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Global exponential stability of delayed Hopfield neural network on time scale

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***Abstract*— In This paper, by using the theory of calculus on time scales and constructing some suitable Liapunov functions, we obtained the existence, uniqueness and global exponential stability of equilibrium point of delayed Hopfield neural network with impulses on time scale.The conditions can be easily checked in practice by simple algebraic methods.**

I. INTRODUCTION AND SYSTEM DESCRIPTION

HOPFIELD proposed Hopfield neural networks (HNNs) model based on the assumption that the elements in the network communicate with each other instantaneously without time delays in 1980s [1-2]. During the past several years, the convergence dynamics of HNNs have been extensively studied because of the wider application in information processing, optimization problems, etc. Stability results that impose constraint conditions on the network parameters will be dependent on the intended applications in investigating the stability properties of neural networks.

The theory of time scale was initiated by S. Hilger in 1988, which has recently received a lot of attention. The books on the subject of time scale, by Agarwal [3], Bohner and Peterson [4-5], summarize and organize much of time scale calculus. Its novel and fascinating type of mathematics is more general and versatile than the traditional theories of differential and difference equations as it can, under one framework, mathematically describe continuous and discrete hybrid processes and hence is the optimal way forward for accurate and malleable mathematical modelling. As well known, both continuous and discrete systems are very important in implementing and applications. So, it is very meaningful to study the stability of neural networks. In recent years, dynamic equations on time scale have received much attention. Several authors have expounded on various aspects of this new theory(see [3, 6-8] and the references cited therein).

In this paper, we study a new Hopfield neural network on time scale, which is defined by the following system of dynamic equation on time scale.

$$x_i^{\triangleleft}(t) = -c_i x_i(t) + \sum_{j=1}^{n} a_{ij} f_j(x_j(t-\tau_{ij})) + J_i \quad i = 1, 2, \cdots, n \tag{1.1}$$

for $t \in \mathbb{T}_0^+$, $\mathbb{T}_0^+$ is the $\mathbb{T}$-interval $\{t \in \mathbb{T}, 0 \leq t\}$, where $\mathbb{T}$ denotes a time scale, which is an arbitrary nonempty closed subset of the real number $\mathbb{R}$ with the topology and ordering inherited from $\mathbb{R}$ and with bounded graininess μ. For the simplicity, we assume that $0 \in \mathbb{T}$ and $\mathbb{T}$ is unbounded above, i.e., sup $\mathbb{T} = \infty$. τ_{ij} is positive constants such that the delay function $\tau_{ij}(t) := t - \tau_{ij} < t$ and satisfies $\tau_{ij}(t) := \mathbb{T} \to \mathbb{T}$ and for all $t \in \mathbb{T}$.

In (1.1), $x_i^{\triangleleft}$ expresses the delta derivative of the function $x_i(t)$ (see Definition 3). c_i represents the rate with which the ith neuron will reset their potential to the resting state in isolation when they are disconnected from the network and the external inputs. n corresponds to the number of neurons in layers, $x_i(t)$ is the activations of the ith neuron on time scale $\mathbb{T}$, respectively. a_{ij} is the connection weight, J_i denotes the external input. f_j is the input-output function (the activation function). Time delay τ_{ij} on time scale $\mathbb{T}$ correspond to finite speed of axonal signal transmission, $\tau = max_{1\leq i,j\leq n}(\tau_{ij})$

The initial condition associated with (1.1) is given

$$x_i(s) = \phi_i(s), \qquad s \in [-\tau, 0] \bigcap \mathbb{T}, \tag{1.2}$$

where $\phi_i \in C_{rd}([-\tau, 0] \bigcap \mathbb{T}, \mathbb{R})$ is rd-continuous which are defined in Section 2 (see Definition 5).

