Particle Swarm Optimization

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Instructor

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Presentation Outline

Introduction
The Origins

Particle swarm optimization (PSO):
- developed by Kennedy & Eberhart [11, 22],
- first published in 1995, and
- with an exponential increase in the number of publications since then.

What is PSO?
- a simple, computationally efficient optimization method
- population-based, stochastic search
- individuals follow very simple behaviors:
  - emulate the success of neighboring individuals,
  - but also bias towards own experience of success
- emergent behavior: discovery of optimal regions within a high dimensional search space

What are the origins of PSO?

- In the work of Reynolds on “boids” [36]
  - collision avoidance
  - velocity matching
  - flock centering
- The work of Heppner and Grenander on using a “rooster” as attractor of all birds in the flock [18]

- Original PSO is a simplified social model of determining nearest neighbors and velocity matching
- Initial objective: to simulate the graceful, unpredictable choreography of collision-proof birds in a flock
  - Randomly initializes positions of birds
  - At each iteration, each individual determines its nearest neighbor and replaces its velocity with that of its neighbor
- This resulted in synchronous movement of the flock, but flock settled too quickly on an unanimous, unchanging flying direction

- Random adjustments to velocities (referred to as craziness) prevented individuals to settle too quickly on an unchanging direction
- To further expand the model, roosters were added as attractors:
  - personal best
  - neighborhood best

  → particle swarm optimization

Overview of Basic PSO

Main Components

What are the main components?

- a swarm of particles
- each particle represents a candidate solution
- elements of a particle represent parameters to be optimized

The search process:

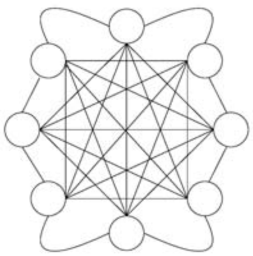
- Position updates

$$\mathbf{x}_i(t+1) = \mathbf{x}_i(t) + \mathbf{v}_i(t+1), \quad \mathbf{x}_{ij}(0) \sim U(x_{min,j}, x_{max,j})$$

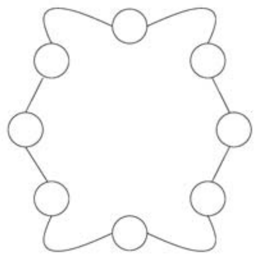
- Velocity (step size)
  - drives the optimization process
  - step size
  - reflects experiential knowledge and socially exchanged information

Overview of Basic PSO
Social Network Structures

Social network structures are used to determine best positions/attractors



: Star Topology

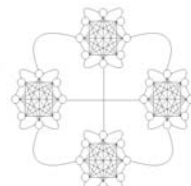


: Ring Topology

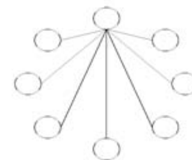
Overview of Basic PSO
Social Network Structures (cont)

(a) Von Neumann

(b) Pyramid



(c) 4 Clusters



(d) Wheel

Overview of Basic PSO
global best (gbest) PSO

- uses the star social network
- velocity update per dimension:

$$v_{ij}(t+1) = v_{ij}(t) + c_1 r_{1j}(t)[y_{ij}(t) - x_{ij}(t)] + c_2 r_{2j}(t)[\hat{y}_j(t) - x_{ij}(t)]$$

- $v_{ij}(0) = 0$ (preferred)
- c_1, c_2 are positive acceleration coefficients
- $r_{1j}(t), r_{2j}(t) \sim U(0,1)$
- note that a random number is sampled for each dimension

Overview of Basic PSO
gbest PSO (cont)

- $\mathbf{y}_i(t)$ is the personal best position calculated as (assuming minimization)

$$\mathbf{y}_i(t+1) = \begin{cases} \mathbf{y}_i(t) & \text{if } f(\mathbf{x}_i(t+1)) \geq f(\mathbf{y}_i(t)) \\ \mathbf{x}_i(t+1) & \text{if } f(\mathbf{x}_i(t+1)) < f(\mathbf{y}_i(t)) \end{cases}$$

- $\hat{\mathbf{y}}(t)$ is the global best position calculated as

$$\hat{\mathbf{y}}(t) \in \{\mathbf{y}_0(t), \ldots, \mathbf{y}_{n_s}(t)\} | f(\hat{\mathbf{y}}(t)) = \min\{f(\mathbf{y}_0(t)), \ldots, f(\mathbf{y}_{n_s}(t))\}$$

or (removing memory of best positions)

$$\hat{\mathbf{y}}(t) = \min\{f(\mathbf{x}_0(t)), \ldots, f(\mathbf{x}_{n_s}(t))\}$$

where n_s is the number of particles in the swarm

Overview of Basic PSO
gbest PSO Algorithm

Create and initialize an n_x-dimensional swarm, S;
repeat
    for *each particle* $i = 1, \ldots, S.n_s$ **do**
        if $f(S.\mathbf{x}_i) < f(S.\mathbf{y}_i)$ **then**
            $S.\mathbf{y}_i = S.\mathbf{x}_i$;
        end
        if $f(S.\mathbf{y}_i) < f(S.\hat{\mathbf{y}})$ **then**
            $S.\hat{\mathbf{y}} = S.\mathbf{y}_i$;
        end
    end
    for *each particle* $i = 1, \ldots, S.n_s$ **do**
        update the velocity;
        update the position;
    end
until *stopping condition is true*;

Overview of Basic PSO
local best (lbest) PSO

- uses the ring social network

$$v_{ij}(t+1) = v_{ij}(t) + c_1 r_{1j}(t)[y_{ij}(t) - x_{ij}(t)] + c_2 r_{2j}(t)[\hat{y}_{ij}(t) - x_{ij}(t)]$$

- $\hat{\mathbf{y}}_i$ is the neighborhood best, defined as

$$\hat{\mathbf{y}}_i(t+1) \in \{\mathcal{N}_i | f(\hat{\mathbf{y}}_i(t+1)) = \min\{f(\mathbf{x})\}, \ \forall \mathbf{x} \in \mathcal{N}_i\}$$

  with the neighborhood defined as

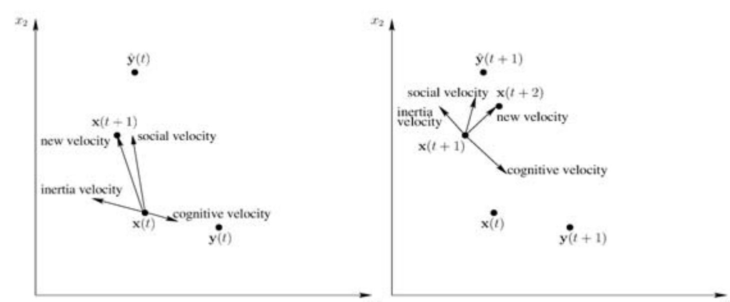
$$\mathcal{N}_i = \{\mathbf{y}_{i-n_{\mathcal{N}_i}}(t), \mathbf{y}_{i-n_{\mathcal{N}_i}+1}(t), \ldots, \mathbf{y}_{i-1}(t), \mathbf{y}_i(t), \mathbf{y}_{i+1}(t), \ldots, \mathbf{y}_{i+n_{\mathcal{N}_i}}(t)\}$$

  where $n_{\mathcal{N}_i}$ is the neighborhood size
- neighborhoods based on particle indices, not spatial information
- neighborhoods overlap to facilitate information exchange

Aspects of Basic PSO
Velocity Components

- previous velocity, $\mathbf{v}_i(t)$
  - inertia component
  - memory of previous flight direction
  - prevents particle from drastically changing direction
- cognitive component, $c_1\mathbf{r}_1(\mathbf{y}_i - \mathbf{x}_i)$
  - quantifies performance relative to past performances
  - memory of previous best position
  - nostalgia
- social component, $c_2\mathbf{r}_2(\hat{\mathbf{y}}_i - \mathbf{x}_i)$
  - quantifies performance relative to neighbors
  - envy

