# Theory of Swarm Intelligence

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## Swarm Intelligence

Collective behavior of a "swarm" of agents.

**Examples from Nature**

- dome construction by termites
- communication of bees
- ant trails
- foraging behavior of fish schools and bird flocks
- swarm robotics

Plenty of inspiration for optimization.





## ACO and PSO

**Ant colony optimization (ACO)**

- inspired by foraging behavior of ants
- artificial ants construct solutions using pheromones
- pheromones indicate attractiveness of solution component

**Particle swarm optimization (PSO)**

- mimics search of bird flocks and fish schools
- particles "fly" through search space
- each particle is attracted by own best position and best position of neighbors

## Theory

### What "theory" can mean

- convergence analysis
- analysis of simplified models of algorithms
- empirical studies on test functions
- runtime analysis / computational complexity analysis
- ...

### Example Question

How long does it take on average until algorithm $A$ finds a target solution on problem $P$?

Notion of time: number of iterations, number of function evaluations





## Content

### What this tutorial is about

- runtime analysis
- simple variants of swarm intelligence algorithms
- insight into their working principles
- impact of parameters and design choices on performance
- what distinguishes ACO/PSO from evolutionary algorithms?
- performance guarantees for combinatorial optimization
- methods and proof ideas

### What this tutorial is not about

- convergence results
- analysis of models of algorithms
- no intend to be exhaustive





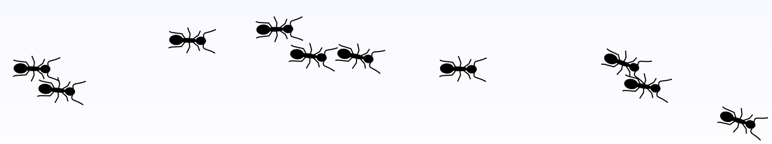
## Overview

1. Introduction
2. ACO in Pseudo-Boolean Optimization
   - MMAS with best-so-far update
   - How MMAS deals with plateaus
   - MMAS with iteration-best update
3. ACO and Shortest Path Problems
   - Single-Destination Shortest Paths
   - All-Pairs Shortest Paths
4. ACO and Minimum Spanning Trees
5. ACO and the TSP
6. Particle Swarm Optimization
   - Binary PSO
   - Continuous Spaces
7. Conclusions





## Ant Colony Optimization (ACO)



Main idea: artificial ants communicate via pheromones.

### Scheme of ACO

Repeat:

- construct ant solutions guided by pheromones
- update pheromones by reinforcing good solutions

# Pseudo-Boolean Optimization

Goal: maximize $f\colon \{0,1\}^n \to \mathbb{R}$.

### Illustrative test functions

$$\textsc{OneMax}(x) = \sum_{i=1}^{n} x_i$$

$$\textsc{LeadingOnes}(x) = \sum_{i=1}^{n} \prod_{j=1}^{i} x_j$$

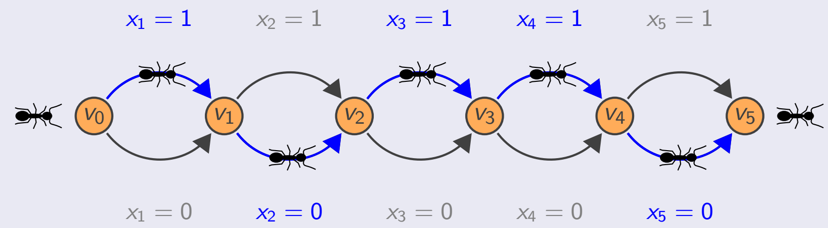
$$\textsc{Needle}(x) = \prod_{i=1}^{n} x_i$$





# ACO in Pseudo-Boolean Optimization

### Solution Construction




Probability of choosing an edge equals pheromone on the edge.

Initial pheromones: $\tau(x_i = 0) = \tau(x_i = 1) = 1/2$.

Note: no linkage between bits. No heuristic information used.

Pheromones $\tau(x_i = 1)$ suffice to describe all pheromones.





# ACO in Pseudo-Boolean Optimization (2)

Pheromone update: reinforce some good solution $x$.

Strength of update determined by evaporation factor $0 \leq \rho \leq 1$:

$$\tau'(x_i = 1) = \begin{cases} (1-\rho)\cdot\tau(x_i = 1) & \text{if } x_i = 0 \\ (1-\rho)\cdot\tau(x_i = 1) + \rho & \text{if } x_i = 1 \end{cases}$$

Pheromone borders as in MAX-MIN Ant System (Stützle and Hoos, 2000):

$$\tau_{\min} \leq \tau' \leq 1 - \tau_{\min}$$

Default choice: $\tau_{\min} := 1/n$ (cf. standard mutation in EAs).





# One Ant?

Most ACO algorithms analyzed: one ant per iteration.

One ant at a time, many ants over time.

### Steady-state GA

- Probabilistic model: Population
- New solutions: selection + variation
- Environmental selection

### Ant Colony Optimization

- Probabilistic model: Pheromones
- New solutions: construction graph
- Selection for reinforcement

# Evolutionary Algorithms vs. ACO

**MMAS* (Gutjahr and Sebastiani, 2008)**

Start with uniform random solution $x^*$ and repeat:

- Construct $x$.
- Replace $x^*$ by $x$ if $f(x) > f(x^*)$.
- Update pheromones w. r. t. $x^*$ (best-so-far update).

