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Distributed LQR Design for Multi-agent Systems on Directed Graph Topologies

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***Abstract*—In this paper, the inverse optimal approach is employed to design distributed cooperative control protocols for identical linear systems that guarantee consensus and global optimality with respect to a positive (semi-) definite quadric performance index. Cooperative control and pinning control problems are considered, where the communication graphs are assumed to be directed and have fixe topology. Simple sufficient conditions are established, which indicate that the global optimality is achieved using local distributed protocols which are designed by the linear quadric regulator (LQR) based optimal control method. Examples are given to show the effectiveness of the proposed methods.**

I. Introduction

Research on the multi-agent network cooperative systems [1], [2] has received immense amounts of attention in the last two decades, due to its wide range of applications including cooperative control of unmanned air vehicles (UAVs), flocking and formation forming, wireless sensor networks, etc. Applications of cooperative control of multi-vehicle systems are summarized by Murray [3].

Consensus problem [4]–[8], or the leaderless following problem, aims to make all the nodes converge to a common, which is not prescribed. This is sufficiently studied, necessary and sufficient condition for the distributed systems has already been proposed. On the other hand, that consensus problem with a leader, or leader following problem, which pins all the nodes to track the desired leader node trajectories, this is also known as cooperative tracking control [9], or model reference consensus [10]. In literature [11], the distributed state feedback derived using local linear quadric regulator (LQR) based method solved the leader following problem for a broad class of communication graphs.

Cooperative optimal control problems have attracted many researchers [13]–[19]. The difficulty is that the globally optimal problems require complete state information, which cannot be observed generally in real applications [17]. For multi-agent systems, the graph topology interplays with the system dynamics, hence the global optimal control problems are fairly complicated. In the existing work, [18] provides distributed optimal scheme which each agent minimizes its own local performance index. The distributed games on graphs are studied in [14], where the agent only minimizes its local performance index, either. In the case of agents with identical linear time-invariant dynamics, a suboptimal design is presented in [13].

The inverse optimality method [20] does provide an effective way to the LQ regulator design, and the parameterizations of the optimal regulators are proposed [21]. Using the inverse optimality method, an optimality criterion is established related to the graph topology to obtain the distributed optimal control [19], [22]. In [23], the LQR based optimal design method is proposed to obtain the distributed optimal control by constructing a globally optimal performance index. However, the results in [23] are not sufficient, and the lower bound of the coupling gain are computationally complex. In this paper, we propose a new lower bound of the coupling gain, which holds for any weighting matrices $Q > 0$ and $R > 0$. This greatly increase the generality (no re-computation of the coupling gain for different Q and R and reduce the computational complexity. The lower bound only depends on the Laplacian matrix and the pinning matrix.

The paper is organized as follows. In section II, we firstly show some concepts of the graph theory, and then we generalize the inverse optimality of the linear systems. The main results for the cooperative optimal control are given in section III. In section IV, numerical examples are given to illustrate the proposed methods.

Notations: The Kronnecker product is denoted by "$\otimes$". The transposition of matrix A is denoted by A^T. I_n denotes the identity matrix in $\mathbb{R}^{n\times n}$. $1_n \in \mathbb{R}^n$ is the vector with all elements 1. Matrix $A > 0 (\geq 0)$ means A is positive definite (semi-definite), $A < 0 (\leq 0)$ means A is negative definite (semi-definite). $\ker A$ denotes the null space of matrix A.