Aspects of Basic PSO
Geometric Illustration



(a) Time Step t (b) Time Step $t+1$

Aspects of Basic PSO
Velocity Clamping

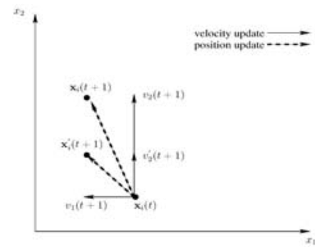
- the problem: velocity quickly explodes to large values
- solution:

$$v_{ij}(t+1) = \begin{cases} v_{ij}(t+1) & \text{if } |v_{ij}(t+1)| < V_{max,j} \\ \text{sgn}(v_{ij}) V_{max,j} & \text{if } |v_{ij}(t+1)| \geq V_{max,j} \end{cases}$$

- controlling the global exploration of particles
- does not confine the positions, only the step sizes

Aspects of Basic PSO
Velocity Clamping (cont)

- Issues with velocity clamping:
  - dimensions with ranges smaller than V_{max} will never be clamped
  - changes search direction – normalized clamping



: Change in Search Direction Due to Velocity Clamping

Aspects of Basic PSO
Velocity Clamping (cont)

- problem-dependent
  - dynamically changing V_{max} when *gbest* does not improve over τ iterations [38]

$$V_{max,j}(t+1) = \begin{cases} \beta V_{max,j}(t) & \text{if } f(\hat{\mathbf{y}}(t)) \geq f(\hat{\mathbf{y}}(t-t')),\ \forall\, t' = 1, \ldots, \tau \\ V_{max,j}(t) & \text{otherwise} \end{cases}$$

  β decreases from 1.0 to 0.01
  - exponentially decaying V_{max} [16]

$$V_{max,j}(t+1) = (1 - (t/n_t)^{\alpha}) V_{max,j}(t)$$

Aspects of Basic PSO
Inertia Weight

- to control exploration and exploitation
- controls the momentum
- velocity update changes to

$$v_{ij}(t+1) = wv_{ij}(t) + c_1 r_{1j}(t)[y_{ij}(t) - x_{ij}(t)] + c_2 r_{2j}(t)[\hat{y}_j(t) - x_{ij}(t)]$$

- for $w \geq 1$
  - velocities increase over time
  - swarm diverges
  - particles fail to change direction towards more promising regions
- for $0 < w < 1$
  - particles decelerate, depending on c_1 and c_2
- exploration–exploitation
  - large values – favor exploration
  - small values – promote exploitation
- problem-dependent

Aspects of Basic PSO
Inertia Weight (cont)

Dynamically changing inertia weights

- $w \sim N(0.72, \sigma)$
- linear decreasing [39]

$$w(t) = (w(0) - w(n_t))\frac{(n_t - t)}{n_t} + w(n_t)$$

- non-linear decreasing [44]

$$w(t+1) = \alpha w(t'), \quad w(0) = 1.4$$

- based on relative improvement [6]

$$w_i(t+1) = w(0) + (w(n_t) - w(0))\frac{e^{m_i(t)} - 1}{e^{m_i(t)} + 1}$$

where the relative improvement, m_i, is estimated as

$$m_i(t) = \frac{f(\hat{\mathbf{y}}_i(t)) - f(\mathbf{x}_i(t))}{f(\hat{\mathbf{y}}_i(t)) + f(\mathbf{x}_i(t))}$$

Aspects of Basic PSO
Constriction Coefficient

- to ensure convergence to a stable point without the need for velocity clamping

$$\begin{aligned} v_{ij}(t+1) &= \chi[v_{ij}(t) + \phi_1(y_{ij}(t) - x_{ij}(t)) \\ &\quad + \phi_2(\hat{y}_j(t) - x_{ij}(t))] \end{aligned}$$

where

$$\chi = \frac{2\kappa}{|2 - \phi - \sqrt{\phi(\phi - 4)}|}$$

with

$$\begin{aligned} \phi &= \phi_1 + \phi_2 \\ \phi_1 &= c_1 r_1 \\ \phi_2 &= c_2 r_2 \end{aligned}$$

Aspects of Basic PSO
Constriction Coefficient (cont)

- if $\phi \geq 4$ and $\kappa \in [0, 1]$, then the swarm is guaranteed to converge
- $\chi \in [0, 1]$
- κ controls exploration–exploitation
  $\kappa \approx 0$: fast convergence, exploitation
  $\kappa \approx 1$: slow convergence, exploration
- effectively equivalent to inertia weight for specific χ:
  $w = \chi, \phi_1 = \chi c_1 r_1$ and $\phi_2 = \chi c_2 r_2$

Aspects of Basic PSO
Iteration Strategies

- Synchronous interation strategy
  - personal best and neighborhood bests updated separately from position and velocity vectors
  - slower feedback of new best positions
- Asynchronous iteration strategy
  - new best positions updated after each particle position update
  - immediate feedback of new best positions
  - lends itself well to parallel implementation

Aspects of Basic PSO
Iteration Strategies (cont)

Synchronous Iteration Strategy

Create and initialize the swarm;
repeat
    for *each particle* **do**
        Evaluate particle's fitness;
        Update particle's personal best position;
        Update particle's neighborhood best position;
    end
    for *each particle* **do**
        Update particle's velocity;
        Update particle's position;
    end
until *stopping condition is true*;

Asynchronous Iteration Strategy

Create and initialize the swarm;
repeat
    for *each particle* **do**
        Update the particle's velocity;
        Update the particle's position;
        Evaluate particle's fitness;
        Update the particle's personal best position;
        Update the particle's neighborhood best position;
    end
until *stopping condition is true*;

Aspects of Basic PSO
Acceleration Coefficients

- $c_1 = c_2 = 0$?
- $c_1 > 0, c_2 = 0$:
  - particles are independent hill-climbers
  - local search by each particle
  - cognitive-only PSO
- $c_1 = 0, c_2 > 0$:
  - swarm is one stochastic hill-climber
  - social-only PSO
- $c_1 = c_2 > 0$:
  - particles are attracted towards the average of $\mathbf{y}_i$ and $\hat{\mathbf{y}}_i$
- $c_2 > c_1$:
  - more beneficial for unimodal problems
- $c_1 < c_2$:
  - more beneficial for multimodal problems

Aspects of Basic PSO
Acceleration Coefficients (cont)

- low c_1 and c_2:
  - smooth particle trajectories
- high c_1 and c_2:
  - more acceleration, abrupt movements
- problem dependent
  - adaptive acceleration coefficients [35]

$$c_1(t) = (c_{1,min} - c_{1,max})\frac{t}{n_t} + c_{1,max}$$

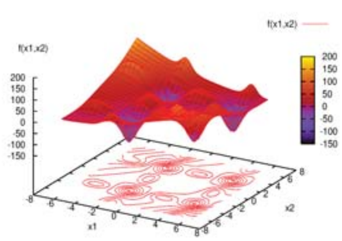
$$c_2(t) = (c_{2,max} - c_{2,min})\frac{t}{n_t} + c_{2,min}$$

Some Basic Applications of PSO
Function Optimization

Minimize the 2-D Bird function

$$f(\mathbf{x}) = \sin(x_1)e^{(1-cos(x_2))^2} + \cos(x_2)e^{(1-sin(x_1))^2} + (x_1 - x_2)^2$$

with $x_j \in [-2\pi, 2\pi]$



Some Basic Applications of PSO
Training A Feedforward Neural Network

- Objective is to find weight and bias values that minimizes an error function, e.g. sum-squared error
- Representation: particle represents weight vector and biases
- Fitness function: Sean-squared error, classification error
- Initialization:
  - Small initial weights to prevent velocity from growing too fast
  - Zero initial velocity, to start with as small as possible step sizes
  - Small V_{max} to prevent too fast growth in velocity

Some Basic Applications of PSO
Data Clustering

- Objective is to find centroids such that intra-cluster distances are minimized and inter-cluster distances maximized
- Representation of centroid vectors:

$$\mathbf{x}_i = (\mathbf{m}_{i1}, \ldots, \mathbf{m}_{ik}, \ldots, \mathbf{m}_{iK})$$

- Fitness function: Quantization error

$$J_{e,i} = \frac{\sum_{k=1}^{K}[\sum_{\forall \mathbf{z}_p \in C_{ki}} \mathcal{E}(\mathbf{z}_p, \mathbf{m}_{ki})]/n_{ki}}{K}$$

PSO Issues
About Convergence

Particles are guaranteed under certain conditions to converge to an equilibrium [8, 40, 9]:

- Particles will converge to

$$\frac{\phi_1 y + \phi_2 \hat{y}}{\phi_1 + \phi_2}$$

- This is not necessarily even a local minimum
- It has been proven that standard PSO is not a local minimizer [10]

Potential dangerous property:

- when $\mathbf{x}_i = \mathbf{y}_i = \hat{\mathbf{y}}_i$
- then the velocity update depends only on $w\mathbf{v}_i$
- if this condition persists for a number of iterations,

$$w\mathbf{v}_i \to 0$$

PSO Issues
Roaming Particles

- Empirical analysis [15] and theoretical proofs [17] showed that particles leave search boundaries very early during the optimization process
- Potential problems:
  - **Infeasible solutions**: Should better positions be found outside of boundaries, and no boundary constraint method employed, personal best and neighborhood best positions are pulled outside of search boundaries
  - **Wasted search effort**: Should better positions not exist outside of boundaries, particles are eventually pulled back into feasible space.
  - **Incorrect swarm diversity calculations**: As particles move outside of search boundaries, diversity increases

PSO Issues
Roaming Particles (cont)

Goal of this experiment: To illustrate

- particle roaming behavior, and
- infeasible solutions may be found

Experimental setup:

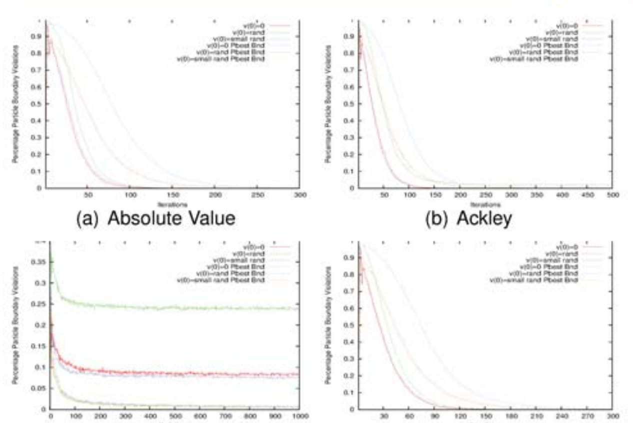
- A standard gbest PSO was used
- 30 particles
- $w = 0.729844$
- $c_1 = c_2 = 1.496180$
- Memory-based global best selection
- Synchronous position updates
- 50 independent runs for each initialization strategy

PSO Issues
Roaming Particles (cont)

: Functions Used for Empirical Analysis to Illustrate Roaming Behavior

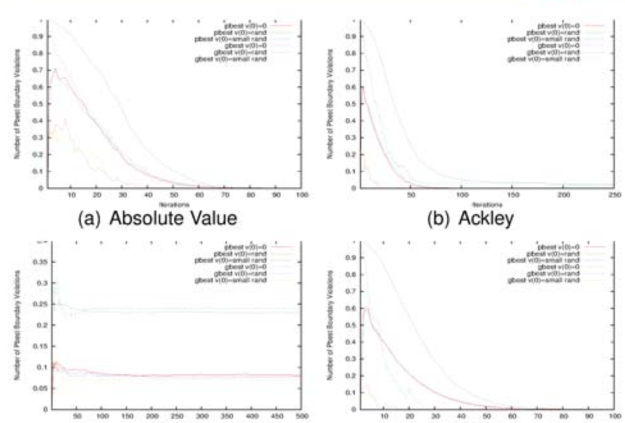
Function	Definition	Domain
AbsValue	$f(\mathbf{x}) = \sum_{j=1}^{n_x} \lvert x_j \rvert$	[-100,100]
Ackley	$f(\mathbf{x}) = -20e^{-0.2\sqrt{\frac{1}{n_x}\sum_{j=1}^{n_x} x_j^2}} - e^{\frac{1}{n_x}\sum_{j=1}^{n_x}\cos(2\pi x_j)} + 20 + e$	[-32.768,32.768]
Bukin 6	$f(\mathbf{x}) = 100\sqrt{\lvert x_2 - 0.01x_1^2 \rvert} + 0.01\lvert x_1 + 10 \rvert$	[-15,5],[-3,3]
Griewank	$f(\mathbf{x}) = 1 + \frac{1}{4000}\sum_{j=1}^{n_x} x_j^2 - \prod_{j=1}^{n_x}\cos\left(\frac{x_j}{\sqrt{j}}\right)$	[-600,600]
Quadric	$f(\mathbf{x}) = \sum_{l=1}^{n_x}\left(\sum_{j=1}^{l} x_j\right)^2$	[-100,100]
Rastrigin	$f(\mathbf{x}) = 10n_x + \sum_{j=1}^{n_x}\left(x_j^2 - 10\cos(2\pi x_j)\right)$	[-5.12,5.12]
Rosenbrock	$f(\mathbf{x}) = \sum_{j=1}^{n_x-1}\left(100(x_{j+1} - x_j^2)^2 + (x_j - 1)^2\right)$	[-2.048,2.048]

PSO Issues: Roaming
Percentage Particles that Violate Boundaries



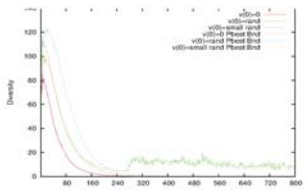
(a) Absolute Value (b) Ackley (c) Bukin 6 (d) Griewank

PSO Issues: Roaming
Percentage Best Position Boundary Violations

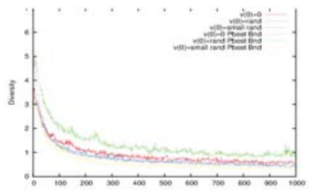


(a) Absolute Value (b) Ackley (c) Bukin 6 (d) Griewank

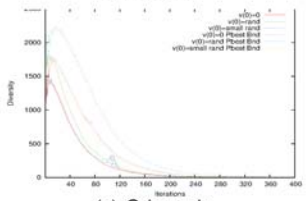
PSO Issues: Roaming
Diversity Profiles



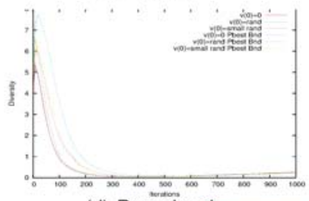
(a) Ackley



(b) Bukin 6



(c) Griewank



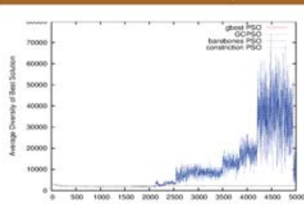
(d) Rosenbrock

PSO Issues: Roaming
Finding Infeasible Solutions

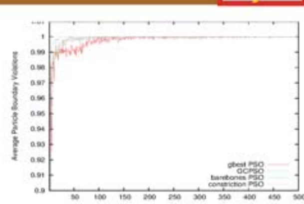
: Functions Used for Empirical Analysis to Illustrate Finding of Infeasible Solutions

Function	Domain	Function Definition
Ackley	[10,32.768]	$f(\mathbf{x}) = -20e^{-0.2\sqrt{\frac{1}{n}\sum_{j=1}^{n_x} x_j^2}} - e^{\frac{1}{n}\sum_{j=1}^{n_x}\cos(2\pi x_j)} + 20 + e$
Ackley^{SR}	[-32.768,0]	$f(\mathbf{x}) = -20e^{-0.2\sqrt{\frac{1}{n}\sum_{j=1}^{n_x} z_j^2}} - e^{\frac{1}{n}\sum_{j=1}^{n_x}\cos(2\pi z_j)} + 20 + e$
Eggholder	[-512,512]	$f(\mathbf{x}) = \sum_{j=1}^{n_x-1}\left(-(x_{j+1}+47)\sin(\sqrt{\lvert x_{j+1}+x_j/2+47\rvert}) + \sin(\sqrt{\lvert x_j-(x_{j+1}+47)\rvert})(-x_j)\right)$
Griewank^{SR}	[0,600]	$f(\mathbf{x}) = 1 + \frac{1}{4000}\sum_{j=1}^{n_x} z_j^2 - \prod_{j=1}^{n_x}\cos\left(\frac{z_j}{\sqrt{j}}\right)$
Norwegian^S	[-1.1,1.1]	$f(\mathbf{x}) = \prod_{j=1}^{n_x}\left(\cos(\pi z_j^3)\left(\frac{99+z_j}{100}\right)\right)$
Rosenbrock^S	[-30,30]	$f(\mathbf{x}) = \sum_{j=1}^{n_x-1}\left(100(z_{j+1}-z_j^2)^2 + (z_j-1)^2\right)$
Schwefel1.2^S	[0,100]	$f(\mathbf{x}) = \sum_{j=1}^{n_x}\left(\sum_{k=1}^{j} z_k\right)^2$
Schwefel2.26	[-50,50]	$f(\mathbf{x}) = -\sum_{j=1}^{n_x}\left(x_j\sin\left(\sqrt{\lvert x_j\rvert}\right)\right)$
Spherical^S	[0,100]	$f(\mathbf{x}) = \sum_{j=1}^{n_x} z_j^2$
Salomon	[-100,5]	$f_{14}(\mathbf{x}) = -\cos(2\pi\sum_{j=1}^{n_x} x_j^2) + 0.1\sqrt{\sum_{j=1}^{n_x} x_j^2} + 1$