Note: best-so-far solution $x^*$ is constantly reinforced.

**(1+1) EA**

Start with uniform random solution $x^*$ and repeat:

- Create $x$ by flipping each bit in $x^*$ independently with probability $1/n$.
- Replace $x^*$ by $x$ if $f(x) \geq f(x^*)$.

(1+1) EA: Probability of setting bit to 1 is in $\{1/n, 1-1/n\}$.

MMAS*: Probability of setting bit to 1 is in $[1/n, 1-1/n]$ (unless $\rho \approx 1$).





# Overview

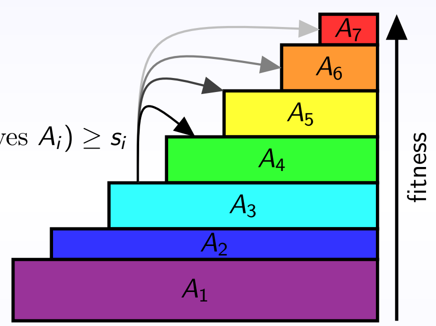






# Fitness-level Method for the (1+1) EA

$\Pr((1+1)\text{ EA leaves } A_i) \geq s_i$



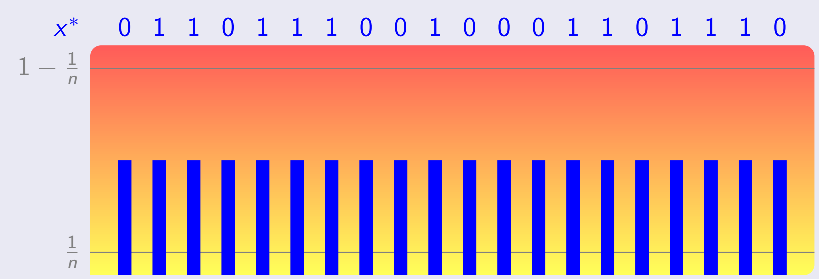

Expected optimization time of (1+1) EA at most $\sum_{i=1}^{m-1} \frac{1}{s_i}$.





# MMAS*

**Pheromones on 1-edges**




After $(\ln n)/\rho$ reinforcements of $x^*$ MMAS* temporarily behaves like (1+1) EA.

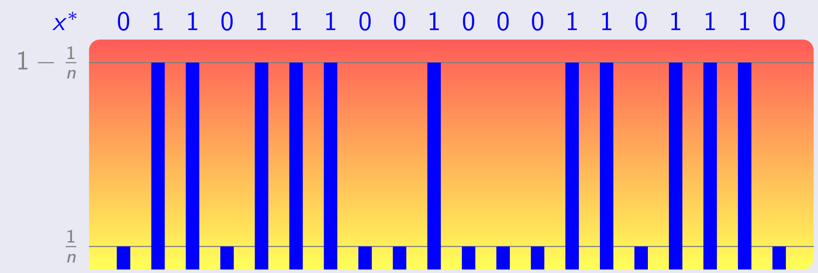
**Fitness-Level Method with $A_i$ = search points with $i$-th fitness value**

(1+1) EA: $\leq \sum_{i=1}^{m-1} \frac{1}{s_i}$ MMAS*: $\leq \sum_{i=1}^{m-1} \frac{1}{s_i} + m \cdot \frac{\ln n}{\rho}$

Upper bounds: time for finding improvements + time for pheromone adaptation.

# MMAS*

## Pheromones on 1-edges

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Upper bounds: time for finding improvements + time for pheromone adaptation.





# Bounds with Fitness Levels

LeadingOnes 11110010

$$s_i \geq \frac{1}{n} \cdot \left(1 - \frac{1}{n}\right)^{n-1} \geq \frac{1}{en}$$

**Theorem**

(1+1) EA: $en^2$ MMAS*: $en^2 + n \cdot \frac{\ln n}{\rho} = O(n^2 + (n \log n)/\rho)$

Unimodal functions with $d$ function values:

**Theorem**

(1+1) EA: $end$ MMAS*: $end + d \cdot \frac{\ln n}{\rho} = O(nd + (d \log n)/\rho)$





# Overview







# Strict Selection

Most ACO algorithms replace $x^*$ only if $f(x) > f(x^*)$.

**Drawback**

Cannot explore plateaus.

**Theorem (Neumann, Sudholt, Witt, 2009)**

*Expected time of* MMAS* *on* Needle *is* $\Omega(2^{-n} \cdot n^n) = \Omega((n/2)^n)$.

Define variant MMAS of MMAS* replacing $x^*$ if $f(x) \geq f(x^*)$.
Pheromones on each bit perform a random walk.

**Theorem (Neumann, Sudholt, Witt, 2009 and Sudholt, 2011)**

*Expected time of* MMAS *on* Needle *is* $O(n^2/\rho^2 \cdot \log n \cdot 2^n)$.

**Mixing time estimates (Sudholt, 2011)**

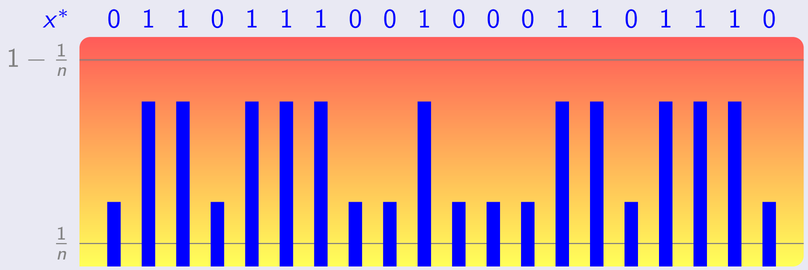
MMAS "forgets" initial pheromones on bits that have been irrelevant for the last $\Omega(n^2/\rho^2)$ steps.