We denote, by $C_{rd}^0 := C_{rd}([-\tau, 0] \bigcap \mathbb{T} \times ... \times [-\tau, 0] \bigcap \mathbb{T} \longrightarrow \mathbb{R}^n)$, the space of rd-continuous function $\phi = (\phi_1, ..., \phi_n)^T$, which is equipped with the norm $\parallel \phi \parallel_\infty = \Sigma_{i=1}^n \sup_{t\in[-\tau,0]\bigcap\mathbb{T}} \mid \phi_i(t) \mid$, then $(C_{rd}^0, \parallel \cdot \parallel_\infty)$ forms a Banach space (see Ref. [6, Example 9]). For any $\phi \in C_{rd}^0$, we say that $x(t)$ is a solution of (1.1) on $[0, \infty] \bigcap \mathbb{T}$ through ϕ and denote by $x(t, \phi)$, if $x(t)$ is a rd-continuous function defined on $[0, \infty] \bigcap \mathbb{T}$ such that $x(t) = \phi$ on $[0, \infty] \bigcap \mathbb{T}$ respectively, and $x(t)$ satisfies (1.1) for $t \in [0, \infty] \bigcap \mathbb{T}$, where $x(t) = x(t, \phi) = (x_1(t, \phi), ..., x_n(t, \phi))^T$.

Throughout the whole paper, we assume that the activation function f_j possesses the following property:

(H): The function $f_j(j = 1, 2, ..., n)$ is bounded function and Lipschitz continuous on $\mathbb{R}$ with the Lipschitz constant L_j, respectively, i.e.,

$$\mid f_j(x) - f_j(y) \mid \leq L_j \mid x - y \mid$$

System (1.1) is quite general and it includes several well known neural networks model as its special cases such as delay differential equations [9-12]:

$$\dot{x}_i(t) = -c_i x_i(t) + \sum_{j=1}^{n} a_{ij} f_j(x_j(t-\tau_{ij})) + J_i \tag{1.3}$$

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$i = 1, 2, \cdots, n$, for $t \in [t_0, \infty]$, and delay difference equation :

$$\bigtriangleup x_i(t) = -c_i x_i(t) + \sum_{j=1}^{n} a_{ij} f_j(x_j(t - \tau_{ij})) + J_i \quad (1.4)$$

$i = 1, 2, \cdots, n$, for $t \in \{n_0, n_0 + 1, ..., \}$, where $\triangle x_i(t) = x_i(t+1) - x_i(t)$ is the forward difference operator. Eqs.(1.3) and (1.4) are extensively investigated by many authors and a variety of computing result have accumulated in the literature concerning the global exponential stability in the past decade . To the best of our knowledge, no paper in the literature has investigated neural network on time scale. In this paper, we use the calculus theory on time scale to unify and improve discrete-time and continuous-time Hopfield neural networks (1.3) and (1.4) establish some sufficient conditions to ensure existence and global exponential stability of equilibrium of Eq. (1.1). This work offers the method to study (1.3) and (1.4) under one framework.

The paper is organized as follows: In Section 2, we present some basic definitions concerning the calculus on time scale. In Section 3, we develop Liapunov functions technique on time scale to give some sufficient conditions of global exponential stability for Eq. (1.1). In Section 4, an example is given to illustrate the effectiveness of our results. In Section 5, we give some conclusions.

II. Some preliminaries

In this section, we will introduce some standard definitions(see [1-3,7,8]).

Definition 1: A time scale $\mathbb{T}$ is arbitrary nonempty closed subset of the real set $\mathbb{R}$ with the topology and ordering inherited from $\mathbb{R}$.

Definition 2: On any time scale $\mathbb{T}$, we define the forward and backward jump operators by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}, \qquad \rho(t) := \sup\{s \in \mathbb{T} : s < t\},$$

we put $\inf \emptyset := \sup \mathbb{T}$ and $\sup \emptyset := \inf \mathbb{T}$, where $\emptyset$ denotes the empty set. A point t is said to be left-dense if $t > \inf \mathbb{T}$ and $\rho(t) = t$, right-dense if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, left-scattered if $\rho(t) < t$ and right-scattered if $\sigma(t) > t$. The graininess function m for a time scale $\mathbb{T}$ is defined by $\mu(t) := \sigma(t) - t$. If $\mathbb{T}$ has a left-scattered maximum m, then we defined $\mathbb{T}^k$ to be $\mathbb{T} - m$. Otherwise $\mathbb{T}^k = \mathbb{T}$.