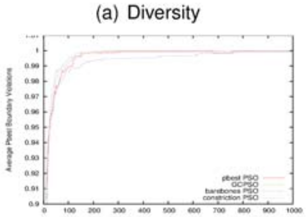
PSO Issues: Roaming
Finding Infeasible Solutions: Ackley



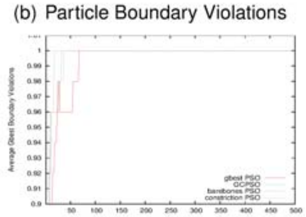
(a) Diversity



(b) Particle Boundary Violations

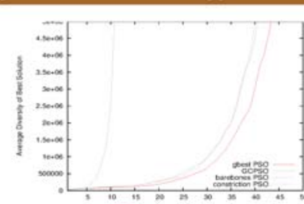


(c) Personal Best Boundary Viola-

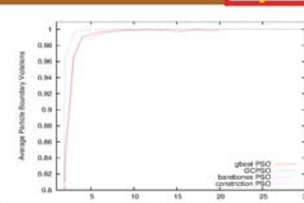


(d) Global Best Boundary Violations

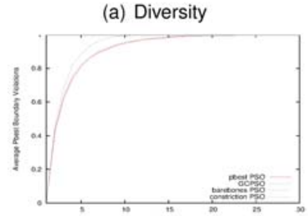
PSO Issues: Roaming
Finding Infeasible Solutions: Eggholder



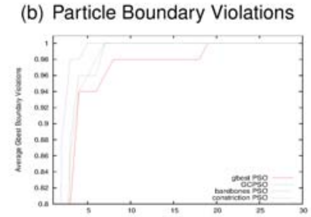
(a) Diversity



(b) Particle Boundary Violations



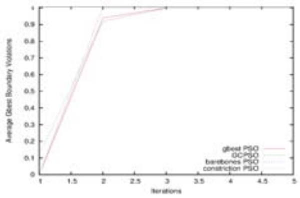
(c) Personal Best Boundary Viola-



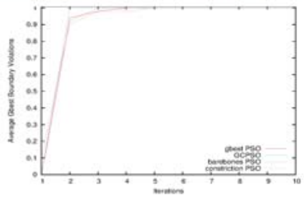
(d) Global Best Boundary Violations

PSO Issues: Roaming

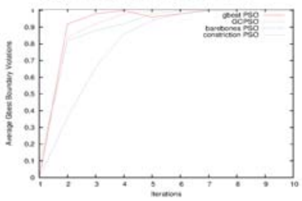
Finding Infeasible Solutions: gbest Boundary Violations



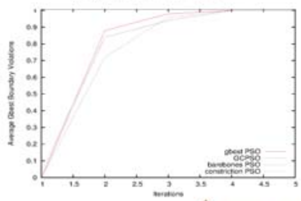
(a) Shifted Schwefel 1.2



(b) Schwefel 2.26



(c) Salomon



(d) Shifted Spherical

PSO Issues

Velocity Initialization

Velocties have been initialized using any of the following:

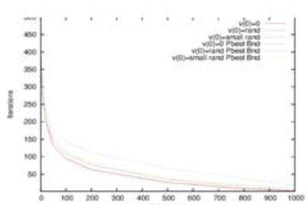
- $\mathbf{v}_i(0) = \mathbf{0}$
  - Critique: Limits exploration ability, therefore extent to which the search space is initially covered
  - Counter argument: Initial positions are uniformly distributed
  - Flocking analogy: Physical objects, in their initial state, do not have any momentum
- $\mathbf{v}_i(0) \sim U(-x_{min}, x_{max})^{n_x}$, where n_x is the problem dimension
  - Argument in favor: Initial random velocities help to improve exploration abilities of the swarm, therefore believed to obtain better solutions, faster
  - Argument against: large initial step sizes cause particles to leave search boundaries:

$$\mathbf{v}_i(0) \sim U(-x_{min}, x_{max})^{n_x} \longrightarrow \mathbf{x}_i(1) \sim U(-2x_{min}, 2x_{max})^{n_x}$$

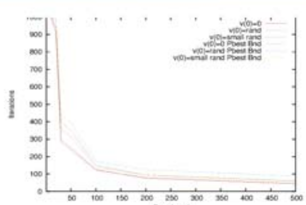
- Initialize to small random values

PSO Issues: Velocity Initialization

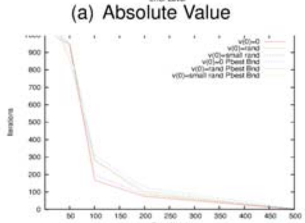
Fitness Reduction Profiles



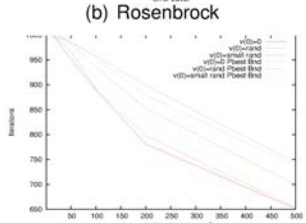
(a) Absolute Value



(b) Rosenbrock



(c) Rastrigin



(d) Quadric

PSO Issues: Velocity Initialization

Fitness After 1000 Iterations

Function	Zero Init No Pbest Bound	Random Init No Pbest Bound
Absolute Value	3.53E-001±2.87E+000	2.46E-001±1.47E+000
Ackley	2.49E+000±1.35E+000	2.68E+000±2.67E+000
Bukin 6	6.20E-002±4.50E-002	6.65E-002±5.56E-002
Griewank	3.72E-002±5.26E-002	3.91E-002±5.57E-002
Quadric	9.04E+001±8.70E+001	1.80E+002±3.15E+002
Rastrigin	6.66E+001±1.71E+001	7.37E+001±2.16E+001
Rosenbrock	2.65E+001±1.53E+001	2.73E+001±1.66E+001

PSO Issues: Velocity Initialization
Observations

The following general observations are made:

- Small random initialization and zero initialization have similar behaviors
- Random initialization
  - slower in improving fitness of best solution
  - resulted in larger diversity
  - had more roaming particles, roaming for longer
  - significantly more best positions left boundaries
  - took longer to reduce number of particle and best position violations
  - very slow in increasing number of converged dimensions

PSO Issues
gbest PSO versus lbestPSO

Current opinions about gbest PSO:

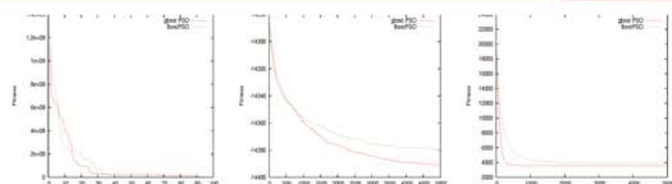
- Gbest PSO should not be used due to premature convergence to local optima as observed for a number of optimization problems [19, 24, 27, 32, 37]
- Gbest PSO converges fast due to the faster transfer of the best position throughout the swarm, and therefore the strong attraction to one best position [2, 13, 14, 19, 24, 25, 27, 28, 29]
- Gbest PSO is more susceptible to being trapped in local minima than lbest PSO [13, 14, 25].
- Gbest PSO is best suited to unimodal problems and should not be used for multimodal problems [2, 7, 21, 24, 32]
- Gbest PSO does not perform well for non-separable problems [25].