II. Formulations

A. Graph Theory

Let's consider a weighted digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ with a nonempty finite set of N nodes $\mathcal{V} = \{v_1, v_2, \ldots, v_N\}$, a set of edges $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ and the associated adjacency matrix $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{N\times N}$. An edge rooted at node j and ended at node i is denoted by (v_j, v_i), which means the information flows from node j to node i. The weight a_{ij} of edge (v_j, v_i) is positive, i.e., $a_{ij} > 0$ if $(v_j, v_i) \in \mathcal{E}$, otherwise, $a_{ij} = 0$. In this paper, assume that there are no repeated edges and no self loops, i.e., $a_{ii} = 0, \forall i \in \mathcal{N}$, where $\mathcal{N} = \{1, 2, \ldots, N\}$. If $(v_j, v_i) \in \mathcal{E}$, then node j is called a neighbor of node i. The set of neighbors of node i is denoted as $\mathcal{N}_i = \{j|(v_j, v_i) \in \mathcal{E}\}$. Define the in-degree matrix as $D = \text{diag}\{d_i\} \in \mathbb{R}^{N\times N}$ with $d_i = \sum_{j\in\mathcal{N}_i} a_{ij}$ and the Laplacian matrix as $L = D - \mathcal{A}$. Obviously, $L1_N = 0$. The graph is said to be connected if every two vertices can be joined by a path. If every two vertices can be joined by a directed path, then the graph is said to be strongly connected. If $\mathcal{G}$ is strongly connected, then the zero eigenvalue of is simple, hence $\ker L = \text{span}\{1_N\}$. If there is a node i_r, such that there is a directed path from the node i_r to every other nodes in the graph, then the digraph is said to contain a spanning tree.

B. Inverse Optimality of Linear Systems

Consider the following continuous-time LQ regulator problem

$$\dot{x} = Ax + Bu, \tag{1a}$$
$$J = \int_0^\infty x^T Qx + u^T Rudt. \tag{1b}$$

The inverse optimal control problem considered in this paper is [20]: for a given stable control

$$u = -Kx, \tag{2}$$

find the condition on A, B and K, such that the control law (2) minimizes the cost (1b) for some symmetric (unknown) $Q \geq 0$ and $R > 0$ and determine them.

The LQR optimal control gain is normally obtained by solving the Algebra Riccati Equation (ARE) and the optimal control gain is given by:

$$A^T P + PA - PBR^{-1}B^T P + Q = 0, \tag{3a}$$
$$K = R^{-1}B^T P. \tag{3b}$$

Assumption 1: The following assumptions will hold throughout in this paper:

1) the pair (A, B) is controllable;
2) the input matrix B is of full column rank m.

Equations (3) are equivalent to

$$A^T P + PA - K^T RK + Q = 0, \tag{4a}$$
$$B^T P = RK. \tag{4b}$$

Now, we establish the necessary and sufficient conditions of the inverse optimality for the multi-inputs LTI systems.

Lemma 1: [21] For the closed-loop system $\dot{x} = (A - BK)x$, the state feedback gain-matrix K is optimal and the corresponding $CARE$ has a symmetric positive (semi-) definite solution P for some symmetric state weighting matrix Q, providing that the input weighting matrix $R > 0$ is given, if and only if:

1) the feedback control gain K is stabilizing,
2) the matrix RKB is a positive definite symmetric matrix.

Theorem 1: For the IOCP (1), the state feedback gain-matrix κK (where $\kappa \geq 2$) is optimal and the corresponding ARE has a symmetric positive definite solution P for some symmetric positive definite (S.P.D. for short) state weighting matrix $\bar{Q}$ and input weighting matrix $\bar{R}$, if the following conditions hold:

1) the feedback control gain K is stabilizing,
2) the matrix KB is a positive definite simple matrix.

Proof: The matrix KB is a positive definite simple matrix, i.e., there is a nonsingular matrix W, s.t., $WKBW^{-1} = \Lambda > 0$. Let the input weighting matrix R be formed as $R = W^T \Psi W$, where the matrix Ψ satisfies: $\Psi = \Psi^T > 0$, and $\Psi\Lambda = \Lambda\Psi$. Apparently, $RKB = W^T \Psi WW^{-1}\Lambda W = W^T \Psi\Lambda W$ is a symmetric positive definite matrix. According to Lemma 1, K is optimal and the corresponding $CARE$ has a positive definite solution P for some symmetric state weighting matrix Q, input weighting matrix $R = W^T \Psi W > 0$.

Let $Q = -K^T RK - (A - BK)^T P - P(A - BK)$, then the $CARE$ is constructed as

$$(A - BK)^T P + P(A - BK) = -Q - K^T RK, \tag{5}$$

and we have $Q + K^T RK > 0$ [21].

To obtain the S.P.D. state matrix Q, we consider the optimal feedback control gain which is of form κK, where $\kappa \geq 2$.