## MMAS and Fitness Levels

Is MMAS as fast as MMAS* on easy functions like OneMax?

Switching between equally fit solutions can prevent freezing.

**Pheromones on 1-edges**




Fitness-level method breaks down!





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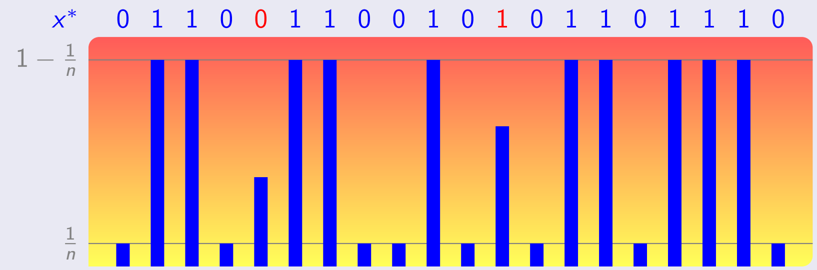



Fitness-level method breaks down!





## Is this Behavior Detrimental?

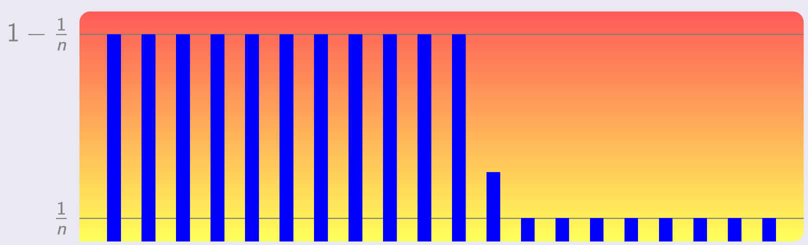
Probably not.

**Theorem (Kötzing, Neumann, Sudholt, and Wagner, 2011)**

*$O(n \log n + n/\rho)$ on OneMax for both MMAS* and MMAS.*

Assuming the sum of pheromones is fixed. Worst possible pheromone distribution for finding improvements on OneMax (Gleser, 1975):

**Pheromones on 1-edges**

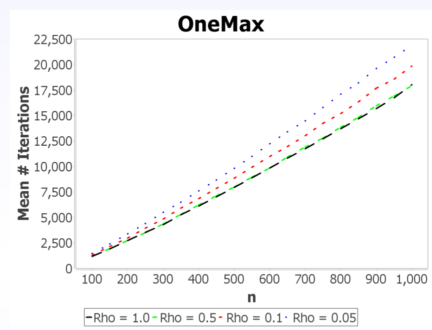



Worst case: all pheromones (but one) at borders.

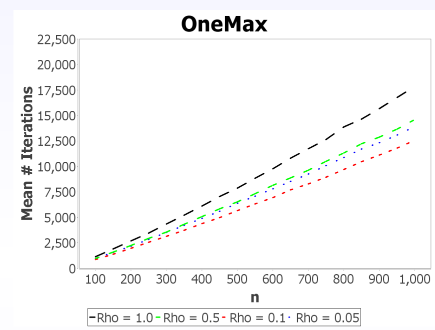




## Experiments (Kötzing et al., 2011)




MMAS*




MMAS

- MMAS better than MMAS*
- MMAS with $\rho < 1$ better than (1+1) EA (=MMAS at $\rho = 1$)!
- does not hold for MMAS*

**Open Problem**

Prove that MMAS with proper $\rho$ is faster than MMAS* and (1+1) EA.

## Overview







## Iteration-Best Update

**$\lambda$-MMAS$_{\text{ib}}$**

Repeat:
- construct $\lambda$ ant solutions
- update pheromones by reinforcing the best of these solutions

Advantages:
- can escape from local optima
- inherently parallel
- simpler ants





## Iteration-Best vs. Comma Strategies

**Rowe and Sudholt, GECCO 2012**

(1,$\lambda$) EA: $\lambda \geq \log_{e/(e-1)} n$ ($\approx 5 \log_{10} n$) necessary, even for OneMax.

If $\lambda \leq \log_{e/(e-1)} n$ ($\approx 5 \log_{10} n$) then (1,$\lambda$) EA needs exponential time.

Reason: (1,$\lambda$) EA moves away from optimum if close and $\lambda$ too small.

Behavior too chaotic to allow for hill climbing!





## Iteration-Best on OneMax

Slow pheromone adaptation effectively eliminates chaotic behavior.

**Theorem (Neumann, Sudholt, and Witt, 2010)**

*If $\rho = 1/(cn^{1/2} \log n)$ for a large constant $c > 0$ then 2-MMAS$_{\text{ib}}$ optimizes OneMax in expected time $O(n \log n)$.*

Two ants are enough!

Proof idea: as long as all pheromones are at least 1/3, the sum of pheromones grows steadily.

Large $\rho$ or small $\lambda$: pheromones come crashing down to $1/n$.

