Theory of Swarm Intelligence

Dirk Sudholt

University of Sheffield, UK

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Swarm Intelligence

Collective behavior of a "swarm" of agents.

Examples from Nature

- dome construction by termites
- communication of bees
- ant trails
- foraging behavior of fish schools and bird flocks
- swarm robotics

Plenty of inspiration for optimization.

ACO and PSO

Ant colony optimization (ACO)

- inspired by foraging behavior of ants
- artificial ants construct solutions using pheromones
- pheromones indicate attractiveness of solution component

Particle swarm optimization (PSO)

- mimics search of bird flocks and fish schools
- particles "fly" through search space
- each particle is attracted by own best position and best position of neighbors

Theory

What "theory" can mean

- convergence analysis
- analysis of simplified models of algorithms
- empirical studies on test functions
- runtime analysis / computational complexity analysis
- ...

Example Question

How long does it take on average until algorithm A finds a target solution on problem P?

Notion of time: number of iterations, number of function evaluations

Content

What this tutorial is about

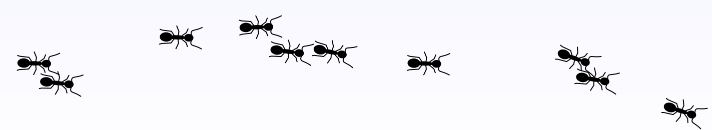
- runtime analysis
- simple variants of swarm intelligence algorithms
- insight into their working principles
- impact of parameters and design choices on performance
- what distinguishes ACO/PSO from evolutionary algorithms?
- performance guarantees for combinatorial optimization
- methods and proof ideas

What this tutorial is not about

- convergence results
- analysis of models of algorithms
- no intend to be exhaustive

Overview

Ant Colony Optimization (ACO)



Main idea: artificial ants communicate via pheromones.

Scheme of ACO

Repeat:

- construct ant solutions guided by pheromones
- update pheromones by reinforcing good solutions

Pseudo-Boolean Optimization

Goal: maximize $f\colon \{0,1\}^n \to \mathbb{R}$.

Illustrative test functions

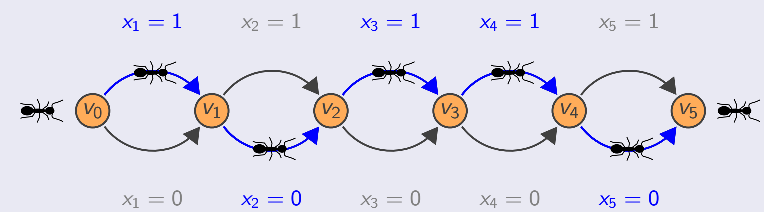
$$\textsc{OneMax}(x) = \sum_{i=1}^{n} x_i$$

$$\textsc{LeadingOnes}(x) = \sum_{i=1}^{n} \prod_{j=1}^{i} x_j$$

$$\textsc{Needle}(x) = \prod_{i=1}^{n} x_i$$

ACO in Pseudo-Boolean Optimization

Solution Construction



Probability of choosing an edge equals pheromone on the edge.

Initial pheromones: $\tau(x_i = 0) = \tau(x_i = 1) = 1/2$.

Note: no linkage between bits. No heuristic information used.

Pheromones $\tau(x_i = 1)$ suffice to describe all pheromones.

ACO in Pseudo-Boolean Optimization (2)

Pheromone update: reinforce some good solution x.

Strength of update determined by evaporation factor $0 \le \rho \le 1$:

$$\tau'(x_i = 1) = \begin{cases} (1-\rho)\cdot\tau(x_i = 1) & \text{if } x_i = 0 \\ (1-\rho)\cdot\tau(x_i = 1) + \rho & \text{if } x_i = 1 \end{cases}$$

Pheromone borders as in MAX-MIN Ant System (Stützle and Hoos, 2000):

$$\tau_{\min} \le \tau' \le 1 - \tau_{\min}$$

Default choice: $\tau_{\min} := 1/n$ (cf. standard mutation in EAs).

One Ant?

Most ACO algorithms analyzed: one ant per iteration.

One ant at a time, many ants over time.

Steady-state GA

- Probabilistic model: Population
- New solutions: selection + variation
- Environmental selection

Ant Colony Optimization

- Probabilistic model: Pheromones
- New solutions: construction graph
- Selection for reinforcement

Evolutionary Algorithms vs. ACO

MMAS* (Gutjahr and Sebastiani, 2008)

Start with uniform random solution x^* and repeat:

- Construct x.
- Replace x^* by x if $f(x) > f(x^*)$.
- Update pheromones w. r. t. x^* (best-so-far update).

Note: best-so-far solution x^* is constantly reinforced.

(1+1) EA

Start with uniform random solution x^* and repeat:

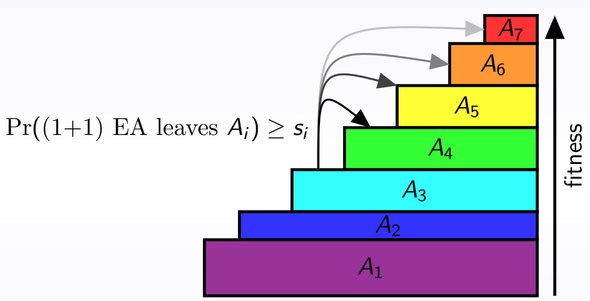
- Create x by flipping each bit in x^* independently with probability $1/n$.
- Replace x^* by x if $f(x) \geq f(x^*)$.