Adding $2(1 - \kappa)K^T RK$ to the both side of (5) yields

$$(A - \kappa BK)^T P + P(A - \kappa BK) = -Q + (1 - 2\kappa)K^T RK. \tag{6}$$

Since that $\kappa \geq 2$, then we have

$$-Q + (1 - 2\kappa)K^T RK \leq -Q - K^T RK < 0, \tag{7}$$

according to the Lypunov stability Theorem, κK is Hurwitz.

Let $\bar{Q} = Q - (1 - \kappa)K^T RK$. Since $\kappa \geq 2$, then

$$\bar{Q} = Q - (1 - \kappa)K^T RK \geq Q + K^T RK > 0.$$

Finally, we see (6) is equivalent to

$$\begin{aligned}
&[A - B(\kappa K)]^T P + P[A - B(\kappa K)] \\
&= - \bar{Q} - \kappa K^T RK \\
&= - \bar{Q} - (\kappa K)^T (\frac{R}{\kappa})(\kappa K).
\end{aligned} \tag{8}$$

Choose the input weighting matrix $\bar{R} = R/\kappa$, then κK is the optimal feedback control gain for the IOCP with some S.P.D. weighting matrices $\bar{Q}$ and $\bar{R}$.

This completes the proof.

Remark 1: Note that $Q = P(A-BK/2)+(A-BK/2)^T P$, if P is positive semi-definite, then $Q \geq 0$. Therefore, Theorem 1 can be modified as: For the IOCP (1), the state feedback gain-matrix κK (where $\kappa \geq 2$) is optimal and the corresponding ARE has a symmetric positive semi-definite solution P for some symmetric positive semi-definite state weighting matrix $\bar{Q}$ and input weighting matrix $\bar{R} > 0$, if the following conditions hold:

1) the feedback control gain K is stabilizing,
2) the matrix KB is a positive definite simple matrix.

This can be easily proved by repeating the proof process of Theorem 1.

Remark 2: Compared with Lemma 1, the information of the inputs weighting matrix R is not required. Another point should be mentioned is that the state weighting matrix Q is positive definite in Theorem 1, which is to be contrasted with the general inverse optimality.

III. Optimal Cooperative Control For Linear Time-Invariant Agent Dynamics

In this section, the globally optimal consensus protocols are considered for the leaderless and pinning control cases for agents with the following identical linear time-invariant dynamics

$$\dot{x}_i = Ax_i + Bu_i, \quad \forall i \in \mathcal{N}, \tag{9}$$

and in global form

$$\dot{x} = (I_N \otimes A)x + (I_N \otimes B)u, \tag{10}$$

where the state $x_i \in \mathbb{R}^n$.

A. Optimal Cooperative Regulator

For the cooperative regulator problem, all agents are to achieve the same state, i.e., $\|x_i - x_j\| \to 0$ as $t \to \infty$, $\forall i, j \in \mathcal{N}$. The local neighborhood error is defined as

$$\varepsilon_i = \sum_{j \in \mathcal{N}} a_{ij}(x_i - x_j) \tag{11}$$

and the global neighborhood error is given as $\xi = (L \otimes I_n)x$. Inspired by [11], the distributed control protocol, which is a state variable feedback (SVBF) control, is given as

$$u_i = -cK\varepsilon_i, \tag{12}$$

where the scalar coupling gain $c > 0$ and the feedback control gain matrix $K \in \mathbb{R}^{m\times n}$. The global form of the distributed control protocol is

$$u = -c(L \otimes K)x, \tag{13}$$

which gives the overall closed-loop system

$$\dot{x} = (I_N \otimes A - cL \otimes BK)x. \tag{14}$$

Lemma 2: [2] Let λ_i $(i \in \mathcal{N})$ be the eigenvalues of the Laplacian matrix L (or the matrix $L+G$). The global closed-loop system (14) is asymptotically stable if and only if all the matrices

$$A - c\lambda_i BK, \quad \forall i \in \mathcal{N} \tag{15}$$

are Hurwitz, i.e., asymptotically stable.