Definition 3: For a function $f : \mathbb{T} \longrightarrow \mathbb{R}$ (the range $\mathbb{R}$ of f may be actually replaced by Banach space) the (delta) derivative is defined by

$$f^{\triangleleft} = \frac{f(\sigma(t)) - f(t))}{\sigma(t) - t}$$

if f is continuous at t and t is right-scattered. If t is not right-scattered then the derivative is defined by

$$f^{\triangleleft} = \lim_{s \longrightarrow t} \frac{f(\sigma(t)) - f(s))}{\sigma(t) - s} = \lim_{s \longrightarrow t} \frac{f(t) - f(s)}{t - s}$$

provided this limit exists.

Definition 4: A function $F : \mathbb{T}^k \longrightarrow \mathbb{R}$ is called a delta-antiderivative of $f : \mathbb{T} \longrightarrow \mathbb{R}$ provided $F^{\triangleleft} = f$ holds for all $t \in \mathbb{T}^k$. In this case we define the integral of f by

$$\int_a^t f(s) \triangle s = F(t) - F(a)$$

for $t \in \mathbb{T}$ and we have the following formula

$$\int_t^{\sigma(t)} f(s) \triangle s = \mu(t) f(t)$$

for $t \in \mathbb{T}^k$

Definition 5: A function $f : \mathbb{T} \longrightarrow \mathbb{R}$ is called right-dense continuous provided it is continuous at right-dense points of $\mathbb{T}$ and the left sided limit exists (finite) at left-dense point of $\mathbb{T}$. The set of all right-dense continuous functions on $\mathbb{T}$ is defined by $C_{rd} = C_{rd}(\mathbb{T}) = C_{rd}(\mathbb{T}, \mathbb{R})$.

Definition 6: We say that a function $p : \mathbb{T} \longrightarrow \mathbb{R}$ is regressive provided $1 + \mu(t)p(t) \neq 0$ for all $t \in \mathbb{T}$. The set of all regressive functions on a time scale $\mathbb{T}$ forms an Abelian group under the addition $\oplus$ defined by

$$p \oplus q := p + q + \mu pq$$

The additive inverse in this group is denoted by $\ominus p := -\dfrac{p}{1 + pq}$. We then define subtraction $\ominus$ on the set of regressive functions by

$$p \ominus q := p \oplus (\ominus q).$$

It can be shown that $p \oplus q(\ominus q) = -\dfrac{p - q}{1 + pq}$.

The set of all regressive and right-dense continuous functions will be denoted by $\Re = \Re(\mathbb{T}) = \Re(\mathbb{T}, \mathbb{R})$.

Definition 7: We define the set $\Re^+$ of all positively regressive elements of $\Re$ by

$$\Re^+ = \Re^+(\mathbb{T}, \mathbb{R}) = \{f \in \Re : 1 + \mu(t) f(t) > 0 \; for \; all \; t \in \mathbb{T}\}$$

Next we give the definition of the exponential function and list some of its properties.

Definition 8: For $h > 0$, we define the function $\xi_h(x) = \dfrac{1}{h}\log(1 + xh)$for any real number x except $\dfrac{-1}{h}$ where Log is the principle Logarithm function. If $h = 0$, we define $\xi_0(x) = x$.

Definition 9: If $p(t) \in \Re$, we define the generalized exponential function as

$$\begin{aligned} e_p(t, s) = & \quad exp(\int_s^t \xi_{\mu(\tau)}(p(\tau)) \bigtriangleup \tau) \\ = & \quad exp(\int_s^t \frac{\log(1 + \mu(\tau)p(\tau))}{\mu(\tau)} \bigtriangleup \tau) \end{aligned}$$

for $\tau \in \mathbb{T}$. Alternately, we can define the exponential function $e_p(\cdot, t_0)$ to be the unique solution of the IVP

$$x^{\triangleleft} = p(t)x, \qquad x(t_0) = 1 \qquad for \quad p(t) \in \Re$$

Lemma 1: If $p, q \in \Re$ then

(i) $e_0(t, s) \equiv 1 \; and \; e_p(t, t) = 1$;

(ii) $e_p(\sigma(t), s) = (1 + \mu(t)p(t)) e_p(t, s)$;

(iii) $e_p(t, s) e_p(s, r) = e_p(t, r)$;

(iv) $e_p(t,s) = 1/e_p(s,t) = e_{\ominus p}(s,t)$;
(v) $e_p(t,s) > 0$, for $p \in \Re^+$;
(vi) $e_p(t,s)e_q(t,s) = e_{p\oplus q}(t,s)$;
(vii) $e_p(t,s)/e_q(t,s) = e_{p\ominus q}(t,s)$.