(1+1) EA: Probability of setting bit to 1 is in $\{1/n, 1 - 1/n\}$.

MMAS*: Probability of setting bit to 1 is in $[1/n, 1 - 1/n]$ (unless $\rho \approx 1$).

Overview

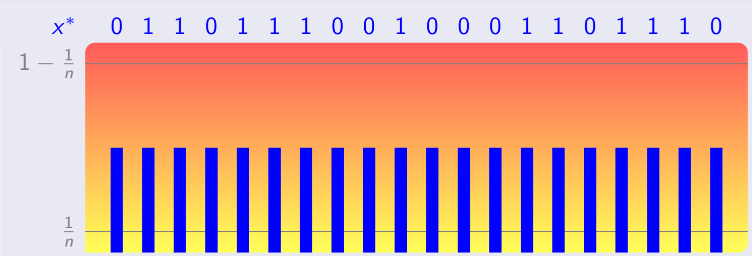
Fitness-level Method for the (1+1) EA



Expected optimization time of (1+1) EA at most $\sum_{i=1}^{m-1} \frac{1}{s_i}$.

MMAS*

Pheromones on 1-edges



After $(\ln n)/\rho$ reinforcements of x^* MMAS* temporarily behaves like (1+1) EA.

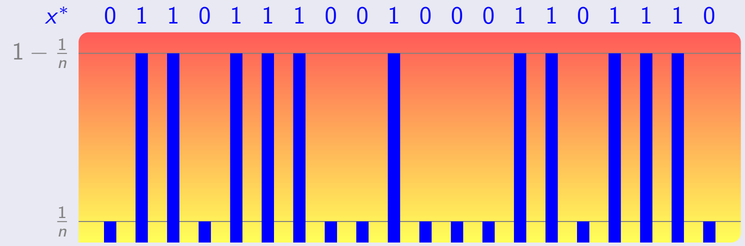
Fitness-Level Method with A_i = search points with i-th fitness value

$$\text{(1+1) EA:} \quad \leq \sum_{i=1}^{m-1} \frac{1}{s_i} \qquad\qquad \text{MMAS*:} \quad \leq \sum_{i=1}^{m-1} \frac{1}{s_i} + m \cdot \frac{\ln n}{\rho}$$

Upper bounds: time for finding improvements + time for pheromone adaptation.

MMAS*

Pheromones on 1-edges



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(1+1) EA: $\leq \sum_{i=1}^{m-1} \frac{1}{s_i}$ MMAS*: $\leq \sum_{i=1}^{m-1} \frac{1}{s_i} + m \cdot \frac{\ln n}{\rho}$

Upper bounds: time for finding improvements + time for pheromone adaptation.

Bounds with Fitness Levels

LeadingOnes 11110010

$$s_i \geq \frac{1}{n} \cdot \left(1 - \frac{1}{n}\right)^{n-1} \geq \frac{1}{en}$$

Theorem

(1+1) EA: en^2 MMAS*: $en^2 + n \cdot \frac{\ln n}{\rho} = O(n^2 + (n \log n)/\rho)$

Unimodal functions with d function values:

Theorem

(1+1) EA: end MMAS*: $end + d \cdot \frac{\ln n}{\rho} = O(nd + (d \log n)/\rho)$

Overview

Strict Selection

Most ACO algorithms replace x^* only if $f(x) > f(x^*)$.

Drawback

Cannot explore plateaus.

Theorem (Neumann, Sudholt, Witt, 2009)

Expected time of MMAS* *on* Needle *is* $\Omega(2^{-n} \cdot n^n) = \Omega((n/2)^n)$.

Define variant MMAS of MMAS* replacing x^* if $f(x) \geq f(x^*)$.
Pheromones on each bit perform a random walk.

Theorem (Neumann, Sudholt, Witt, 2009 and Sudholt, 2011)

Expected time of MMAS *on* Needle *is* $O(n^2/\rho^2 \cdot \log n \cdot 2^n)$.

Mixing time estimates (Sudholt, 2011)

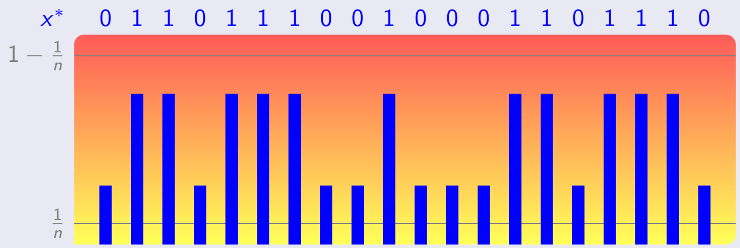
MMAS "forgets" initial pheromones on bits that have been irrelevant for the last $\Omega(n^2/\rho^2)$ steps.

MMAS and Fitness Levels

Is MMAS as fast as MMAS* on easy functions like OneMax?

Switching between equally fit solutions can prevent freezing.

Pheromones on 1-edges



Fitness-level method breaks down!