Lemma 3: [11] Let the matrices $Q = Q^T > 0 \in \mathbb{R}^{n\times n}$ and $R = R^T > 0 \in \mathbb{R}^{m\times m}$ be positive definite. Design the SVFB control gain as

$$K = R^{-1}B^T P, \tag{16}$$

where P is the unique positive definite solution of the Riccati equation

$$A^T P + PA + Q - PBR^{-1}B^T P = 0. \tag{17}$$

Then the global closed-loop system (14) is asymptotically stable if

1) A is Hurwitz,
2) the coupling gain

$$c \geq \frac{1}{2\underline{\lambda}}, \tag{18}$$

where $\underline{\lambda}$ denotes the minimum positive eigenvalue of L.

Theorem 2: The control (13) with the gain K given by (16) is optimal for some performance indexes

$$J = \int_0^\infty x^T \bar{Q} x + u^T \bar{R} u dt, \tag{19}$$

with $\bar{Q} = \bar{Q}^T \geq 0$ and $\bar{R} = \bar{R}^T > 0$, if

1) the Laplacian matrix L is simple positive semi-definite,
2) A is Hurwitz,
3) the coupling gain

$$c \geq \frac{1}{\underline{\lambda}}, \tag{20}$$

where $\underline{\lambda}$ denotes the minimum positive eigenvalue of L.

Proof: The Riccati equation for system (10) with respect to the performance index (19) is

$$\begin{aligned}(I_N \otimes A)^T \bar{P} + \bar{P}(I_N \otimes A) + \bar{Q} \\ - \bar{P}(I_N \otimes B)\bar{R}^{-1}(I_N \otimes B)^T \bar{P} = 0.\end{aligned} \tag{21}$$

Using Theorem 1, the $u = -c(L \otimes K)x$ is optimal, only if

a) the feedback control gain $c(L \otimes K)$ is stabilizing,
b) the matrix $c(L\otimes K)(I_N \otimes B)$ is a positive definite simple matrix.

Since that A is Hurwitz and the coupling gain

$$c \geq \frac{1}{\underline{\lambda}} > \frac{1}{2\underline{\lambda}}, \tag{22}$$

then $c(L \otimes K)$ is stabilizing according to Lemma 3.

The Laplacian matrix L is simple positive semi-definite, i.e., there exists a nonsingular matrix T, such that $L = T^{-1}\Lambda T$, where Λ is a diagonal matrix with all diagonal elements of the eigenvalues of L. Note that the gain K is optimal, then KB is a positive definite simple matrix, which means there exists a nonsingular matrix Y, such that $KB = Y^{-1}\Omega Y$, where Ω is a diagonal positive definite matrix. Then is holds that

$$\begin{aligned}&c(L \otimes K)(I_N \otimes B) \\ =&c(L \otimes BK) \\ =&c(T^{-1}\Lambda T \otimes Y^{-1}\Omega Y) \\ =&c(T^{-1} \otimes Y^{-1})(\Lambda \otimes \Omega)(T \otimes Y) \\ =&c(T \otimes Y)^{-1}(\Lambda \otimes \Omega)(T \otimes Y).\end{aligned}$$

It is easily seen that the matrix $\Lambda \otimes \Omega$ is diagonal, hence the matrix $c(L\otimes K)(I_N \otimes B)$ is a positive definite simple matrix.

Then according to Theorem 1, $u = -c(L \otimes K)x$ is an optimal control and the following relationship arises

$$u = -c(L \otimes K)x = -\bar{R}^{-1}(I_N \otimes B)^T \bar{P}x$$

for some $\bar{Q} = \bar{Q}^T$ and $\bar{R} = \bar{R}^T > 0$.

The existence of the positive semi-definiteness of the state weighting matrices Q is shown as follows. Let

$$\bar{H} = \frac{c}{2}(I_N \otimes B)(L \otimes K) - I_N \otimes A, \tag{23}$$

then using the conclusion in Remark 1, there exists a symmetric state weighting matrix $\bar{Q} \geq 0$ if $-\bar{H}$ is Hurwitz. According to Lemma 2, $-\bar{H}$ is Hurwitz is equivalent to

$$A - \frac{c}{2}\lambda_i BK, \quad \forall i \in \mathcal{N} \tag{24}$$

are Hurwitz, where λ_i $(i \in \mathcal{N})$ be the eigenvalues of the Laplacian matrix L.