Lemma 2: ([13]) Assume that $f, g : \mathbb{T} \to \mathbb{R}$ are delta differential at $t \in \mathbb{T}^k$. then

$$\begin{aligned}(fg)^{\triangleleft}(t) &= f^{\triangleleft}(t)g(t) + f(\sigma(t))g^{\triangleleft}(t) \\ &= f(t)g^{\triangleleft}(t) + f^{\triangleleft}(t)g(\sigma(t))\end{aligned}$$

Lemma 3: ([14-15]) If $H(x) \in C(\mathbb{R}^n, \mathbb{R}^n)$ satisfies following conditions
(i) $H(x)$ is injective on $\mathbb{R}^n$
(ii) $\parallel H \parallel \to +\infty$ as $\parallel x \parallel \to +\infty$
then $H(x)$ is a homeomorphism of $\mathbb{R}^n$ onto itself.

III. Main Results

Firstly, by means of homeomorphism theory, we will study the existence and uniqueness of the equilibrium point of system (1.1). An equilibrium point of system (1.1) is a constant vector $(x_1^*, \cdots, x_n^*) \in R^n$ which satisfies the equation

$$-c_i x_i^* + \sum_{j=1}^{n} a_{ij} f_j(x_j^*) + J_i = 0$$

the existence of equilibrium point of system (1.1) is easily obtained by Brouwer's fixed point theorem.

Theorem 1: Assume that (H) hold, suppose further that for each i=1...n, following inequality is satisfied

$$c_i > \sum_{j=1}^{n} |a_{ji}| L_i, \qquad i = 1, \cdots, n, \tag{3.1}$$

then there exists a unique equilibrium point of system (1.1)

Proof: Consider a mapping $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$\Phi_i(x) = -c_i x_i + \sum_{j=1}^{n} a_{ij} f_j(x_j) + J_i \tag{3.2}$$

where $x = (x_1, \cdots, x_n)^T \in \mathbb{R}^n$, $\Phi(x) = (\Phi_1(x), \cdots, \Phi_n(x))^T \in \mathbb{R}^n$. First, we want to show that Φ is an injective mapping on $\mathbb{R}^n$. By contradiction, suppose that there exists distinct $x, \bar{x} \in \mathbb{R}^n$ such that $\Phi(x) = \Phi(\bar{x})$, Where $x = (x_1, \cdots, x_n)^T$ and $\bar{x} = (\bar{x}_1, \cdots, \bar{x}_n)^T$. then it follows from (3.2) that

$$c_i x_i - c_i \bar{x}_i = \sum_{j=1}^{n} a_{ij}(f_j(x_j) - f_j(\bar{x}_j)) \tag{3.3}$$

It follows from (H) and (3.3) that

$$\begin{aligned}c_i|x_i - \bar{x}_i| &= |\sum_{j=1}^{n} a_{ij}[f_j(x_j) - f_j(\bar{x}_j)]| \\ &\le \sum_{j=1}^{n} |a_{ij}||f_j(x_j) - f_j(\bar{x}_j)| \\ &\le \sum_{j=1}^{n} L_j |a_{ij}||x_j - \bar{x}_j|\end{aligned}$$

Then we have

$$\sum_{i=1}^{n}(c_i - \sum_{j=1}^{n} L_i|a_{ji}|)|x_i - \bar{x}_i| \le 0 \tag{3.4}$$

It follows from (3.1) and (3.4) that $|x_i - \bar{x}_i| = 0,\ i = 1...n$. That is $x = \bar{x}$, which leads to a contradiction. Therefore,Φ is an injective on $\mathbb{R}^n$. So we shall prove Φ is a homeomorphism on $\mathbb{R}^n$. For convenience, we let $\tilde{\Phi}(x) = \Phi(x) - \Phi(0)$,Where

$$\tilde{\Phi}_i(x) = -c_i x_i + \sum_{j=1}^{n} a_{ij}(f_j(x_j) - f_j(0)), \quad i = 1, \cdots, n$$