PSO Issues
gbest PSO versus lbestPSO (cont)

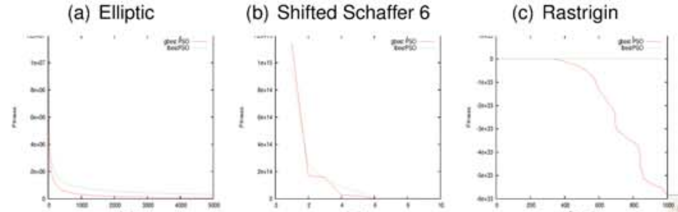
: Outcomes of Statistical Analysis Comparing Gbest with Lbest PSO

Function Class		Number of Functions	Accuracy			Success Rate			Efficiency			Diversity		
			>	=	<	>	=	<	>	=	<	>	=	<
UM	Seperable	7	5	0	2	6	0	1	2	0	5	5	0	2
	Non-separable	3	2	1	0	2	1	0	2	1	0	2	0	1
	Noisy	2	1	0	1	1	1	0	2	0	0	1	0	1
	Shifted	5	2	3	0	2	3	0	2	3	0	1	0	4
	Rotated	1	1	0	0	1	0	0	0	1	0	0	0	1
MM	Seperable	6	1	2	3	2	2	2	3	1	2	6	0	0
	Non-seperable	9	4	1	4	3	4	2	4	3	2	1	0	8
	Shifted	10	3	4	3	5	5	0	8	1	1	1	0	9
	Rotated	4	0	3	1	1	2	1	2	1	1	0	0	4
	Noisy	1	0	1	0	0	1	0	0	1	0	0	0	1
	Composition	11	1	2	8	0	4	7	1	5	5	0	0	11
Overall Total		59	20	17	22	23	23	13	26	17	16	11	0	48
Overall Unimodal		18	11	4	3	12	5	1	8	5	5	9	0	9
Overall Multimodal		41	13	19	14	11	18	12	18	12	11	2	0	39
Overall Seperable		17	7	4	6	9	5	3	12	1	4	5	0	12
Overall Non-seperable		42	13	13	16	14	18	10	11	16	9	6	0	36

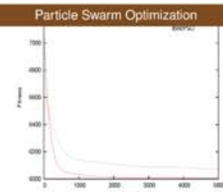
PSO Issues: gbest vs lbest
Fitness Profiles



(a) Elliptic (b) Shifted Schaffer 6 (c) Rastrigin



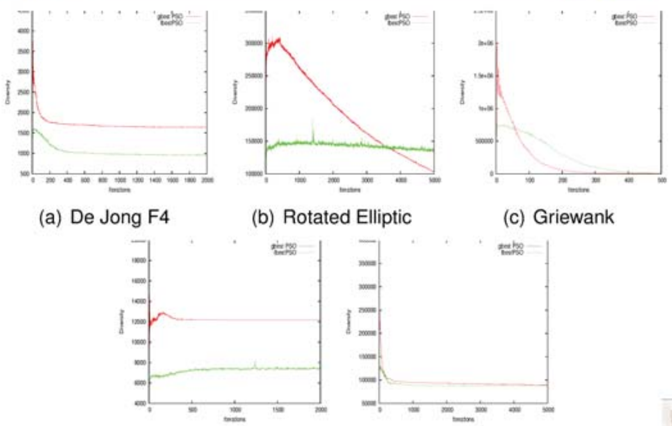
(d) Noisy Shifted Schwefel 1.2 (e) Schwefel 2.22 (f) Shubert



(g) Shifted Rotated Weierstrass

PSO Issues: gbest vs lbest

Diversity Profiles



(a) De Jong F4 (b) Rotated Elliptic (c) Griewank

(d) Rastrigin (e) Shifted Rosenbrock

PSO Issues: gbest vs lbest

Observations

- None of the star or ring topologies can be considered outright best for any of the main function classes
- Very similar performance over 60 functions with respect to solution accuracy
- gbest PSO performed slightly better than lbets PSO with respect to success rate and efficiency
- lbest PSO is slightly more consistent than gbest PSO
- Which topology is best, is function specific

PSO Issues

Iteration Strategies

- Should a synchronous iteration strategy (SIS) or and asynchronous iteration strategy (AIS) be used?
- General opinions:
  - AIS is generally faster and less costly than SIS [4, 26, 20, 33, 38]
  - AIS generally provides better results [26, 20, 33, 38]
  - AIS is better suited for lbest PSO, while SIS is better for gbest PSO [4]
- Recently, it was shown that SIS generally yields better results than AIS, specifically unimodal functions, and equal to AIS or better for multimodal functions [34]
- It was also recently stated that the choice of iteration strategy is very function dependent [45]

PSO Issues: Iteration Strategies

Accuracy Scores

: Ranks based on Final Fitness Values

Function Class		Number of Functions	gbest PSO			lbest PSO			GCPSO			BBPSO		
			>	=	<	>	=	<	>	=	<	>	=	<
UM	Sep	7	0	0	7	0	1	6	0	0	7	0	1	6
	Non-sep	3	1	1	1	0	2	1	0	2	1	0	3	0
	Noisy	2	0	0	2	1	1	0	1	0	1	1	0	1
	Shifted	5	0	5	0	0	4	1	0	5	0	0	5	0
	Rotated	1	0	0	1	0	0	1	0	0	1	0	1	0
MM	Sep	6	0	5	1	0	6	0	0	4	2	0	6	0
	Non-sep	9	0	7	2	0	9	0	1	7	1	0	9	0
	Shifted	10	2	6	2	0	10	0	1	7	2	1	8	1
	Rotated	4	0	1	3	0	4	0	1	0	3	1	1	2
	Noisy	1	1	0	0	0	1	0	1	0	0	1	0	0
	Composition	11	7	4	0	0	11	0	7	3	1	10	0	1
Overall Total		59	11	29	19	1	49	9	12	28	19	14	34	11
Overall UM		18	1	6	11	1	8	9	1	7	10	1	10	7
Overall MM		41	10	23	8	0	41	0	11	21	9	13	24	4
Overall Sep		17	1	7	9	1	10	6	0	7	10	0	10	7
Overall Non-sep		42	10	23	9	0	39	3	12	21	9	13	25	4

PSO Issues: Iteration Strategies
Observation

- Unimodal functions: AIS better accuracy for most functions
- Multimodal functions:
  - No significant difference for most of the functions
  - For the remainder of the functions, no clear winner
  - For lbest PSO not significant difference over all functions – insensitive to iteration strategy
- Separable functions: SIS not the preferred strategy for most of the functions
- Non-separable:
  - AIS bad for BBPSO
  - For lbest PSO AIS slightly better than SIS
  - For gbest PSO, GCPSO, SIS slightly better
  - However, for most functions no significant difference

Particle Trajectories
Theoretical Results

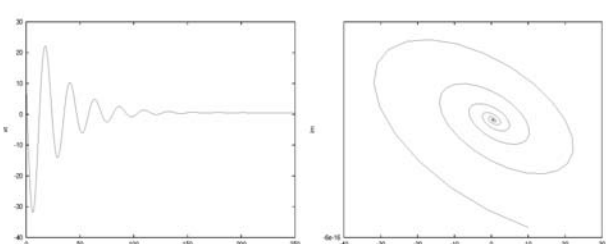
Simplified particle trajectories [41, 9]

- no stochastic component
- single, one-dimensional particle
- using w
- personal best and global best are fixed: $y = 1.0, \hat{y} = 0.0$

Example trajectories:

- Convergence to an equilibrium (figure 9)
- Cyclic behavior (figure 10)
- Divergent behavior (figure 11)

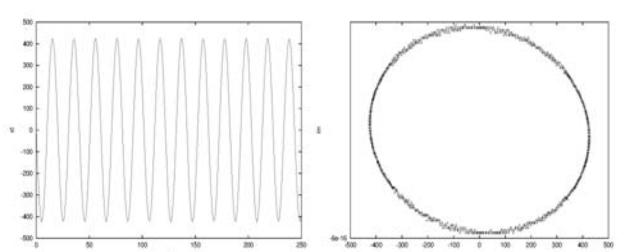
Particle Trajectories
Convergent Trajectory