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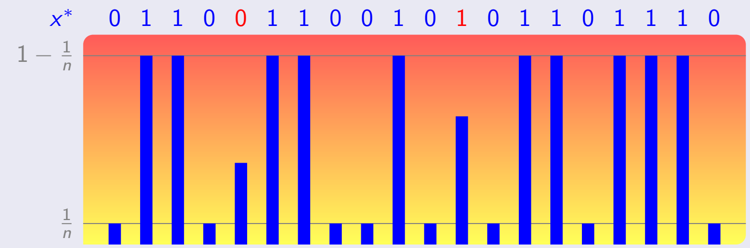


Fitness-level method breaks down!

Is this Behavior Detrimental?

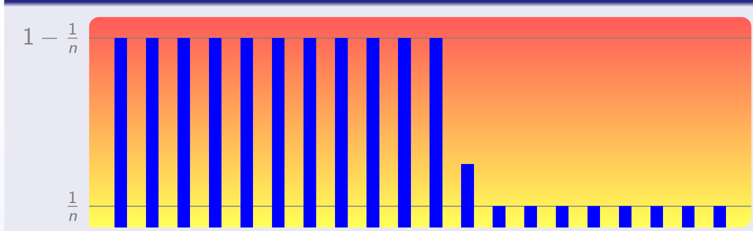
Probably not.

Theorem (Kötzing, Neumann, Sudholt, and Wagner, 2011)

$O(n \log n + n/\rho)$ on OneMax *for both* MMAS* *and* MMAS.

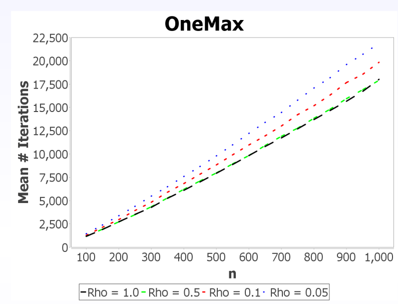
Assuming the sum of pheromones is fixed. Worst possible pheromone distribution for finding improvements on OneMax (Gleser, 1975):

Pheromones on 1-edges

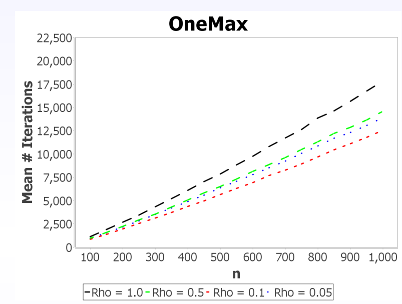


Worst case: all pheromones (but one) at borders.

Experiments (Kötzing et al., 2011)



MMAS*



MMAS

- MMAS better than MMAS*
- MMAS with $\rho < 1$ better than (1+1) EA (=MMAS at $\rho = 1$)!
- does not hold for MMAS*

Open Problem

Prove that MMAS with proper ρ is faster than MMAS* and (1+1) EA.

Overview

Iteration-Best Update

λ-MMAS$_{\text{ib}}$

Repeat:

- construct λ ant solutions
- update pheromones by reinforcing the best of these solutions

Advantages:

- can escape from local optima
- inherently parallel
- simpler ants

Iteration-Best vs. Comma Strategies

Rowe and Sudholt, GECCO 2012

(1,λ) EA: $\lambda \geq \log_{e/(e-1)} n (\approx 5 \log_{10} n)$ necessary, even for OneMax.

If $\lambda \leq \log_{e/(e-1)} n (\approx 5 \log_{10} n)$ then (1,λ) EA needs exponential time.

Reason: (1,λ) EA moves away from optimum if close and λ too small.

Behavior too chaotic to allow for hill climbing!

Iteration-Best on OneMax

Slow pheromone adaptation effectively eliminates chaotic behavior.

Theorem (Neumann, Sudholt, and Witt, 2010)

If $\rho = 1/(cn^{1/2} \log n)$ for a large constant $c > 0$ then 2-MMAS$_{\text{ib}}$ optimizes OneMax in expected time $O(n \log n)$.

Two ants are enough!

Proof idea: as long as all pheromones are at least $1/3$, the sum of pheromones grows steadily.

Large ρ or small λ: pheromones come crashing down to $1/n$.

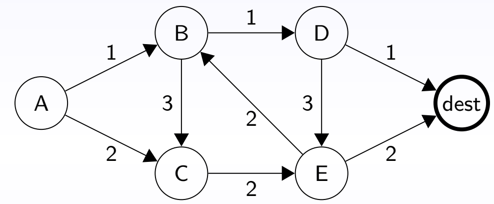
Theorem

Choosing $\lambda/\rho \leq (\ln n)/244$, the optimization time of λ-MMAS$_{\text{ib}}$ on every function with a unique optimum is $2^{\Omega(n^{\varepsilon})}$ for some constant $\varepsilon > 0$ w. o. p.

Overview

ACO System for Single-Destination Shortest Path Problem

From Sudholt and Thyssen (2012), going back to Attiratanasunthron and Fakcharoenphol (2008).