We assert that $\|\tilde{\Phi}\| \to \infty$ as $\|x\| \to \infty$
Clearly,

$$\begin{aligned}\|\tilde{\Phi}\| &= \sum_{i=1}^{n} |\ \Phi(x)\ | = \sum_{i=1}^{n} |\ -c_i x_i + \sum_{j=1}^{n} a_{ij}(f_j(x_j) - f_j(0))\ | \\ &\ge \sum_{i=1}^{n} |\ c_i \mid x_i\ | - |\sum_{j=1}^{n} |a_{ij}|(f_j(x_j) - f_j(0))\ || \\ &\ge \sum_{i=1}^{n} |\ c_i \mid x_i\ | - \sum_{j=1}^{n} |a_{ij}| L_j |x_j|\ | \\ &\ge \sum_{i=1}^{n} |\ (c_i - \sum_{j=1}^{n} |a_{ji}| L_i)|x_i|\ |\end{aligned}$$

So,it follows that Φ satisfies $\|\Phi\| \to \infty$ as $\|x\| \to \infty$. By Lemma 3, Φ is a homeomorphism on $\mathbb{R}^n$ and there exists a unique point $x^* = (x^*_{1,\cdots,}x^*_n)^T$ such that $\Phi(x^*) = 0$.From the definition of Φ, we know that $x^* = (x^*_{1,\cdots,}x^*_n)^T$ is the unique equilibrium point of Eq.(1.1).

Secondly, we study the global exponential stability of the unique equilibrium for Eq. (1.1) on time scale by using Liapunov method.

Theorem 2: Suppose that system (1.1)-(1.2) satisfies (H), if there exist constants λ_i and $p > 0$ such that

$$\begin{aligned}&\lambda_i\{p + [c_i^2\mu(t) - 2c_i + \sum_{j=1}^{n} \mid a_{ij} \mid L_j(1 + c_i\mu(t))] \\ &\times(1 + p\mu(t))\} + \sum_{j=1}^{n} \lambda_j \mid a_{ji} \mid L_i[1 + c_j\mu(t + \tau_{ji}) \\ &+ n \mid a_{ji} \mid L_i](1 + p\mu(t + \tau_{ji}))e_p(t + \tau_{ji}, t) < 0\end{aligned} \tag{3.5}$$

for all $i, j = 1, 2, ..., n, t \in \mathbb{T}_0^+$, then the equilibrium $x^* = (x^*_{1,\cdots,}x^*_n)^T$ of system (1.1)-(1.2) is globally exponentially stable for every J, i.e., every solution $x = (x_1, \cdots, x_n)^T$ of system (1.1)-1.2) satisfy

$$\sum_{i=1}^{n}(x_i(t) - x_i^*)^2 \le \frac{M}{e_p(t,0)} \sum_{i=1}^{n} \sup_{s\in[-\tau,0]} (x_i(s) - x_i^*)^2 \tag{3.6}$$

for all $t \in \mathbb{T}_0^+$, where $M \ge 1$ is a constant.

Proof: Condition (3.5) implies that

$$-2c_i\lambda_i + \sum_{j=1}^{n}(\lambda_i \mid a_{ij} \mid L_j + \lambda_j \mid a_{ji} \mid L_i) < 0, \tag{3.7}$$

$i = 1, 2, \cdots, n$. By using the theorem 1, we can prove system (1.1)-1.2) possesses a unique equilibrium $x^* = (x^*_{1,\cdots,}x^*_n)^T$. Let $u_i(t) = x_i(t) - x^*_i, i = 1, 2, ..., n$. Then we can rewrite Eq. (1.1) into

$$u_i(t)^{\triangleleft} = -c_iu_i(t) + \textstyle\sum_{j=1}^n a_{ij}\big(f_j(x_j(t-\tau_{ij})) - f_j(x^*_j)\big), \tag{3.8}$$

$i = 1, 2, \cdots, n$, for all $t \in \mathbb{T}^+_0$. To prove Eq. (3.6) is equivalent to prove

$$\sum_{i=1}^n u_i(t)^2 \le \frac{M}{e_p(t,0)} \sum_{i=1}^n \sup_{s\in[-\tau,0]} u_i(s)^2 \tag{3.9}$$

for all $t \in \mathbb{T}^+_0$, where $M \ge 1$ is a constant.