(a) Time domain (b) Phase space

: $w = 0.5$ and $\phi_1 = \phi_2 = 1.4$

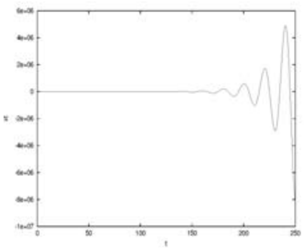
Particle Trajectories
Cyclic Trajectory



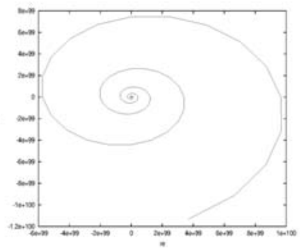
(a) Time domain (b) Phase space

: $w = 1.0$ and $\phi_1 = \phi_2 = 1.999$

Particle Trajectories
Divergent Trajectory



(a) Time domain

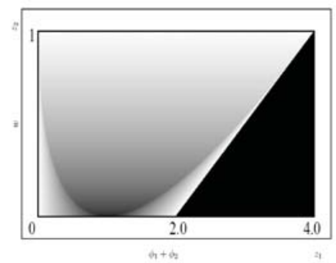


(b) Phase space

: $w = 0.7$ and $\phi_1 = \phi_2 = 1.9$

Particle Trajectories
Convergence Conditions

- What do we mean by the term convergence?
- Convergence map for values of w and $\phi = \phi_1 + \phi_2$, where $\phi_1 = c_1 r_1, \phi_2 = c_2 r_2$



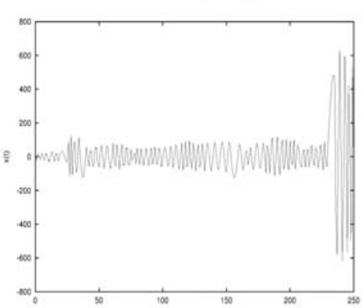
Convergence conditions on values of w, c_1 and c_2:

$$1 > w > \frac{1}{2}(\phi_1 + \phi_2) - 1 \geq 0$$

: Convergence Map for Values of w and $\phi = \phi_1 + \phi_2$

Particle Trajectories
Stochastic Trajectories

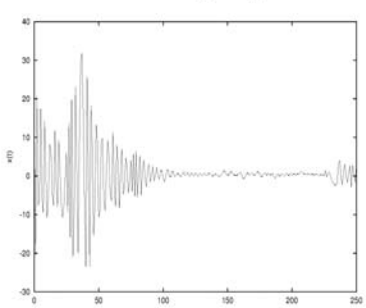
: $w = 1.0$ and $c_1 = c_2 = 2.0$



- violates the convergence condition
- for $w = 1.0$, $c_1 + c_2 < 4.0$ to validate the condition

Particle Trajectories
Stochastic Trajectories (cont)

: $w = 0.9$ and $c_1 = c_2 = 2.0$



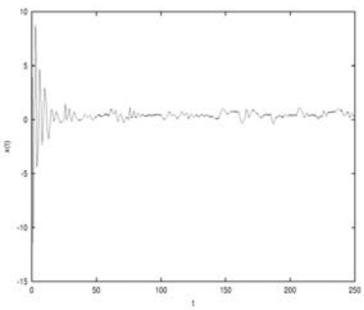
- violates the convergence condition
- for $w = 0.9$, $c_1 + c_2 < 3.8$ to validate the condition

What is happening here?

- since $0 < \phi_1 + \phi_2 < 4$,
- and $r_1, r_2 \sim U(0, 1)$,
- $\text{prob}(c_1 + c_2 < 3.8) = \frac{3.8}{4} = 0.95$

Particle Trajectories
Good Convergent Parameter Choices

: $w = 0.7$ and $c_1 = c_2 = 1.4$



- validates the convergence condition

PSO as Universal Optimizer

- Many different classes of optimization problems exist, for example,
  - Discrete-valued versus continuous-valued
  - Boundary constrained versus constrained
  - Single versus multi-objective
  - Static versus dynamic and noisy
  - Large scale
  - Unimodal versus multimodal
- Original PSO was developed to solve boundary constrained, single-objective, static, continuous-valued optimization problems
- Can PSO be used to solve optimization problems of these different problem classes, without changing the main principles of PSO?

PSO as Universal Optimizer (cont)
Exploration vs Exploitation

- For each problem type, what are the issues, and how can PSO be adapted to address these issues, while still maintaining the behavioral principles of PSO?
- An issue that relates to all of these problems: Exploration–exploitation tradeoff
  - **exploration**
    - ability to explore the search space
    - need to maintain swarm diversity
  - **exploitation**
    - ability to concentrate the search around a promising area to refine a candidate solution
    - need ways to ensure that all particles converge on the same point

Discrete-Valued Optimization Problems
Binary PSO

- What is the problem?
  - PSO originally developed for optimizing continuous-valued variables
  - Uses vector algebra on floating-point vectors
- How to adapt PSO for binary-valued variables?
  - Binary PSO (binPSO) of Kennedy and Eberhart [23]
  - Velocity remains a floating-point vector, but meaning changes
  - Velocity is no longer a step size, but is used to determine a probability of selecting bit 0 or bit 1
  - Position is a bit vector, i.e. $x_{ij} \in \{0, 1\}$
  - How to interpret velocity as a probability?

$$p_{ij}(t) = \frac{1}{1 + e^{-v_{ij}(t)}}$$

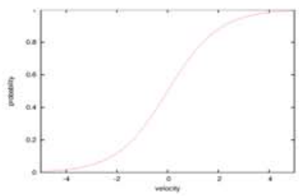
  - Then, position update changes to

$$x_{ij}(t+1) = \begin{cases} 1 & \text{if } U(0,1) < p_{ij}(t+1)) \\ 0 & \text{otherwise} \end{cases}$$

Discrete-Valued Optimization Problems
Binary PSO (cont)

Issues:

- Interpretation of control parameters changes
  - w: small values facilitate longer exploration
  - V_{max}: smaller values promote exploration
- Initial velocities should be zero
- Velocities should never move to zero, but to $\pm\infty$
- Curse of dimensionality
- What happens if binary representations of consecutive numbers have a large Hamming distance?



: Sigmoid Function

Discrete-Valued Optimization Problems
Angle Modulated PSO

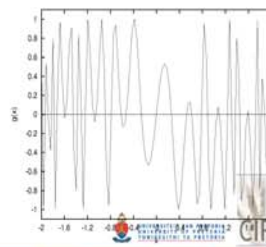
- Velocities and positions remain floating-point vectors
- Find a bitstring generating function to generate bitstring solution
- The generating function:

$$g(x) = \sin(2\pi(x-a) \times b \times \cos(2\pi(x-a) \times c)) + d$$

sampled at evenly spaced positions, x

The coefficients determine the shape of the generating function:

- a: horizontal shift of generating function
- b: maximum frequency of the sin function
- c: frequency of the cos function
- d: vertical shift of generating function

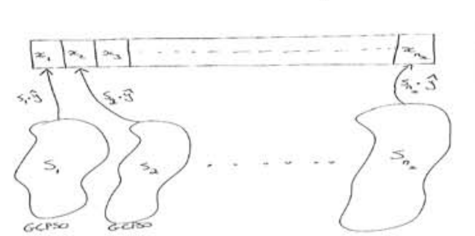


Discrete-Valued Optimization Problems
Angle Modulated PSO (cont)

Use a standard PSO to find the best values for these coefficients
Generate a swarm of 4-dimensional particles;
repeat
  Apply any PSO for one iteration;
  for *each particle* **do**
    Substitute values for coefficients a, b, c and d into generating function;
    Produce n_x bit-values to form a bit-vector solution;
    Calculate the fitness of the bit-vector solution in the original bit-valued space;
  end
until *a convergence criterion is satisfied*;

Large Scale Optimization Problems
Cooperative PSO

- Each particle is split into K separate parts of smaller dimension [41, 42, 43]
- Each part is then optimized using a separate sub-swarm
- If $K = n_x$, each dimension is optimized by a separate sub-swarm
- What are the issues?
  - Problem if there are strong dependencies among variables
  - How should the fitness of sub-swarm particles be evaluated?