$\text{MMAS}_{\text{SDSP}}$

For each vertex u the ant
- memorizes and keeps track of its best-so-far path
- constructs a simple path from u to the destination
- updates pheromones on edges $(u, \cdot)$ (local pheromone update)

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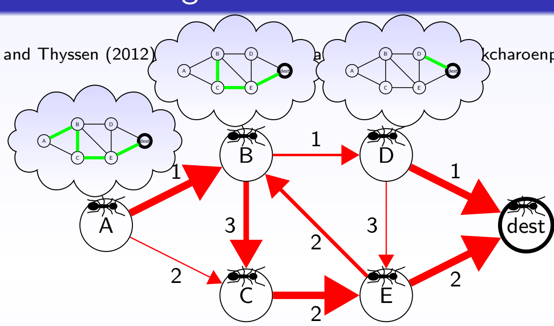

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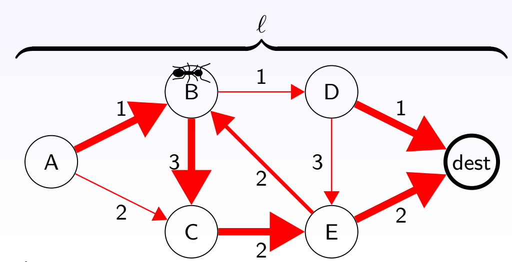
Shortest Paths Propagate Through the Graph



Let $\tau_{\min} := 1/(\Delta\ell)$. Consider vertex u such that all ants on its shortest paths have found shortest paths and adapted their pheromones.

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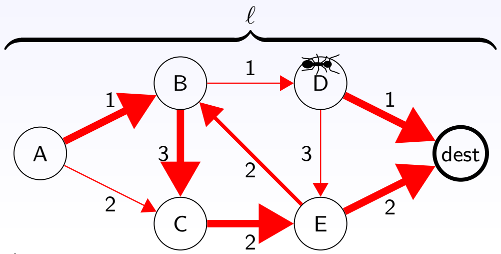

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- probability of ant at u choosing the first edge correctly $\geq \tau(e)/2 \geq \tau_{\min}/2$

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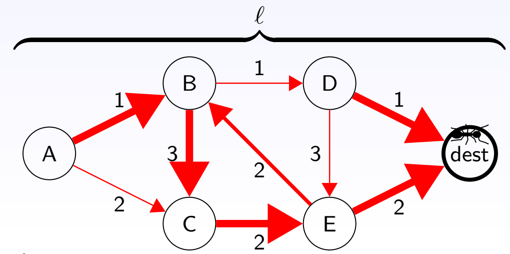

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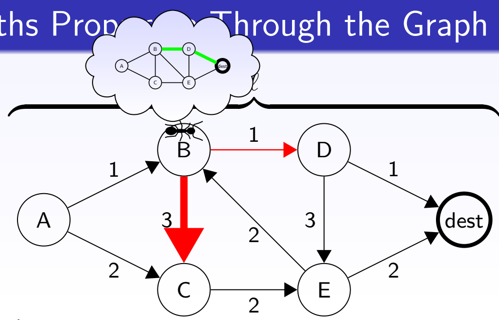

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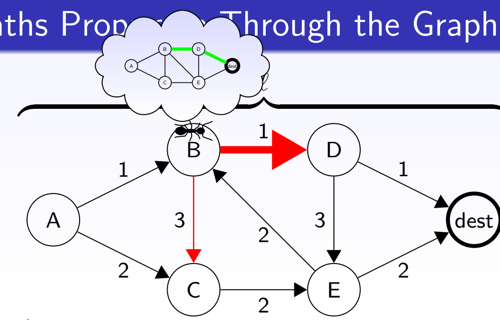

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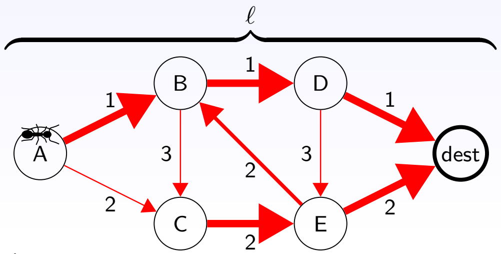
Expected time until ant at u has done the same $\leq 2e/\tau_{\min} + \ln(1/\tau_{\min})/\rho$.

Shortest Paths Propagate Through the Graph



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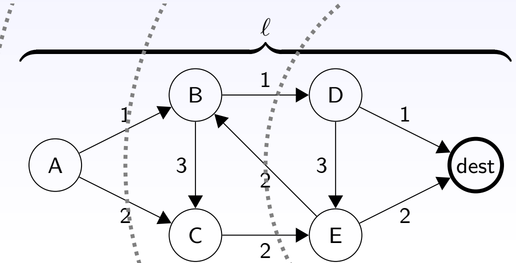
Upper bounds for $\text{MMAS}_{\text{SDSP}}$ (Sudholt and Thyssen, 2012)

- Consider all vertices sequentially: $O(n\Delta\ell + n\ln(\Delta\ell)/\rho)$.

Shortest Paths Propagate Through the Graph



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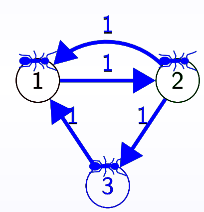
Upper bounds for $\text{MMAS}_{\text{SDSP}}$ (Sudholt and Thyssen, 2012)

- Consider all vertices sequentially: $O(n\Delta\ell + n\ln(\Delta\ell)/\rho)$.
- Slice graph into "layers" and exploit parallelism: $O(\Delta\ell^2 + \ell/\rho)$.