**Theorem**

*Choosing $\lambda/\rho \leq (\ln n)/244$, the optimization time of $\lambda$-MMAS$_{\text{ib}}$ on every function with a unique optimum is $2^{\Omega(n^{\varepsilon})}$ for some constant $\varepsilon > 0$ w. o. p.*

# Overview

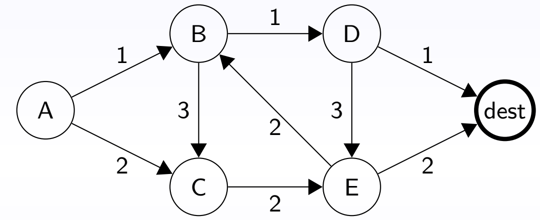






# ACO System for Single-Destination Shortest Path Problem

From Sudholt and Thyssen (2012), going back to Attiratanasunthron and Fakcharoenphol (2008).




**$\text{MMAS}_{\text{SDSP}}$**

For each vertex $u$ the ant
- memorizes and keeps track of its best-so-far path
- constructs a simple path from $u$ to the destination
- updates pheromones on edges $(u, \cdot)$ (local pheromone update)





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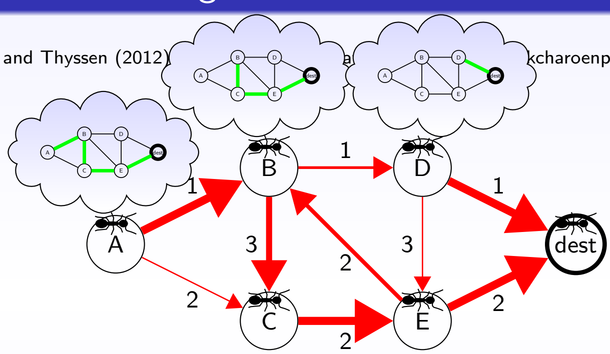


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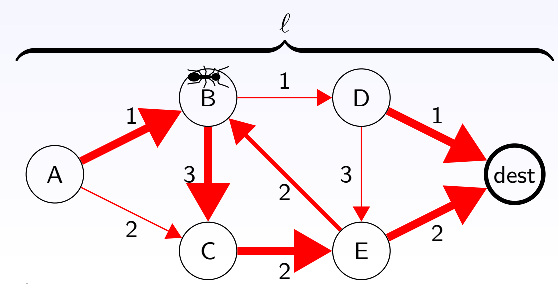
# Shortest Paths Propagate Through the Graph

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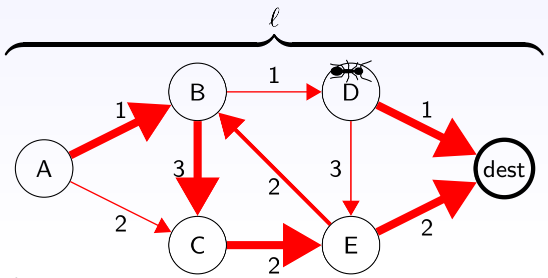


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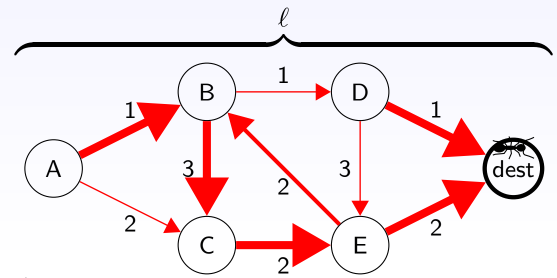


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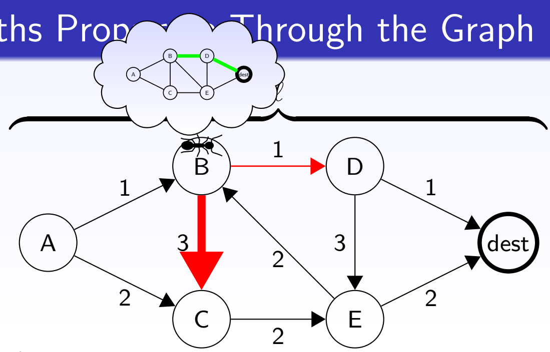


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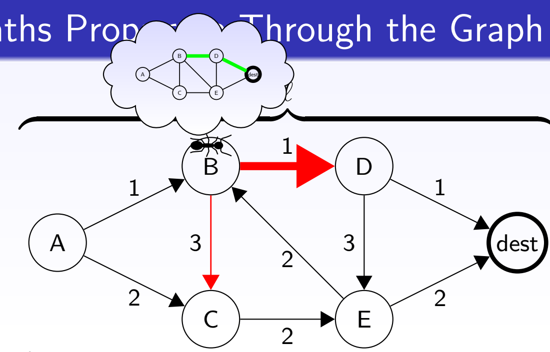


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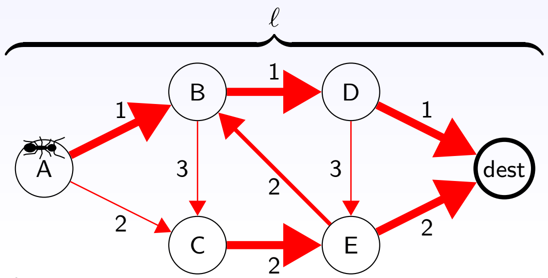
Expected time until ant at $u$ has done the same $\leq 2e/\tau_{\min} + \ln(1/\tau_{\min})/\rho$.