Now, we construct the Liapunov functional $V(t)$ as follows

$$\begin{aligned}
V(t) &= V_1(t) + V_2(t)\\
V_1(t) &= \sum_{i=1}^n \lambda_i u_i(t)^2 e_p(t,0),\\
V_2(t) &= \sum_{i=1}^n\sum_{j=1}^n \lambda_i L_j \mid a_{ij} \mid \int_{t-\tau_{ij}}^t (1 + c_i\mu(s+\tau_{ij})\\
&\quad +n \mid a_{ij} \mid L_j)(1 + p\mu(s+\tau_{ij}))\\
&\quad \times u_j(s)^2 e_p(s+\tau_{ij},0) \triangle s.
\end{aligned}$$

Calculating $V(t)^{\triangleleft}$-derivative of $V(t)$ along the solution of $Eq.$(3.8), we have

$$\begin{aligned}
&V_1(t)^{\triangleleft}\\
&= \sum_{i=1}^n \lambda_i[(u_i(t)^2)^{\triangleleft} e_p(\sigma(t),0) + u_i(t)^2(e_p(t,0))^{\triangleleft}]\\
&= \sum_{i=1}^n \lambda_i\Big[\big(2u_i(t)(-c_iu_i(t)\\
&\quad + \textstyle\sum_{j=1}^n a_{ij}(f_j(x_j(t-\tau_{ij})) - f_j(x^*_j)))\\
&\quad +\mu(t)(-c_iu_i(t) + \displaystyle\sum_{j=1}^n a_{ij}(f_j(x_j(t-\tau_{ij}))\\
&\quad -f_j(x^*_j)))^2\big)e_p(\sigma(t),0) + u_i(t)^2 p e_p(t,0)\Big]\\
&\le \sum_{i=1}^n \Big[\lambda_i e_p(\sigma(t),0)(-2c_i(u_i(t))^2\\
&\quad +2\sum_{j=1}^n \mid a_{ij} \mid L_j \mid x_j(t-\tau_{ij}) - x^*_j \mid\mid u_i(t) \mid\\
&\quad +\mu(t)((c_i)^2(u_i(t))^2 + 2c_i \mid u_i(t) \mid\\
&\quad \times \sum_{j=1}^n \mid a_{ij} \mid L_j \mid x_j(t-\tau_{ij}) - x^*_j \mid\\
&\quad +n\sum_{j=1}^n (a_{ij})^2 L_j^2 (x_j(t-\tau_{ij}) - x^*_j)^2))\\
&\quad +\lambda_i(u_i(t))^2 p e_p(t,0)\Big]\\
&\le \sum_{i=1}^n \Big[\lambda_i(u_i(t))^2 p e_p(t,0)\\
&\quad +\lambda_i e_p(\sigma(t),0)((c_i^2\mu(t) - 2c_i)(u_i(t))^2\\
&\quad +\sum_{j=1}^n \mid a_{ij} \mid L_j(1+c_i)(u_j(t-\tau_{ij})^2\\
&\quad +(u_i(t))^2) + n\sum_{j=1}^n a_{ij})^2 L_j^2(u_j(t-\tau_{ij}))^2))\Big]\\
&= \sum_{i=1}^n \lambda_i\Big[p + (c_i^2\mu(t) - 2c_i\\
&\quad +\sum_{j=1}^n \mid a_{ij} \mid L_j(1 + c_i\mu(t)))(1 + p\mu(t))\Big]\\
&\quad \times e_p(t,0)(u_i(t))^2\\
&\quad +\sum_{i=1}^n\sum_{j=1}^n \lambda_i \mid a_{ij} \mid L_j(1 + c_i\mu(t)\\
&\quad +n \mid a_{ij} \mid^2 L_j^2)(1 + p\mu(t))\\
&\quad \times(u_j(t-\tau_{ij}))^2 e_p(t,0).
\end{aligned}$$