Overview

All-Pairs Shortest Path Problem

Use distinct pheromone function $\tau_v \colon E \to \mathbb{R}_0^+$ for each destination v:



A Simple Interaction Mechanism

Path construction with interaction

For each ant traveling from u to v

- with prob. $1/2$
  - use τ_v to travel from u to v
- with prob. $1/2$
  - choose an intermediate destination $w \in V$ uniformly at random
  - uses τ_w to travel from u to w
  - uses τ_v to travel from w to v

Speed-up by Interaction

Theorem

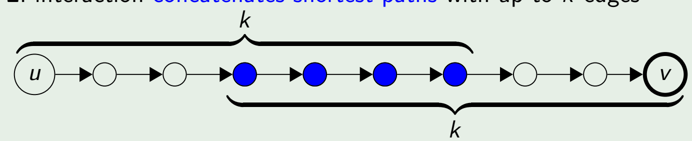
If $\tau_{\min} = 1/(\Delta\ell)$ and $\rho \le 1/(23\Delta \log n)$ the number of iterations using interaction w. h. p. is $O(n \log n + \log(\ell) \log(\Delta\ell)/\rho)$.

Possible improvement: $O(n^3) \to O(n \log^3 n)$

Proof Sketch

Phase 1: find all shortest paths with one edge
slow evaporation $\longrightarrow$ near-uniform search

Phase 2: interaction concatenates shortest paths with up to k edges



$\longrightarrow$ find shortest paths with up to $3/2 \cdot k$ edges.

Note: slow adaptation helps!

Stochastic and Dynamic Shortest Path Problems

Sudholt and Thyssen, Algorithmica 2012

Unmodified $\text{MMAS}_{\text{SDSP}}$ on noisy SDSP: ants can become risk-seeking.

Doerr, Hota, and Kötzing, GECCO 2012

Re-evaluating best-so-far paths removes risk-seeking behavior.

Lissovoi and Witt, GECCO 2013

How effective is ACO in tracking dynamically changing shortest paths?

Lissovoi and Witt, GECCO 2014

MMAS vs. Population-Based EA on a Family of Dynamic Fitness Functions

Friedrich, Kötzing, Krejca, Sutton, GECCO 2015

Robustness of Ant Colony Optimization to Noise

Overview

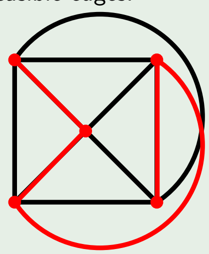
Construction Based on Broder's Algorithm

Based on Neumann and Witt (2010).

Problem: Minimum Spanning Trees (tree of minimum weight spanning all nodes)

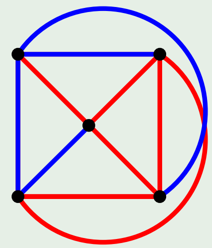
Broder-based Construction

Ants construct spanning tree by random walk (Broder, 1989).
Skip infeasible edges.



Component-based Construction

Add edges in arbitrary order based on attractiveness.
Exclude those closing a cycle.



Algorithm

1-ANT: (following Neumann/Witt, 2010)

- two pheromone values
- value h: if edge has been rewarded
- value ℓ: otherwise
- heuristic information η, $\eta(e) = \frac{1}{w(e)}$ (used before for TSP)
- Let v_k the current vertex and N_{v_k} be its neighborhood.
- Prob(to choose neighbor y of v_k) $= \frac{[\tau_{(v_k,y)}]^\alpha \cdot [\eta_{(v_k,y)}]^\beta}{\sum_{y \in N(v_k)} [\tau_{(v_k,y)}]^\alpha \cdot [\eta_{(v_k,y)}]^\beta}$ with $\alpha, \beta \geq 0$.
- Consider special cases where either $\beta = 0$ or $\alpha = 0$.

Results for Pheromone Updates

Case $\alpha = 1$, $\beta = 0$: proportional influence of pheromone values

Theorem (Broder-based construction graph)

Choosing $h/\ell = n^3$, the expected time until the 1-ANT with the Broder-based construction graph has found an MST is $O(n^6(\log n + \log w_{max}))$.

Theorem (Component-based construction graph)

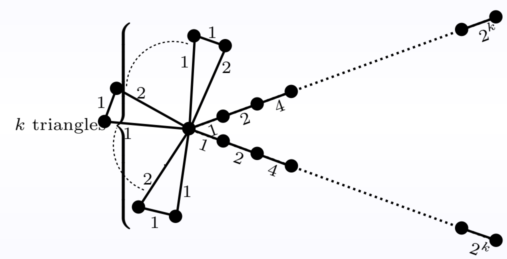
Choosing $h/\ell = (m - n + 1) \log n$, the expected time until the 1-ANT with the component-based construction graph has found an MST is $O(mn(\log n + \log w_{max}))$.

Better than (1+1) EA!

Broder Construction Graph: Heuristic Information

Example graph G^* with $n = 4k + 1$ vertices.

- k triangles of weight profile $(1, 1, 2)$
- two paths of length k with exponentially increasing weights.



Theorem (Broder-based construction graph)

Let $\alpha = 0$ and β be arbitrary, then the probability that the 1-ANT using the Broder construction procedure does not find an MST in polynomial time is $1 - 2^{-\Omega(n)}$.