## Shortest Paths Propagate Through the Graph

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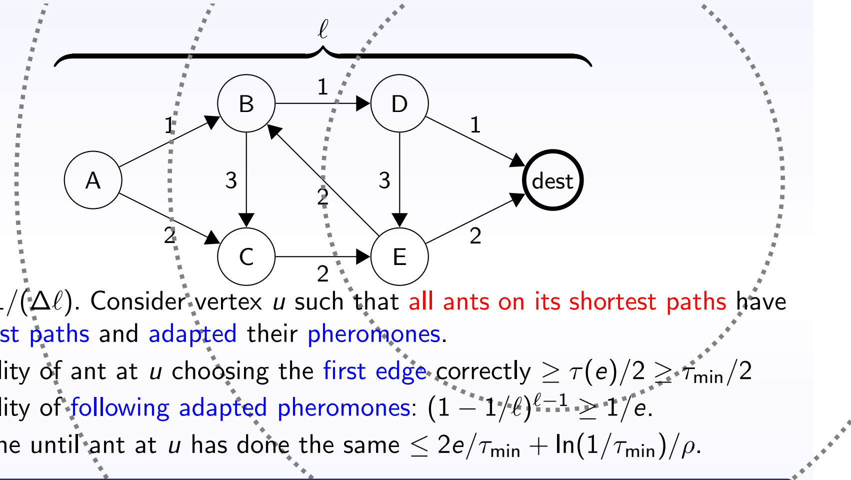
**Upper bounds for MMAS$_{\text{SDSP}}$ (Sudholt and Thyssen, 2012)**

- Consider all vertices sequentially: $O(n\Delta\ell + n\ln(\Delta\ell)/\rho)$.





## Shortest Paths Propagate Through the Graph

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**Upper bounds for MMAS$_{\text{SDSP}}$ (Sudholt and Thyssen, 2012)**

- Consider all vertices sequentially: $O(n\Delta\ell + n\ln(\Delta\ell)/\rho)$.
- Slice graph into "layers" and exploit parallelism: $O(\Delta\ell^2 + \ell/\rho)$.





## Overview

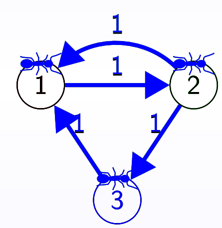






## All-Pairs Shortest Path Problem

Use distinct pheromone function $\tau_v \colon E \to \mathbb{R}_0^+$ for each destination $v$:



## A Simple Interaction Mechanism

**Path construction with interaction**

For each ant traveling from $u$ to $v$

- with prob. 1/2
  - use $\tau_v$ to travel from $u$ to $v$
- with prob. 1/2
  - choose an intermediate destination $w \in V$ uniformly at random
  - uses $\tau_w$ to travel from $u$ to $w$
  - uses $\tau_v$ to travel from $w$ to $v$





## Speed-up by Interaction

**Theorem**

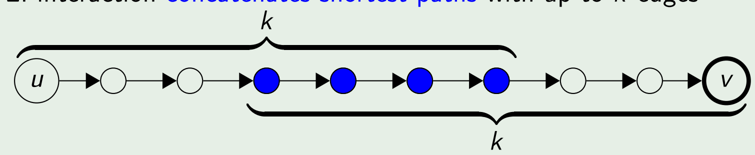
If $\tau_{\min} = 1/(\Delta\ell)$ and $\rho \leq 1/(23\Delta \log n)$ the number of iterations using interaction w. h. p. is $O(n \log n + \log(\ell) \log(\Delta\ell)/\rho)$.

Possible improvement: $O(n^3) \to O(n \log^3 n)$

**Proof Sketch**

**Phase 1**: find all shortest paths with one edge
slow evaporation $\longrightarrow$ near-uniform search

**Phase 2**: interaction concatenates shortest paths with up to $k$ edges




$\longrightarrow$ find shortest paths with up to $3/2 \cdot k$ edges.

Note: slow adaptation helps!





## Stochastic and Dynamic Shortest Path Problems

**Sudholt and Thyssen, Algorithmica 2012**

Unmodified $\text{MMAS}_{\text{SDSP}}$ on noisy SDSP: ants can become risk-seeking.

**Doerr, Hota, and Kötzing, GECCO 2012**

Re-evaluating best-so-far paths removes risk-seeking behavior.

**Lissovoi and Witt, GECCO 2013**

How effective is ACO in tracking dynamically changing shortest paths?





## Overview

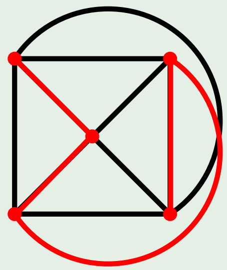
# Construction Based on Broder's Algorithm

Based on Neumann and Witt (2010).

Problem: Minimum Spanning Trees (tree of minimum weight spanning all nodes)

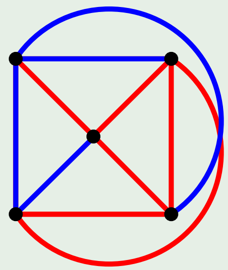
**Broder-based Construction**

Ants construct spanning tree by random walk (Broder, 1989).
Skip infeasible edges.



**Component-based Construction**

Add edges in arbitrary order based on attractiveness.
Exclude those closing a cycle.







