algorithm (2):

$$err(t) = \sum_i |x_i(t) - s|. \tag{9}$$

If $err(t)$ is small enough, we say the pinning algorithm is stable. Fig 1. shows the dynamics of states $x_i(t)$ and error $err(t)$ with respect to time, which implies the states of agents converge to given state s.

B. Stochastic case

[18] defined a new pinning control algorithm, spatial pinning control, where agents moving in the planar space. Once agent enters a fixed region, it will be pinned. Here, we consider spatial pinning control of networks with mobile agents moving in the planar space $\Gamma \in \mathbb{R}^2$ according to the random waypoint model.

According to the movements of agents, a linking graph can be constructed by the following linking rule: Two agents are considered to be linked if the distance between them is less than a given interaction threshold. Suppose two agents are coupled with coefficient 1 if they have a link, which induces a coupling matrix L. Spatial pinning is activated to an agent once it enters a given region Γ_c. Hence, pinned sets $\mathcal{D}$ can be obtained from the movements of agents.

The random waypoint model is one of the most widely used model to evaluate protocols of ad hoc networks, which was first proposed by [19]. In this model, suppose there are N mobile agents moving in a two-dimensional space Γ. The movement of each agent is stochastically independent of the other ones but follows one identical distribution. The agent moves towards a randomly selected target with a random velocity. After approaching the target, the agent waits for a random time length and then continues the process. This leads to two states for every agents: moving state and waiting state.

Consider the location and state of agent i as a stochastic process: $\gamma_t^{(i)} = \{\alpha_t^{(i)}, \beta_t^{(i)}, v_t^{(i)}, w_t^{(i)}, \xi_t^{(i)}\}$. $\alpha_t^{(i)}$ denotes the location at time t, $\beta_t^{(i)}$ is the location of the target, $v_t^{(i)}$ is the velocity of the movement and it is zero if i is waiting, $w_t^{(i)}$ is the current waiting time period if i is waiting and it is zero if i is moving, and $\xi_t^{(i)}$ is the past time of current wait if i is waiting and zero if i is moving.

The next motion of each agent depends on its current location but not on its previous locations, which implies the motion variation of locations of i can be modeled as a homogeneous Markov chain and so as are the links between agents and the pinned sets.

Denote $x_i(t)$ the state of agent i at time t. Each agent updates its state $x_i(t)$ according to the states of its neighbors, through a linear coupling on their states. In particular, an agent j is considered a neighbor of agent i at time t, if $d(\alpha_t^i, \alpha_t^j) \leq r$, where r is the interaction radius. A control input is applied to agent i if its location is in Γ at time t, i.e. $\alpha_t^i \in \Gamma_c$.

The node distribution of the RWP model was studied in [20]. Their result implies that the node distribution of RWP is ergodic and its stationary pdf is always positive everywhere in the moving region. Note the motions of agents are independent, all agents have a positive probability to be in the transmission range, despite of not entering this disc at the same time. Therefore, the network has a positive probability to be a complete network, which implies the expectation is a complete graph. Similarly, for a pre-given spatial pinning region, the pinned set has a positive probability to be in it, which implies the expectation of the pinned set is the set of all agents. Therefore, the conditions in theorem 3 are satisfied, which implies the algorithm is L_p stable to the pre-given consensus value s.

We realize the random waypoint model with 20 mobile agents moving in $\Gamma = [0, 1000] \times [0, 1000](m^2)$ and spatial pinning control region $\Gamma_c \subset \Gamma$. The velocity of movement is randomly chosen in $(10, 20)$(m/sec) with uniform distribution. After approaching the target, agent waits for a random time period, with a uniform distribution in $(1, 5)$(sec). Pick the transmission range $r = 120$. If the distance between locations