$$\begin{aligned}
&V_2(t)^{\triangleleft}\\
&= \Big(\sum_{i=1}^n\sum_{j=1}^n \lambda_i \mid a_{ij} \mid L_j \int_{t-\tau_{ij}}^t (1 + c_i\mu(s+\tau_{ij})\\
&\quad +n \mid a_{ij} \mid L_j)(1 + p\mu(s+\tau_{ij}))\\
&\quad \times(u_j(s))^2 e_p(s+\tau_{ij},0) \triangle s\Big)^{\triangleleft}\\
&= \sum_{i=1}^n\sum_{j=1}^n \lambda_i \mid a_{ij} \mid L_j(1 + c_i\mu(t+\tau_{ij})\\
&\quad +n \mid a_{ij} \mid L_j)\\
&\quad \times(1 + p\mu(t+\tau_{ij}))(u_j(t))^2 e_p(t+\tau_{ij},0)\\
&\quad -\sum_{i=1}^n\sum_{j=1}^n \lambda_i \mid a_{ij} \mid L_j(1 + c_i\mu(t) + n \mid a_{ij} \mid\\
&\quad \times L_j)(1 + p\mu(t))(u_j(t-\tau_{ij}))^2 e_p(t,0).
\end{aligned}$$

$$\begin{aligned}
&V(t)^{\triangleleft}\\
&= V_1(t)^{\triangleleft} + V_2(t)^{\triangleleft}\\
&\le \sum_{i=1}^n e_p(t,0)\Big\{\lambda_i\Big[p + (c_i^2\mu(t) - 2c_i\\
&\quad +\sum_{j=1}^n \mid a_{ij} \mid L_j(1 + c_i\mu(t)))(1 + p\mu(t))\Big]\\
&\quad +\sum_{j=1}^n \lambda_j \mid a_{ji} \mid L_i(1 + c_j\mu(t+\tau_{ji}) + n \mid a_{ji} \mid\\
&\quad \times L_i)(1 + p\mu(t+\tau_{ji}))e_p(t+\tau_{ji},t)\Big\}(u_i(t))^2
\end{aligned}$$

By using (3.5), we can conclude that $V(t)^{\triangleleft} \le 0$, for $t \in \mathbb{T}^+_0$, which implies that $V(t) \le V(0)$,for $t \in \mathbb{T}^+_0$.

$$\begin{aligned}
&V(0)\\
&= V_1(0) + V_2(0) = \sum_{i=1}^n \lambda_i u_i(0)^2 e_p(0,0)\\
&\quad +\sum_{i=1}^n\sum_{j=1}^n \lambda_i L_j \mid a_{ij} \mid \int_{-\tau_{ij}}^0 (1 + c_i\mu(s+\tau_{ij})\\
&\quad +n \mid a_{ij} \mid L_j)\\
&\quad \times(1 + p\mu(s+\tau_{ij}))u_j(s)^2 e_p(s+\tau_{ij},0) \triangle s\\
&\le \max_{1\le i\le n}\Big[\lambda_i + \sum_{j=1}^n \lambda_j L_i \mid a_{ji} \mid (1 + c_j\mu + n \mid a_{ji} \mid\\
&\quad \times L_i)(1 + p\mu)\tau e_p(\tau,0)\Big] \sum_{i=1}^n \sup_{-\tau\le s\le 0} u_i(s)^2.
\end{aligned} \tag{3.10}$$

where $\mu(t) \leq \mu$. Observe that

$$V(t) \geq \min_{1\leq i\leq n} \lambda_i \sum_{i=1}^{n} u_i(t)^2 e_p(t,0) \qquad (3.11)$$

Then it follows from (3.10) and (3.11) that

$$\sum_{i=1}^{n} u_i(t)^2 \leq \frac{M}{e_p(t,0)} \sum_{i=1}^{n} \sup_{s\in[-\tau,0]} u_i(s)^2$$

for $t \in \mathbb{T}_0^+$, where $M \geq 1$ is constant. This completes the proof.

Remark 1: If the time scale $\mathbb{T} = \mathbb{R}$, $\mu(t) = 0$ .Then , from Theorem 2, we can immediately derive the following result which is similar to the proof of Ref. [16].