# Algorithm

1-ANT: (following Neumann/Witt, 2010)

- two pheromone values
- value $h$: if edge has been rewarded
- value $\ell$: otherwise
- heuristic information $\eta$, $\eta(e) = \frac{1}{w(e)}$ (used before for TSP)
- Let $v_k$ the current vertex and $N_{v_k}$ be its neighborhood.
- Prob(to choose neighbor $y$ of $v_k$) $= \frac{[\tau_{(v_k,y)}]^\alpha \cdot [\eta_{(v_k,y)}]^\beta}{\sum_{y \in N(v_k)} [\tau_{(v_k,y)}]^\alpha \cdot [\eta_{(v_k,y)}]^\beta}$ with $\alpha, \beta \geq 0$.
- Consider special cases where either $\beta = 0$ or $\alpha = 0$.





# Results for Pheromone Updates

Case $\alpha = 1$, $\beta = 0$: proportional influence of pheromone values

**Theorem (Broder-based construction graph)**

*Choosing $h/\ell = n^3$, the expected time until the 1-ANT with the Broder-based construction graph has found an MST is $O(n^6(\log n + \log w_{max}))$.*

**Theorem (Component-based construction graph)**

*Choosing $h/\ell = (m - n + 1) \log n$, the expected time until the 1-ANT with the component-based construction graph has found an MST is $O(mn(\log n + \log w_{max}))$.*

Better than (1+1) EA!



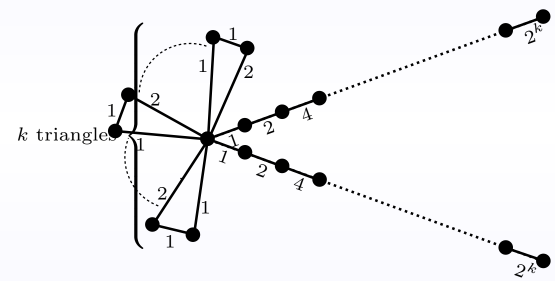




# Broder Construction Graph: Heuristic Information

Example graph $G^*$ with $n = 4k + 1$ vertices.

- $k$ triangles of weight profile $(1, 1, 2)$
- two paths of length $k$ with exponentially increasing weights.




**Theorem (Broder-based construction graph)**

*Let $\alpha = 0$ and $\beta$ be arbitrary, then the probability that the 1-ANT using the Broder construction procedure does not find an MST in polynomial time is $1 - 2^{-\Omega(n)}$.*

# Component-based Construction Graph/Heuristic Information

**Theorem (Component-based construction graph)**

*Choosing $\alpha = 0$ and $\beta \geq 6w_{\max} \log n$, the expected time of the 1-ANT with the component-based construction graph to find an MST is constant.*

**Proof Idea**

- Choose edges as Kruskal's algorithm.
- Calculation shows: probability of choosing a lightest edge is at least $1 - 1/n$.
- $n - 1$ steps $\Longrightarrow$ probability for an MST is $\Omega(1)$.





# Overview



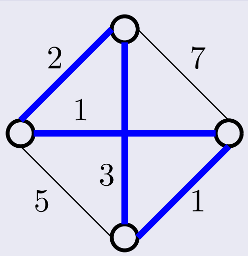




# Traveling Salesman Problem

Based on Kötzing, Neumann, Röglin and Witt (2010).

**Traveling Salesman Problem (TSP)**




- Input: weighted complete graph $G = (V, E, w)$ with $w : E \to \mathbb{R}$.
- Goal: Find Hamiltonian cycle of minimum weight.





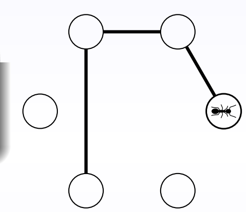
# MMAS for TSP (Kötzing, Neumann, Röglin, Witt 2010)

Best-so-far pheromone update with $\tau_{\min} := 1/n^2$ and $\tau_{\max} := 1 - 1/n$.

Initialization: same pheromone on all edges.

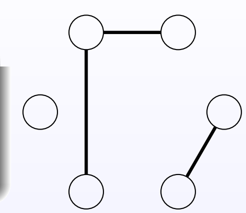
**"Ordered" tour construction**

Append a feasible edge chosen with probability proportional to pheromones.



**"Arbitrary" tour construction**

Add an edge chosen with probability proportional to pheromones as long as no cycle is closed or a vertex gets degree at least 3.



# ACO Simulating 2-OPT

Zhou (2009): ACO can simulate 2-OPT.

Probability of particular 2-Opt step (for constant $\rho$):

$\text{MMAS}^*_{Ord}$: $\Theta(1/n^3)$

$\text{MMAS}^*_{Arb}$: $\Theta(1/n^2)$





# Average Case Analysis

Assume that $n$ points placed independently, uniformly at random in the unit hypercube $[0,1]^d$.

**Theorem [Englert, Röglin, Vöcking 2007]**

2-Opt finds after $O(n^{4+1/3} \cdot \log n)$ iterations with probability $1 - o(1)$ a solution with approximation ratio $O(1)$.

**Theorem**

For $\rho = 1$, $\text{MMAS}^*_{Arb}$ finds after $O(n^{6+2/3})$ iterations with probability $1 - o(1)$ a solution with approximation ratio $O(1)$.

**Theorem**

For $\rho = 1$, $\text{MMAS}^*_{Ord}$ finds after $O(n^{7+2/3})$ iterations with probability $1 - o(1)$ a solution with approximation ratio $O(1)$.