For $\lambda_i = 0$, A is Hurwitz.

For $\lambda_i > 0$, according to (16) and (17), one has

$$\begin{aligned}(A - \frac{c}{2}\lambda_i BK)^T P + &P(A - \frac{c}{2}\lambda_i BK) \\ &= -Q - (c\lambda_i - 1)K^T RK.\end{aligned} \tag{25}$$

Since that $P > 0$ and $Q > 0$, by Lyapunov theory, the matrix $A - c\lambda_i BK/2$ is Hurwitz if 3) holds.

Remark 3: For the undirected graphs, the Laplacian matrix $L = L^T \geq 0$, so L is simple positive semi-definite, i.e., condition 1) is always met.

B. Optimal Cooperative Tracker

The dynamics of the leader or control node, labeled by 0, is given by

$$\dot{x}_0 = Ax_0, \tag{26}$$

where $x_0 \in \mathbb{R}^n$ is the state. If node i observes the leader, an edge (v_0, v_i) is said to exist with weighting gain $g_i > 0$. The node with $g_i > 0$ is referred as a pinned or controlled node. Denote the pinning matrix as $G = \mathrm{diag}\{g_1, \cdots, g_N\}$.

The local neighborhood error is defined as

$$\varepsilon_i = \sum_{j \in \mathcal{N}} a_{ij}(x_i - x_j) + g_i(x_0 - x_i). \tag{27}$$

and the overall neighborhood tracking error is $\xi = (L+G) \otimes I_n \delta$, where the global disagreement error is $\delta = x - 1_N \otimes x_0$. The distributed control protocol, which is a state variable feedback (SVBF) control, is given as

$$u_i = -cK\varepsilon_i, \tag{28}$$

where the scalar coupling gain $c > 0$ and the feedback control gain matrix $K \in \mathbb{R}^{m \times n}$. The global form of the distributed control protocol is

$$u = -c(L+G) \otimes K\delta, \tag{29}$$

where $G = \mathrm{diag}\{g_1, \cdots, g_N\}$ is the matrix of pinning gain. The overall closed-loop system is given as

$$\dot{x} = I_N \otimes Ax - c(L+G) \otimes BK\delta. \tag{30}$$

The global disagreement closed-loop system with the control $u = -c(L+G) \otimes K\delta$ is

$$\dot{\delta} = (I_N \otimes A - c(L+G) \otimes BK)\delta. \tag{31}$$

To achieve the synchronization, (31) must be asymptotically stabilized to the origin.

Assumption 2: The digraph $\mathcal{G}$ contains a spanning tree and the root node i_r can observe information from the leader node, i.e., $g_{i_r} > 0$.

Remark 4: Under Assumption 2, all the eigenvalues of graph matrix $L+G$ have positive real part.

Theorem 3: The control (29) with the gain K given by (16) is optimal for some performance indexes

$$J = \int_0^\infty \delta^T \bar{Q}\delta + u^T \bar{R}u dt, \tag{32}$$

with $\bar{Q} = \bar{Q}^T > 0$ and $\bar{R} = \bar{R}^T > 0$, if

1) the matrix $L+G$ is simple positive definite,

2) the coupling gain

$$c \geq \frac{1}{\underline{\lambda}}, \tag{33}$$

where $\underline{\lambda} = \min_{i \in \mathcal{N}} \lambda_i$, λ_i denote the eigenvalues of $L+G$.

Proof: Similar to the analysis as the proof in Theorem 2, the matrix $L+G$ is simple implies that the matrix $c((L+G) \otimes K)(I_N \otimes B)$ is a positive definite simple matrix. Then according to Theorem 1 and Lemma 3, the coupling gain $c \geq 1/\underline{\lambda}$ indicates that $u = -c(L+G) \otimes K\delta$ is an optimal control and the following relationship arises

$$u = -c(L+G) \otimes K\delta = -\bar{R}^{-1}(I_N \otimes B)^T \bar{P}\delta.$$

for some $\bar{Q} = \bar{Q}^T$ and $\bar{R} = \bar{R}^T > 0$.