Large Scale Optimization Problems
Cooperative PSO (cont)

$K_1 = n_x \bmod K$ and $K_2 = K - (n_x \bmod K)$;
Initialize K_1 $\lceil n_x/K \rceil$-dimensional and K_2 $\lfloor n_x/K \rfloor$-dimensional swarms;
repeat
  for *each sub-swarm* $S_k, k = 1, \ldots, K$ **do**
    for *each particle* $i = 1, \ldots, S_k.n_s$ **do**
      if $f(\mathbf{b}(k, S_k.\mathbf{x}_i)) < f(\mathbf{b}(k, S_k.\mathbf{y}_i))$ **then**
        $S_k.\mathbf{y}_i = S_k.\mathbf{x}_i$;
      end
      if $f(\mathbf{b}(k, S_k.\mathbf{y}_i)) < f(\mathbf{b}(k, S_k.\hat{\mathbf{y}}))$ **then**
        $S_k.\hat{\mathbf{y}} = S_k.\mathbf{y}_i$;
      end
    end
    Apply velocity and position updates;
  end
until *stopping condition is true*;

Multiple Solutions to Multimodal Problems

Niching capability of PSO:

- Can the *gbest* PSO find more than one solution?
  - Formal proofs showed that all particles converge to a weighted average of their personal best and global best positions
  - Therefore, only one solution can be found
  - What if we re-run the algorithm? No guarantee to find different solutions
- What about *lbest* PSO?
  - Neighborhoods may converge to different solutions
  - However, due to overlapping neighborhoods, particles are still attracted to one solution

Multiple Solutions to Multimodal Problems
Objective Function Stretching

Sequential niching, stretching the function to remove found minima [30, 31]

Create and initialize a n_x-dimensional swarm, S, and $\mathcal{X} = \emptyset$;
repeat
  Perform a single PSO iteration;
  if $f(S.\hat{\mathbf{y}}) \leq \epsilon$ **then**
    Isolate $S.\hat{\mathbf{y}}$;
    Perform a local search around $S.\hat{\mathbf{y}}$;
    if *a minimizer* $\mathbf{x}^*_{\mathcal{N}}$ *is found by the local search* **then**
      $\mathcal{X} \leftarrow \mathcal{X} \cup \{\mathbf{x}^*_{\mathcal{N}}\}$;
      Let $f(\mathbf{x}) \leftarrow H(\mathbf{x})$;
    end
  end
  Reinitialize the swarm S;
until *stopping condition is true*;
Return $\mathcal{X}$ as the set of multiple solutions;

Multiple Solutions to Multimodal Problems
NichePSO

Parallel niching PSO [3]

Create and initialize a n_x-dimensional *main* swarm, S;
repeat
  Train main swarm, S, for one iteration using *cognition-only* model;
  Update the fitness of each main swarm particle, $S.\mathbf{x}_i$;
  for *each sub-swarm* S_k **do**
    Train sub-swarm particles, $S_k.\mathbf{x}_i$, using a full model PSO;
    Update each particle's fitness;
    Update the swarm radius $S_k.R$;
  endFor
  If possible, merge sub-swarms;
  Allow sub-swarms to absorb any particles from the main swarm that moved into the sub-swarm;
  If possible, create new sub-swarms;
until *stopping condition is true*;
Return $S_k.\hat{\mathbf{y}}$ for each sub-swarm S_k as a solution;

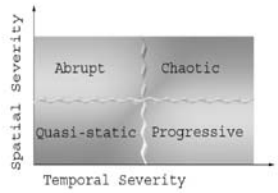
Dynamic Environments

- Objective: To find and track solutions in dynamically changing search spaces

$$\mathbf{x}^*(t) = \min_{\mathbf{x}} f(\mathbf{x}, \varpi(t))$$

where $\mathbf{x}^*(t)$ is the optimum found at time step t, and $\varpi(t)$ is a vector of time-dependent objective function control parameters

- Environment types:
  - Location of optima may change
  - Value of optima may change
  - Optima may disappear and new ones appear
  - Change frequencey
  - Change severity



: Environment Classes

Dynamic Environments
Consequences for PSO

- PSO can not be applied to dynamic environments without any changes to maintain swarm diversity
- Recall that particles converge to a weighted average of their personal best and global best positions
- At the point of convergence, $\mathbf{v}_i = 0$, and the contributions of the cognitive and social components are approximately zero
- New velocities are zero, therefore no change in position
- When the environment changes, personal best positions becomes stale, and will cause particles to be attrackted to old best positions
- Small inertia weight values limit exploration
- Velocity clamping limits exploration

Dynamic Environments
Consequences for PSO (cont)

- Environment change detection:
  - Optimization algorithm needs to react when a change is detected in order to increase diversity
  - Use sentry particles [5]
  - Gbest versus pbest versus arbitrary positions as sentries
- How to respond to environment changes?
  - Change the inertia update
    - $w \sim N(0.72, \sigma)$ [12], using no velocity clamping
    - If decreasing inertia is used, reset w to larger value

Dynamic Environments
Consequences for PSO (cont)

- Reinitialize particle positions [12]:
  - Reinitialize the entire swarm
  - Reinitialize parts of the swarm
  - Total reinitialization versus keeping previous personal best positions
- Limit memory
  - Reinitialize the personal best position to the particle's current position – only effective if swarm has not yet converged
  - Reset personal best positions only if significant change in fitness is observed
  - Recalculate global best after resetting personal best positions
- Do a local search around the previous optimum [46]

Dynamic Environments
Charged PSO

- Some particles attract one another, and others repell one another
- Velocity changes to

$$v_{ij}(t+1) = wv_{ij}(t)+c_1r_1(t)[y_{ij}(t)-x_{ij}(t)]+c_2r_2(t)[\hat{y}_j(t)-x_{ij}(t)]+a_{ij}(t)$$

where $\mathbf{a}_i$ is the particle acceleration, determining the magnitude of inter-particle repulsion [1]

$$\mathbf{a}_i(t) = \sum_{l=1,i\neq l}^{n_s} \mathbf{a}_{il}(t)$$

- The repulsion force between particles i and l is

$$\mathbf{a}_{il}(t) = \begin{cases} \left(\frac{Q_iQ_l}{d_{il}^3}\right)(\mathbf{x}_i(t)-\mathbf{x}_l(t)) & \text{if } R_c \leq d_{il} \leq R_p \\ \left(\frac{Q_iQ_l(\mathbf{x}_i(t)-\mathbf{x}_l(t))}{R_c^2 d_{il}}\right) & \text{if } d_{il} < R_c \\ 0 & \text{if } d_{il} > R_p \end{cases}$$

Dynamic Environments
Charged PSO (cont)

- $d_{il} = ||\mathbf{x}_i(t) - \mathbf{x}_l(t)||$
  Q_i is the particle's charged magnitude
  R_c is the core radius
  R_p is the perception limit of each particle
- If $Q_i = 0$, particles are neutral and there is no repelling
- If $Q_i \neq 0$, particles are charged, and repel from each other
- Inter-particle repulsion occurs only when the separation between two particles is within the range $[R_c, R_p]$
- The smaller the separation, the larger the repulsion between the corresponding particles
- Acceleration is fixed at the core radius to prevent too severe repelling

Dynamic Environments
Quantum PSO

- Based on quantum model of an atom, where orbiting electrons are replaced by a quantum cloud which is a probability distribution governing the position of the electron
- Developed as a simplified and less expensive version of the charged PSO
- Swarm contains
  - neutral particles following standard PSO updates
  - charged, or quantum particles, randomly placed within a multi-dimensional sphere

$$\mathbf{x}_i(t+1) = \begin{cases} \mathbf{x}_i(t) + \mathbf{v}_i(t+1) & \text{if } Q_i = 0 \\ \mathbf{B}_{\hat{\mathbf{y}}}(r_{cloud}) & \text{if } Q_i \neq 0 \end{cases}$$

Constrained Optimization Problems
Definition

Constrained optimization problem:

$$\begin{aligned} \text{minimize} \quad & f(\mathbf{x}), \quad \mathbf{x} = (x_1, \ldots, x_{n_x}) \\ \text{subject to} \quad & g_m(\mathbf{x}) \leq 0, \; m = 1, \ldots, n_g \\ & h_m(\mathbf{x}) = 0, \; m = n_g+1, \ldots, n_g+n_h \\ & x_j \in \text{dom}(x_j) \end{aligned}$$

where n_g and n_h are the number of inequality and equality constraints resp ectively