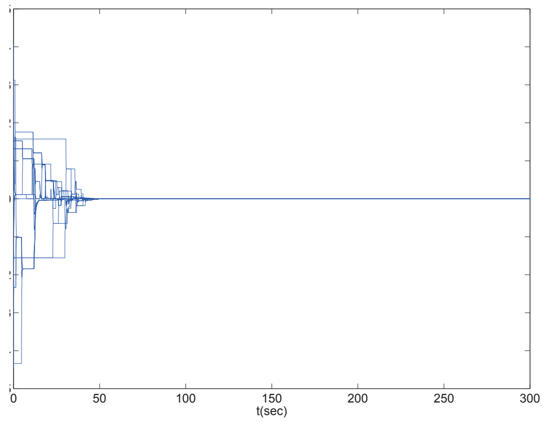

Fig. 2. The dynamical behaviors of $x_i(t) - s$ with $\Gamma_c = [0, 200] \times [0, 200](m^2)$.

of two agents is less than r, then these two agents are coupled with weight 1. If an agent enters the pinning control region, then it will be pinned with weight 1. The numerical average network is complete and average pinned set is the entire agent set, which coincides the analysis given above. The coupling matrices and pinned sets are collected with a step length of 0.1. Randomly pick the initial states of agents in $(-5, 5)$, i.e. , $x_i(t) \in (-5, 5), i = 1, \cdots, 20$. For any pre-given consensus value s, the ordinary differential equations (5) are solved by the euler formula with a step length of 0.01. Randomly pick $s \in (-5, 5)$. Fig.2 indicates that the spatial pinning control is stable.

[22] provides the plots of the stationary node distribution of a two-dimensional RWP model in a $[0, 10] \times [0, 10]$ square region. It seems that the stationary distribution has a larger probability in the middle area of the moving region. Hence, the authors wonder the choice of pinning region might influence the convergent speed. In the following, we pick several different pinning regions and compare their convergent speed. Fig. 3 tells that the convergent speed of $err(t)$ is fasten if the pinning region is close to the center of moving region; Fig. 4 shows the convergence of $err(t)$ is speed up if the area of the pinning region enlarges.

VI. Conclusion

In this paper, we have studied the stability of pinning control algorithms of networks with general directed time varying topologies, including determinately and stochastically switching topologies. For networks with determinately switching topologies, we have shown that the pinned vertex set can stabilize the network if the pinned vertex set can access all other vertices in the network. For networks with stochastically switching topologies, we have shown that if the conditional expected pinned vertex set can access all other vertices in the graph of conditional expected Laplacian matrix, then the network is stable. As applications, numerical simulations

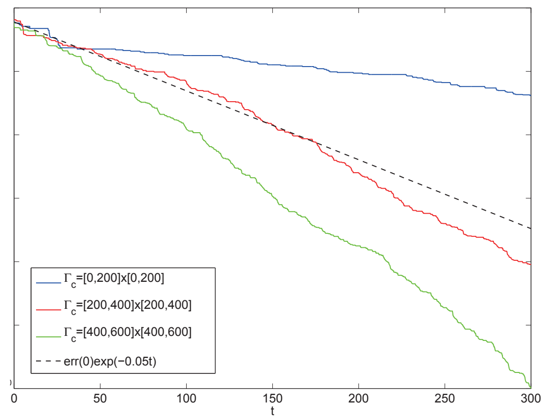

Fig. 3. Dependence of convergent speed on position of pinning region.

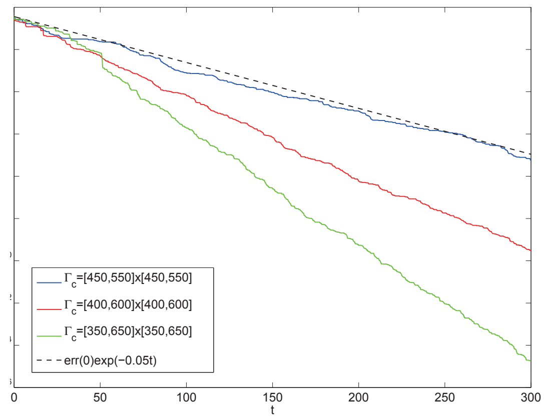

Fig. 4. Dependence of convergent speed on the area of pinning region.

based on the random waypoint model are given to verify our theoretical results.

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