Let

$$-\bar{H} = I_N \otimes A - \frac{c}{2}((L+G) \otimes BK), \tag{34}$$

then using Theorem 1, there exists a S.P.D. state weighting matrix $\bar{Q}$ if

$$-\bar{H} = I_N \otimes A - c((L+G) \otimes BK)/2$$

is Hurwitz.

According to Lemma 2, it is equivalent to

$$A - \frac{c}{2}\lambda_i BK, \quad \forall i \in \mathcal{N} \tag{35}$$

are Hurwitz, where λ_i $(i \in \mathcal{N})$ be the eigenvalues of the matrix $L+G$.

According to (16) and (17), one has

$$\begin{aligned}(A - \frac{c}{2}\lambda_i BK)^T P + &P(A - \frac{c}{2}\lambda_i BK) \\ &= -Q - (c\lambda_i - 1)K^T RK.\end{aligned} \tag{36}$$

Since that $P > 0$ and $Q > 0$, by Lyapunov theory, the matrix $A - c\lambda_i BK/2$ is Hurwitz if the coupling gain $c \geq 1/\lambda_i$, $\forall i \in \mathcal{N}$, which is true if condition (33) holds. This completes the proof.

Remark 5: It should be mentioned that the lower bound of the coupling gain (33) obtained here is only depends on the eigenvalue of the Laplacian matrix and the pinning matrix, which indicates that this bound is a uniform lower bound for any weighting matrices $Q > 0$ and $R > 0$. Although such a bound of the coupling gain trades off a possible lower gain for a certain choice of Q and R [23], but may greatly increases the generality (no re-computation of c for different Q and R and reduces the computational complexity.

Remark 6: For the leader following problem on undirected graphs, the Laplacian matrix L is symmetric positive semi-definite, then $L + G$ is symmetric positive definite, hence Theorem 3 holds for the undirected graphs situation, either. The results also intuitively show that the result L is simple in Theorem 5 of [23] is not sufficient.

IV. Simulations

In this section, two examples are given to demonstrate the effectiveness of the proposed LQR based optimal distributed protocols.

Example 1: (Leaderless case) Consider the following multi-agent systems with three nodes:

$$\dot{x}_i = Ax_i + Bu_i,\ i = 1, 2, 3, \tag{37}$$

where

$$A = \begin{bmatrix} -0.5 & -2 \\ 3 & -0.2 \end{bmatrix},\quad B = \begin{bmatrix} -1 \\ 1 \end{bmatrix}. \tag{38}$$

The initial states of the subsystems are

$$x_{1,0} = \begin{bmatrix} 6 \\ 8 \end{bmatrix},\ x_{2,0} = \begin{bmatrix} 2 \\ -2 \end{bmatrix},\ x_{3,0} = \begin{bmatrix} -6 \\ -8 \end{bmatrix}. \tag{39}$$

The in-degree matrix D and the associated adjacency matrix $\mathcal{A}$ are given as

$$\mathcal{A} = \begin{bmatrix} 0 & 0.5 & 1.5 \\ 0.5 & 0 & 1.5 \\ 1 & 1 & 0 \end{bmatrix},\quad D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \tag{40}$$

so the Laplacian matrix $L = D - \mathcal{A}$, which is

$$L = \begin{bmatrix} 2 & -0.5 & -1.5 \\ -0.5 & 2 & -1.5 \\ -1 & -1 & 2 \end{bmatrix}, \tag{41}$$

the eigenvalues of L are $3.5,\ 2.5,\ 0$ and L is simple.

By choosing weighting matrices $Q = I_2$, $R = 1$ and coupling gains $c = 2 > 0.4$, the optimal feedback gains are given by (16) as $K = [-0.6088 \quad 0.8405]$. The evolutionary process of consensus are shown in Fig. 1, the consensus is achieved within 4s. The computation of the coupling gain c is very simple compared with [23].