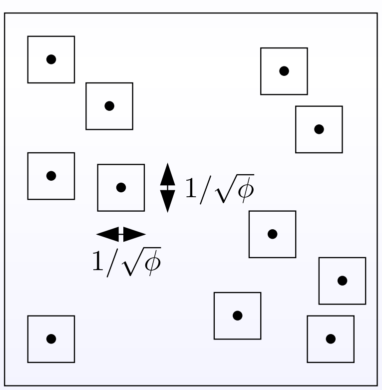




# Smoothed Analysis

**Smoothed Analysis**

Each point $i \in \{1, \ldots, n\}$ is chosen independently according to a probability density $f_i : [0,1]^d \to [0, \phi]$.




2-Opt:
$O(\sqrt[d]{\phi})$-approximation in $O(n^{4+1/3} \cdot \log(n\phi) \cdot \phi^{8/3})$ steps

$\text{MMAS}^*_{Ord}$: $O(\sqrt[d]{\phi})$-approximation in $O(n^{7+2/3} \cdot \phi^3)$ steps

$\text{MMAS}^*_{Arb}$: $O(\sqrt[d]{\phi})$-approximation in $O(n^{6+2/3} \cdot \phi^3)$ steps





# ACO: Summary and Open Questions

**Shortest Paths**

Natural and interesting test-bed for the robustness of ACO algorithms.

- global pheromone updates?
- how to deal with noise and dynamic changes?
- where does slow pheromone adaptation help?
- average-case analyses with heuristic information

**Strength of ACO**

Problem-specific construction procedures can make ACO more powerful.

- how to find a fruitful combination of metaheuristic search and problem-specific components?

**Main Challenge in Analysis of ACO**

Understand dynamics of pheromones within borders.

- results for MST and TSP with more natural pheromone models

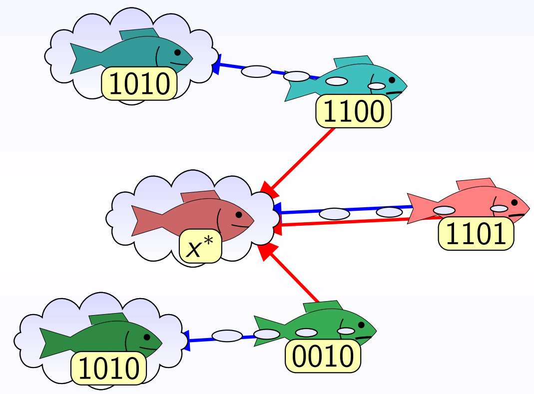
# Overview







# Particle Swarm Optimization




Binary PSO (Kennedy und Eberhart, 1997)

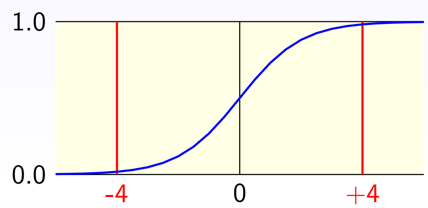




# Creating New Positions

Probabilistic construction using velocity $v$ and sigmoid function $s(v)$:

$$\text{Prob}(x_j = 1) = s(v_j) = \frac{1}{1+e^{-v_j}}$$




Restrict velocities to $v_j \in [-v_{\max}, +v_{\max}]$.

- Common practice: $v_{\max} = 4$ (good for $n \in [50, 500]$)
- Sudholt and Witt (2010): $v_{\max} := \ln(n-1)$ (good across all $n$):

$$\frac{1}{n} \leq \text{Prob}(x_j = 1) \leq 1 - \frac{1}{n}.$$







# Updating Velocities

Update current velocity vector according to

- cognitive component → towards own best: $x^{*(i)} - x^{(i)}$ and
- social component → towards global best: $x^* - x^{(i)}$.

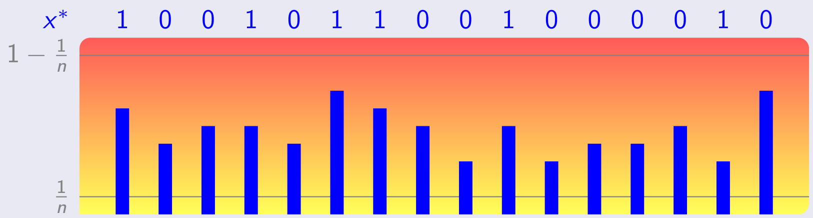
Learning rates $c_1$, $c_2$ affect weights for the two components.

Random scalars $r_1 \in U[0, c_1]$, $r_2 \in U[0, c_2]$ chosen anew in each generation:

$$v^{(i)} = v^{(i)} + r_1(x^{*(i)} - x^{(i)}) + r_2(x^* - x^{(i)})$$

# Velocity Freezing

**Particle with best-so-far solution: own best = global best**




**Lemma**

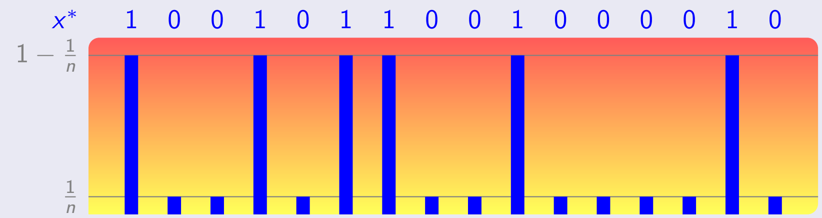
*Expected freezing time to $v_{\max}$ or $-v_{\max}$ is $O(n)$ for single bits and $O(n \log n)$ for $n$ or $\mu n$ bits if $\mu = poly(n)$.*