Component-based Construction Graph/Heuristic Information

Theorem (Component-based construction graph)

Choosing $\alpha = 0$ and $\beta \geq 6w_{\max} \log n$, the expected time of the 1-ANT with the component-based construction graph to find an MST is constant.

Proof Idea

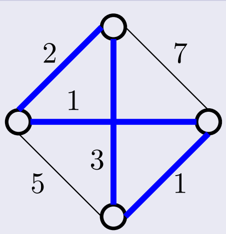
- Choose edges as Kruskal's algorithm.
- Calculation shows: probability of choosing a lightest edge is at least $1 - 1/n$.
- $n - 1$ steps $\Longrightarrow$ probability for an MST is $\Omega(1)$.

Overview

Traveling Salesman Problem

Based on Kötzing, Neumann, Röglin and Witt (2010).

Traveling Salesman Problem (TSP)



- Input: weighted complete graph $G = (V, E, w)$ with $w : E \to \mathbb{R}$.
- Goal: Find Hamiltonian cycle of minimum weight.

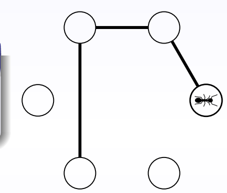
MMAS for TSP (Kötzing, Neumann, Röglin, Witt 2010)

Best-so-far pheromone update with $\tau_{\min} := 1/n^2$ and $\tau_{\max} := 1 - 1/n$.

Initialization: same pheromone on all edges.

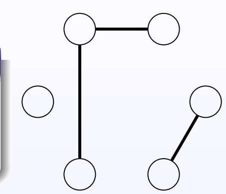
"Ordered" tour construction

Append a feasible edge chosen with probability proportional to pheromones.



"Arbitrary" tour construction

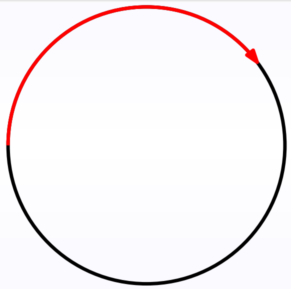
Add an edge chosen with probability proportional to pheromones as long as no cycle is closed or a vertex gets degree at least 3.



Locality

Lemma

MMAS^*_{Ord} with saturated pheromones exchanges $\Omega(\log(n))$ edges in expectation.



Length of unseen part roughly halves each time.

Lemma

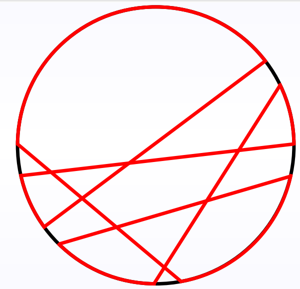
For any constant k: MMAS^*_{Arb} with saturated pheromones creates exactly k new edges with probability $\Theta(1)$.

TSP

Locality

Lemma

MMAS^*_{Ord} with saturated pheromones exchanges $\Omega(\log(n))$ edges in expectation.



Length of unseen part roughly halves each time.

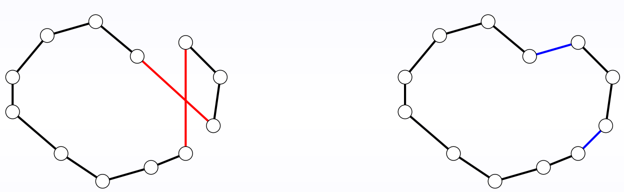
Lemma

For any constant k: MMAS^*_{Arb} with saturated pheromones creates exactly k new edges with probability $\Theta(1)$.

Dirk Sudholt (University of Sheffield) Theory of Swarm Intelligence 44 / 74

ACO Simulating 2-OPT

Zhou (2009): ACO can simulate 2-OPT.



Probability of particular 2-Opt step (for constant ρ):

MMAS^*_{Ord}: $\Theta(1/n^3)$

MMAS^*_{Arb}: $\Theta(1/n^2)$

Average Case Analysis

Assume that n points placed independently, uniformly at random in the unit hypercube $[0,1]^d$.

Theorem [Englert, Röglin, Vöcking 2007]

2-Opt finds after $O(n^{4+1/3} \cdot \log n)$ iterations with probability $1-o(1)$ a solution with approximation ratio $O(1)$.

Theorem

For $\rho = 1$, MMAS^*_{Arb} finds after $O(n^{6+2/3})$ iterations with probability $1-o(1)$ a solution with approximation ratio $O(1)$.

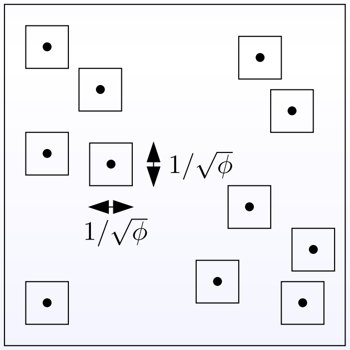
Theorem

For $\rho = 1$, MMAS^*_{Ord} finds after $O(n^{7+2/3})$ iterations with probability $1-o(1)$ a solution with approximation ratio $O(1)$.

Smoothed Analysis

Smoothed Analysis

Each point $i \in \{1, \ldots, n\}$ is chosen independently according to a probability density $f_i : [0,1]^d \to [0, \phi]$.