Example 2: (Leader case) Consider the following multi-agent systems with three nodes:

$$\dot{x}_i = Ax_i + Bu_i,\ i = 1, 2, 3, \tag{42}$$

where

$$A = \begin{bmatrix} -0.5 & -1 \\ 1 & -0.4 \end{bmatrix},\quad B = \begin{bmatrix} -1 \\ 1 \end{bmatrix}. \tag{43}$$

The initial states of the subsystems are

$$x_{1,0} = \begin{bmatrix} 6 \\ 8 \end{bmatrix},\ x_{2,0} = \begin{bmatrix} 2 \\ -2 \end{bmatrix},\ x_{3,0} = \begin{bmatrix} -6 \\ -8 \end{bmatrix}. \tag{44}$$

The pinning matrix $G = 3I_3$, and the leader node is given as

$$\dot{x_0} = Ax_0,\quad x_0(0) = [5\ \ -5]^T. \tag{45}$$

The initial states of the subsystems, the in-degree matrix D and the associated adjacency matrix $\mathcal{A}$ are given as same as in Example 1. The eigenvalues of $L+G$ are $4.5,\ 3.5,\ 1$, and $L+G$ is simple.

By choosing weighting matrices $Q = I_2$, $R = 1$ and coupling gains $c = 2 > 1$, the optimal feedback gains are given by (16) as $K = [-0.6784 \quad 0.4785]$. The evolutionary process of consensus are shown in Fig. 2, the consensus is achieved within 15s.

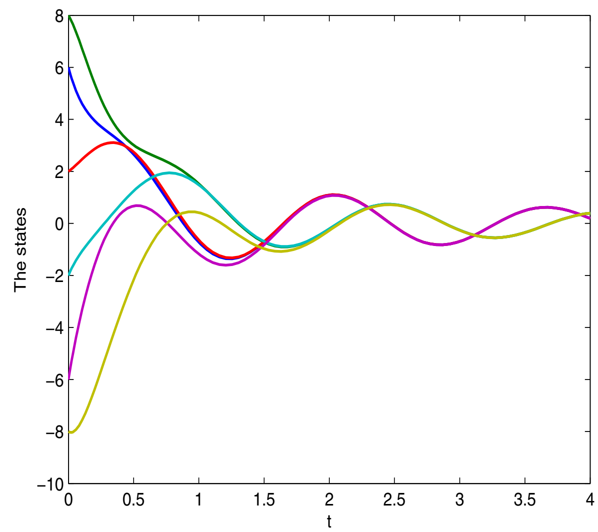

Fig. 1: Consensus process of leaderless case using LQR optimal distributed protocols.

V. Conclusion

In this paper, the inverse optimal approach is employed to design distributed cooperative control protocols for identical linear systems that guarantee consensus and global optimality with respect to a positive definite quadric performance index. Cooperative control and pinning control problems are considered, where the communication graphs are assumed to be directed and have fixe topology. Simple sufficient conditions are established, which indicate that the global optimality is achieved using local distributed protocols which are designed by the linear quadric regulator (LQR) based optimal control method. Examples have been given to demonstrate the effectiveness of the proposed method.

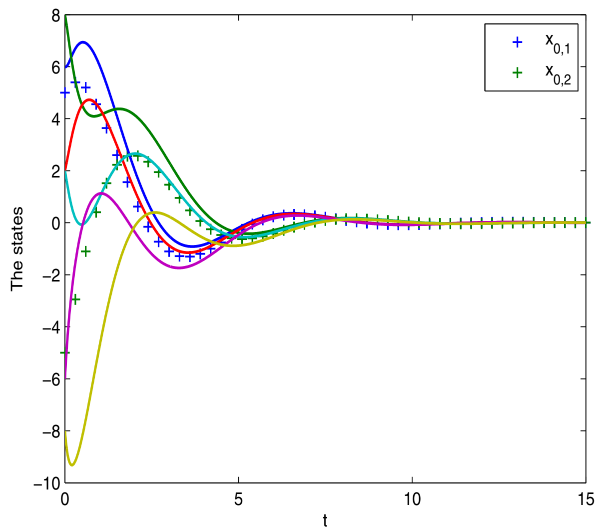

Fig. 2: Consensus process of leader case using LQR optimal distributed protocols.

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