- Optimization problem is reformulated as

$$\text{minimize} \quad F(\mathbf{x}, t) = f(\mathbf{x}, t) + \lambda p(\mathbf{x}, t)$$

λ is the penalty coefficient
$p(\mathbf{x}, t)$ is the (possibly) time-dependent penalty function

- How to find the best penalty coefficients?
- And the penalty?

$$p(\mathbf{x}_i, t) = \sum_{m=1}^{n_g+n_h} \lambda_m(t) p_m(\mathbf{x}_i)$$

where

$$p_m(\mathbf{x}_i) = \begin{cases} \max\{0, g_m(\mathbf{x}_i)^\alpha\} & \text{if } m \in [1, \ldots, n_g] \\ |h_m(\mathbf{x}_i)|^\alpha & \text{if } m \in [n_g + 1, \ldots, n_g + n_h] \end{cases}$$

α is a positive constant, representing the power of the penalty

- Define the Lagrangian for the constrained problem
- The Lagrangian:

$$L(\mathbf{x}, \lambda_g, \lambda_h) = f(\mathbf{x}) + \sum_{m=1}^{n_g} \lambda_{gm} g_m(\mathbf{x}) + \sum_{m=n_g+1}^{n_g+n_h} \lambda_{hm} h_m(\mathbf{x})$$

$\lambda_g \in \mathbb{R}^{n_g}$ and $\lambda_h \in \mathbb{R}^{n_h}$ are the Lagrangian multipliers

- The new optimization problem (the primal problem):

$$\begin{array}{ll} \text{maximize}_{\lambda_g, \lambda_h} & L(\mathbf{x}, \lambda_g, \lambda_h) \\ \text{subject to} & \lambda_{gm} \geq 0, \; m = 1, \ldots, n_g + n_h \end{array}$$

- The vector $\mathbf{x}^*$ that solves the primal problem, as well as the Lagrange multiplier vectors, λ_g^* and λ_h^*, can be found by solving the min-max problem,

$$\min_{\mathbf{x}} \max_{\lambda_g, \lambda_h} L(\mathbf{x}, \lambda_g, \lambda_h)$$

- A coevolutionary PSO approach to solve the above min-max problem uses two swarms
- Swarm S_1 uses fitness function

$$f(\mathbf{x}) = \max_{\lambda_g, \lambda_h \in S_2} L(\mathbf{x}, \lambda_g, \lambda_h)$$

- Swarm S_2 uses fitness function

$$f(\lambda_g, \lambda_h) = \min_{\mathbf{x} \in S_1} L(\mathbf{x}, \lambda_g, \lambda_h)$$

Create and initialize two swarms, S_1 and S_2, where S_1 is n_x-dimensional and S_2 is $n_g + n_h$ dimensional;
repeat
  Run a PSO algorithm on swarm S_1 for $S_1.n_t$ iterations;
  Re-evaluate $S_2.\mathbf{y}_i(t), \forall i = 1, \ldots, S_2.n_s$;
  Run a PSO algorithm on swarm S_2 for $S_2.n_t$ iterations;
  Re-evaluate $S_1.\mathbf{y}_i(t), \forall i = 1, \ldots, S_1.n_s$;
until *stopping condition is true*;

Constrained Optimization Problems
Formulate as a MOP

- Reformulate constraints as additional sub-objective(s)
- Solve using a multi-objective PSO

Multi-Objective Optimization
Definition

Multi-objective problem:

$$\begin{array}{ll} \text{minimize} & \mathbf{f}(\mathbf{x}) \\ \text{subject to} & g_m(\mathbf{x}) \leq 0, \quad m = 1, \ldots, n_g \\ & h_m(\mathbf{x}) = 0, \quad m = n_g + 1, \ldots, n_g + n_h \\ & \mathbf{x} \in [\mathbf{x}_{min}, \mathbf{x}_{max}]^{n_x} \end{array}$$

where $\mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_{n_k}(\mathbf{x})) \in \mathcal{O} \subseteq \mathbb{R}^{n_k}$

$\mathcal{O}$ is referred to as the *objective space*
The search space, $\mathcal{S}$, is also referred to as the *decision space*

Multi-Objective Optimization
Issues

- Can PSO be used to simultaneously optimize more than one, possibly conflicting objective?
- How can PSO be used to find a set of solutions which optimally balances the trade-offs among these conflicting objectives?
- The task is to find a set of non-dominating solutions
- Formal definition of domination:
  A decision vector, $\mathbf{x}_1$ dominates a decision vector, $\mathbf{x}_2$ (denoted by $\mathbf{x}_1 \prec \mathbf{x}_2$), if and only if
  - $\mathbf{x}_1$ is not worse than $\mathbf{x}_2$ in all objectives, i.e. $f_k(\mathbf{x}_1) \leq f_k(\mathbf{x}_2), \forall k = 1, \ldots, n_k$, and
  - $\mathbf{x}_1$ is strictly better than $\mathbf{x}_2$ in at least one objective, i.e. $\exists k = 1, \ldots, n_k : f_k(\mathbf{x}_1) < f_k(\mathbf{x}_2)$

Multi-Objective Optimization
Aggregation-based Methods

- The objective is redefined as

$$\begin{array}{ll} \text{minimize} & \sum_{k=1}^{n_k} \omega_k f_k(\mathbf{x}) \\ \text{subject to} & g_m(\mathbf{x}) \leq 0, \quad m = 1, \ldots, n_g \\ & h_m(\mathbf{x}) = 0, \quad m = n_g + 1, \ldots, n_g + n_h \\ & \mathbf{x} \in [\mathbf{x}_{min}, \mathbf{x}_{max}]^{n_x} \\ & \omega_k \geq 0, k = 1, \ldots, n_k \end{array}$$

  where $\sum_{k=1}^{n_k} \omega_k = 1$
- Problem with getting the best values for ω_k
- Has to be applied repeatedly to get more than one solution

A multi-swarm approach:

- Assume K sub-objectives
- K sub-swarms are used, where each optimizes one of the objectives
- Need a knowledge transfer strategy (KTS) to transfer information about best positions between sub-swarms
- Exchanged information are via selection of global guides, replacing the global best positions in the velocity updates
- Standard KTS: the ring KTS
  - Sub-swarms are arranged in a ring topology
  - Global guide of swarm S_k is swarm $S_{(k+1) \mod K}$

- Assume two objectives

$$
\begin{aligned}
S_1.v_{ij}(t+1) &= wS_1.v_{ij}(t) + c_1 r_{1j}(t)(S_1.y_{ij}(t) - S_1.x_{ij}(t)) \\
&+ c_2 r_{2j}(t)(S_2.\hat{y}_i(t) - S_1.x_{ij}(t)) \\
S_2.v_{ij}(t+1) &= wS_2.v_{ij}(t) + c_1 r_{1j}(t)(S_2.y_{ij}(t) - S_2.x_{ij}(t)) \\
&+ c_2 r_{ij}(t)(S_1.\hat{y}_j(t) - S.x_{2j}(t))
\end{aligned}
$$

  where sub-swarm S_1 evaluates individuals on the basis of objective $f_1(\mathbf{x})$, and sub-swarm S_2 uses objective $f_2(\mathbf{x})$
- Local guide selection:
  - Local guide replaces the personal best
  - Update personal best position only if the new particle position dominates the previous personal best position
- Alternative KTS: random

- Objective of archive is to keep track of all non-dominated solutions
- Non-dominated solutions added to archive after each iteration
- Fixed-sized archives versus unlimited sizes
- Local versus global guides

Let $t = 0$;
Initialize the swarm, $S(t)$, and archive, $A(t)$;
repeat
  Evaluate $(S(t))$;
  $A(t+1) \leftarrow$ Update$(S(t), A(t))$;
  $S(t+1) \leftarrow$ Generate$(S(t), A(t))$;
  $t = t+1$;
until *stopping condition is true*;

- Dynamically changing constraints, in
  - static and dynamic environments
  - single- and multi-objectives
- Tracking multiple optima in dynamic environments
- Dynamic multi-objective optimization problems

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