2-Opt:
$O(\sqrt[d]{\phi})$-approximation in
$O(n^{4+1/3} \cdot \log(n\phi) \cdot \phi^{8/3})$ steps

MMAS^*_{Ord}: $O(\sqrt[d]{\phi})$-approximation
in $O(n^{7+2/3} \cdot \phi^3)$ steps

MMAS^*_{Arb}: $O(\sqrt[d]{\phi})$-approximation
in $O(n^{6+2/3} \cdot \phi^3)$ steps

ACO: Summary and Open Questions

Shortest Paths

Natural and interesting test-bed for the robustness of ACO algorithms.

- global pheromone updates?
- how to deal with noise and dynamic changes?
- where does slow pheromone adaptation help?
- average-case analyses with heuristic information

Strength of ACO

Problem-specific construction procedures can make ACO more powerful.

- how to find a fruitful combination of metaheuristic search and problem-specific components?

Main Challenge in Analysis of ACO

Understand dynamics of pheromones within borders.

- results for MST and TSP with more natural pheromone models

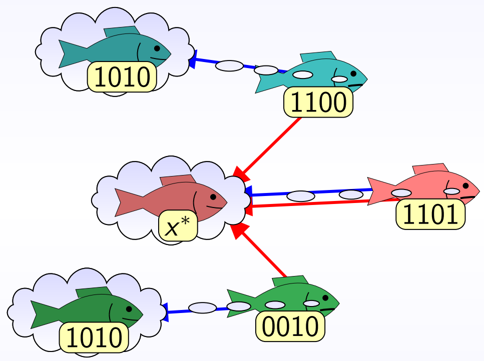
Overview

Particle Swarm Optimization

Particle Swarm Optimization

- Bio-inspired optimization principle developed by Kennedy and Eberhart (1995).
- Mostly applied in continuous spaces.
- Swarm of particles, each moving with its own velocity.
- Velocity is updated according to
  - own best position and
  - position of the best individual in its neighborhood (here: swarm).

Particle Swarm Optimization

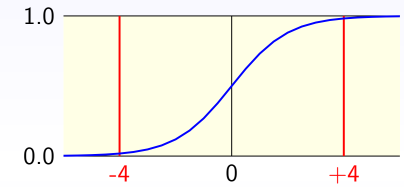


Binary PSO (Kennedy und Eberhart, 1997)

Creating New Positions

Probabilistic construction using velocity v and sigmoid function $s(v)$:

$$\text{Prob}(x_j = 1) = s(v_j) = \frac{1}{1+e^{-v_j}}$$



Restrict velocities to $v_j \in [-v_{\max}, +v_{\max}]$.

- Common practice: $v_{\max} = 4$ (good for $n \in [50, 500]$)
- Sudholt and Witt (2010): $v_{\max} := \ln(n-1)$ (good across all n):

$$\frac{1}{n} \leq \text{Prob}(x_j = 1) \leq 1 - \frac{1}{n}.$$

Updating Velocities

Update current velocity vector according to

- cognitive component $\rightarrow$ towards own best: $x^{*(i)} - x^{(i)}$ and
- social component $\rightarrow$ towards global best: $x^* - x^{(i)}$.

Learning rates c_1, c_2 affect weights for the two components.

Random scalars $r_1 \in U[0, c_1]$, $r_2 \in U[0, c_2]$ chosen anew in each generation:

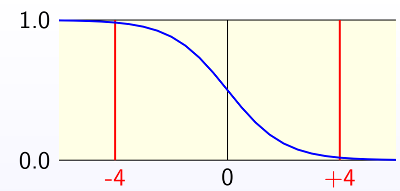
$$v^{(i)} = v^{(i)} + r_1(x^{*(i)} - x^{(i)}) + r_2(x^* - x^{(i)})$$

Understanding Velocities

Assume bit i is 1 in global best and own best. Create x.

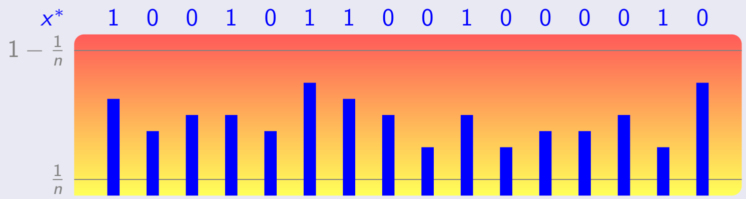
- **ACO**: reinforce bit value 1 in probabilistic model if $x_i = 1$
- **PSO**: reinforce bit value 1 in probabilistic model if $x_i = 0$

Probability of increasing v_i is $1 - s(v_i)$:



Velocity Freezing

Particle with best-so-far solution: own best = global best

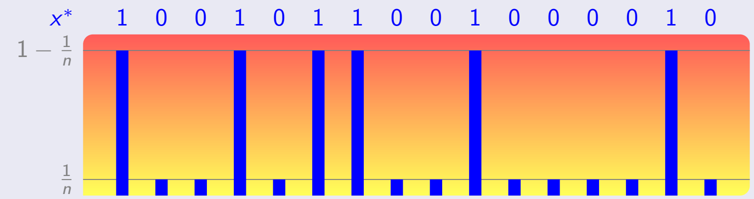


Lemma

Expected freezing time to $v_{\max}$ or $-v_{\max}$ is $O(n)$ for single bits and $O(n \log n)$ for n or μn bits if $\mu = poly(n)$.