PSO Binary PSO

# Velocity Freezing

**Particle with best-so-far solution: own best = global best**




**Lemma**

*Expected freezing time to $v_{\max}$ or $-v_{\max}$ is $O(n)$ for single bits and $O(n \log n)$ for $n$ or $\mu n$ bits if $\mu = poly(n)$.*





# Fitness-Level Method for Binary PSO

**Upper bound for the (1+1) EA**

$$\sum_{i=0}^{m-1} \frac{1}{s_i}$$

**Upper bound for #generations of Binary PSO**

$$\sum_{i=0}^{m-1} \frac{1}{s_i} + O(m \cdot n \log n)$$

**Upper bound for #generations of "social" Binary PSO, i. e., $c_1 := 0$**

$$O\left(\frac{1}{\mu} \sum_{i=0}^{m-1} \frac{1}{s_i} + m \cdot n \log n\right)$$







# Overview

## Continuous PSO

Search space: (bounded subspace of) $\mathbb{R}^n$.

Objective function: $f: \mathbb{R}^n \to \mathbb{R}$.

Particles represent positions $x^{(i)}$ in this space.

Particles fly at certain velocity: $x^{(i)} := x^{(i)} + v^{(i)}$.

Velocity update with inertia weight $\omega$:

$$v^{(i)} = \omega v^{(i)} + r_1(x^{*(i)} - x^{(i)}) + r_2(x^* - x^{(i)})$$





## Convergence of PSO

Swarm can collapse to points or other low-dimensional subspaces.

**Convergence results for standard PSO, $\omega < 1$ (Jiang, Luo, and Yang, 2007)**

PSO converges ... somewhere.

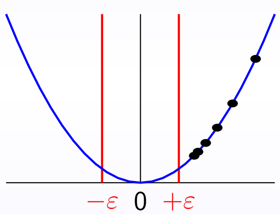




## Stagnation of Standard PSO

**Lehre and Witt, 2013**

Standard PSO with one/two particles stagnates even on one-dimensional Sphere!




Expected first hitting time of $\varepsilon$-ball around optimum is infinite (caveat: for atypically small $\omega$).

**Noisy PSO (Lehre and Witt, 2013)**

Adding noise $U[-\delta/2, \delta/2]$ for $\delta \leq \varepsilon$: finite expected hitting time on (half-)Sphere.





## Convergence of Standard PSO

**Convergence in 1D (Schmitt and Wanka, GECCO 2013/TCS 2015)**

PSO with "good" parameters: for every function in 1 dimension, the best fitness converges to the value of a local minimum.

**Convergence for $n$ dimensions (Schmitt and Wanka, GECCO 2013/TCS 2015)**

- PSO modification: pick random velocities when swarm converges.
- Convergence detected by a potential function: all velocities plus distance to global best $\leq \delta$.
- Modified PSO converges to local optima almost surely.

**Raß, Schmitt, and Wanka, GECCO 2015 Poster**

Explanation of Stagnation at Points that are not Local Optima in Particle Swarm Optimization by Potential Analysis

# PSO Extensions

## Extensions of standard PSO

- Bare-bones PSO (Kennedy, 2003)
- PSO with mutation (several variants)
- PSO using gradient information (several variants)
- Guaranteed Convergence PSO (GCPSO) (van den Bergh and Engelbrecht, 2002)
  - Make a cube mutation of a particle's position by adding $p \in U[-\ell, \ell]^n$.
  - Adapt "step size" $\ell$ in the course of the run by doubling or halving it, depending on the number of successes.

$\longrightarrow$ 1/5-rule known from evolution strategies!





# GCPSO with 1 Particle (Witt, 2009)

GCPSO with one particle is basically a (1+1) ES with cube mutation.

Can be analyzed like classical (1+1) ES (Jägersküpper, 2007)

$$\text{SPHERE}(x) := ||x|| = x_1^2 + x_2^2 + \cdots + x_n^2$$

## Theorem (Witt, 2009)

*Consider the GCPSO$_1$ on* SPHERE. *If $\ell = \Theta(||x^*||/n)$ for the initial solution $x^*$, the runtime until the distance to the optimum is no more than $\varepsilon||x^*||$ is $O(n \log(1/\varepsilon))$ with probability at least $1 - 2^{-\Omega(n)}$ provided that $2^{-n^{O(1)}} \leq \varepsilon \leq 1$.*

Same result as for (1+1) ES using Gaussian mutations in Jägersküpper, 2007.





# PSO: Summary and Open Questions

## Summary

- analysis of Binary PSO and its probabilistic model
- initial results on runtime of GCPSO and convergence of modified PSO
- results on expected first hitting time of $\varepsilon$-ball for Standard PSO & Noisy PSO

## Neighborhood topologies

- ring topology, etc. instead of global best of swarm
- where does a restricted topology help?

## Swarm dynamics

- analyze combined impact of cognitive and social components
- more results on swarms in continuous spaces





# Overview

# Conclusions

## Summary

- Insight into probabilistic models underlying ACO and PSO
- How design choices and parameters affect (bounds on) running times
- How simple ACO algorithms optimize unimodal functions and plateaus
- Results for ACO in combinatorial optimization
- First analyses of basic PSO algorithms in discrete and continuous spaces

## Future Work

- A unified theory of randomized search heuristics?
- More results on multimodal problems
- When and how diversity and slow adaptation help





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**Thank you!**

**Questions?**