Corollary 1: Suppose that system (1.3) satisfies condition (H) and if there exist constants $\lambda_i > 0, i = 1, 2, ..., n$ such that

$$\lambda_i(-2c_i + \sum_{j=1}^{n} \mid a_{ij} \mid L_j) + \sum_{j=1}^{n} \lambda_j \mid a_{ji} \mid L_i < 0$$

then the equilibrium $x^* = (x_1^*, ..., x_n^*)^T$ of Eq. (1.3) is globally exponentially stable for every J.

Remark 2: If the time scale $\mathbb{T} = \mathbb{Z}$, then $\mu(t) = 1$ and Eq. (1.1) becomes Eq. (1.4). From Theorem 2, we can obtain the following result.

Corollary 2: Suppose that Eq.(1.4) satisfies condition (H) and if there exist constants $\lambda_i > 0, i = 1, 2, ..., n$ such that

$$\lambda_i[c_i^2 - 2c_i + \sum_{j=1}^{n} \mid a_{ij} \mid L_j(1 + c_i)]$$
$$+ \sum_{j=1}^{n} \lambda_j \mid a_{ji} \mid L_i(1 + \lambda_j + n \mid a_{ji} \mid L_j) < 0$$

then the equilibrium $x^* = (x_1^*, ..., x_n^*)^T$ of Eq. (1.4) is globally exponentially stable for every J.

Remark 3: The result of Theorem 2 unifies the previous literatures on Hopfield neural networks of discrete-time and continuous-time, and reveals the discrepancies of results of continuous-time ($\mu(t) = 0$) and discrete-time ($\mu(t) = 1$) Hopfield neural network.

IV. An Example

In this section, an example is shown to verify the effectiveness of the result obtained in the previous section. Consider the following simple Hopfield neural network with delays on time scale $\mathbb{T}$:

$$x_i^{\triangleleft}(t) = -c_i x_i(t) + \sum_{j=1}^{n} a_{ij} f_j(x_i(t-\tau_{ij})) + J_i \quad i = 1, 2, \cdots, n \qquad (4.1)$$

for $t \in \mathbb{T}_0^+$, $(c_1, c_2) = (0.1, 0.1)$ $\tau_{ij} = \frac{1}{2}$ $J_i = 2(i, j = 1, 2)$, where

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 0.01 & 0.02 \\ 0.03 & 0.04 \end{bmatrix}$$

Taking $f_j(x) = \frac{1}{2}(\mid x + 1 \mid - \mid x - 1 \mid)$,we have $L_i = L_j = 1(i, j = 1, 2)$. Again choosing $\lambda_i = \lambda_j = 1(i = 1, 2)$, we can easily verify that the conditions of Corollary 1 and 2 are all satisfied, respectively:

$$\lambda_1(-2c_1 + \sum_{j=1}^{2} \mid a_{1j} \mid L_j) + \sum_{j=1}^{2} \lambda_j \mid a_{j1} \mid L_1 < -0.13$$

$$\lambda_2(-2c_2 + \sum_{j=1}^{2} \mid a_{2j} \mid L_j) + \sum_{j=1}^{2} \lambda_j \mid a_{j2} \mid L_2 < -0.07$$

$$\lambda_1[c_1^2 - 2c_1 + \sum_{j=1}^{2} \mid a_{1j} \mid L_j(1 + c_1)]$$
$$+ \sum_{j=1}^{2} \lambda_j \mid a_{j1} \mid L_1(1 + \lambda_j + 2 \mid a_{j1} \mid L_j) < -0.858$$

$$\lambda_2[c_2^2 - 2c_2 + \sum_{j=1}^{2} \mid a_{2j} \mid L_j(1 + c_2)]$$
$$+ \sum_{j=1}^{2} \lambda_j \mid a_{j2} \mid L_2(1 + \lambda_j + 2 \mid a_{j2} \mid L_j) < -0.736$$

Thus, it follows from Corollary 1 and 2 that system (4.1) has a unique equilibrium point which is globally exponentially stable.

V. Conclusions

In this letter, Global exponential stability of delayed Hopfield neural network on time scale have been studied. Some sufficient conditions for global exponential stability of the equilibrium point have been established.The conditions possess highly important significance and are easily checked in practice by simple algebraic methods. These obtained results are new and they complement previously known results. Moreover,an example is given to illustrate the effectiveness of our results.

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