Velocity Freezing

Particle with best-so-far solution: own best = global best



Lemma

Expected freezing time to $v_{\max}$ or $-v_{\max}$ is $O(n)$ for single bits and $O(n \log n)$ for n or μn bits if $\mu = poly(n)$.

Fitness-Level Method for Binary PSO

Upper bound for the (1+1) EA

$$\sum_{i=0}^{m-1} \frac{1}{s_i}$$

Upper bound for #generations of Binary PSO

$$\sum_{i=0}^{m-1} \frac{1}{s_i} + O(m \cdot n \log n)$$

Upper bound for #generations of "social" Binary PSO, i. e., $c_1 := 0$

$$O\left(\frac{1}{\mu} \sum_{i=0}^{m-1} \frac{1}{s_i} + m \cdot n \log n\right)$$

1-PSO vs. (1+1) EA on OneMax

More detailed analysis: average adaptation time of $384 \ln n$ is sufficient.

Theorem (Sudholt and Witt, 2010)

The expected optimization time of the 1-PSO *on* OneMax *is $O(n \log n)$.*

Experiments: 1-PSO 15% slower than (1+1) EA on OneMax.

Overview

Continuous PSO

Search space: (bounded subspace of) $\mathbb{R}^n$.

Objective function: $f\colon \mathbb{R}^n \to \mathbb{R}$.

Particles represent positions $x^{(i)}$ in this space.

Particles fly at certain velocity: $x^{(i)} := x^{(i)} + v^{(i)}$.

Velocity update with inertia weight ω:

$$v^{(i)} = \omega v^{(i)} + r_1(x^{*(i)} - x^{(i)}) + r_2(x^* - x^{(i)})$$

Convergence of PSO

Swarm can collapse to points or other low-dimensional subspaces.

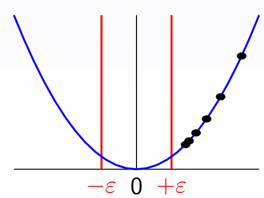
Convergence results for standard PSO, $\omega < 1$ (Jiang, Luo, and Yang, 2007)

PSO converges ... somewhere.

Stagnation of Standard PSO

Lehre and Witt, 2013

Standard PSO with one/two particles stagnates even on one-dimensional Sphere!



Expected first hitting time of ε-ball around optimum is infinite.

Noisy PSO (Lehre and Witt, 2013)

Adding noise $U[-\delta/2, \delta/2]$ for $\delta \le \varepsilon$: finite expected hitting time on (half-)Sphere.

Similar result for n dimensions (Schmitt and Wanka, GECCO 2013)

PSO modification: pick random velocities when swarm converges (all velocities plus distance to global best $\le \delta$). Converges to local optima almost surely.

PSO Extensions

Extensions of standard PSO

- Bare-bones PSO (Kennedy, 2003)
- PSO with mutation (several variants)
- PSO using gradient information (several variants)
- Guaranteed Convergence PSO (GCPSO) (van den Bergh and Engelbrecht, 2002)
  - Make a cube mutation of a particle's position by adding $p \in U[-\ell, \ell]^n$.
  - Adapt "step size" ℓ in the course of the run by doubling or halving it, depending on the number of successes.

  $\longrightarrow$ 1/5-rule known from evolution strategies!

GCPSO with 1 Particle (Witt, 2009)

GCPSO with one particle is basically a (1+1) ES with cube mutation.

Can be analyzed like classical (1+1) ES (Jägersküpper, 2007)

$$\textsc{Sphere}(x) := ||x|| = x_1^2 + x_2^2 + \cdots + x_n^2$$

Theorem (Witt, 2009)

Consider the GCPSO$_1$ on $\textsc{Sphere}$. *If* $\ell = \Theta(||x^*||/n)$ *for the initial solution* x^*, *the runtime until the distance to the optimum is no more than* $\varepsilon||x^*||$ *is* $O(n\log(1/\varepsilon))$ *with probability at least* $1 - 2^{-\Omega(n)}$ *provided that* $2^{-n^{O(1)}} \leq \varepsilon \leq 1$.

Same result as for (1+1) ES using Gaussian mutations in Jägersküpper, 2007.

PSO: Summary and Open Questions

Summary

- analysis of Binary PSO and its probabilistic model
- initial results on runtime of GCPSO and convergence of modified PSO
- results on expected first hitting time of ε-ball for Standard PSO & Noisy PSO

Neighborhood topologies

- ring topology, etc. instead of global best of swarm
- where does a restricted topology help?

Swarm dynamics

- analyze combined impact of cognitive and social components
- more results on swarms in continuous spaces

Overview

Conclusions

Summary

- Insight into probabilistic models underlying ACO and PSO
- How design choices and parameters affect (bounds on) running times
- How simple ACO algorithms optimize unimodal functions and plateaus
- Results for ACO in combinatorial optimization
- First analyses of basic PSO algorithms in discrete and continuous spaces

Future Work

- A unified theory of randomized search heuristics?
- More results on multimodal problems
- When and how diversity and slow adaptation help

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Thank you!

Questions?