Runtime Analysis of Population-based Evolutionary Algorithms[1]

Introductory Tutorial at GECCO 2017

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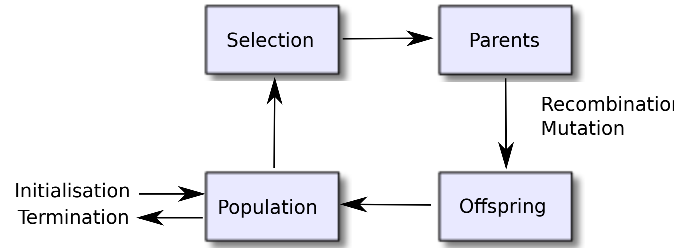
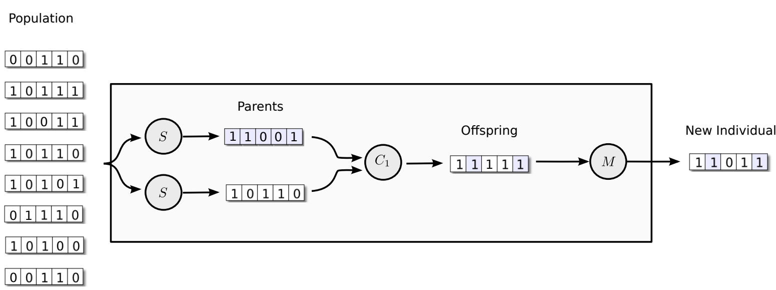
[1]For the latest version of these slides, please go to
http://www.cs.bham.ac.uk/~lehrepk/populations

Evolutionary Algorithms



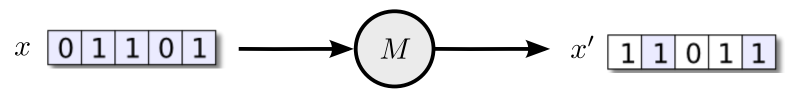
General Scheme for Evolutionary Algorithms[2]



1: **initialise** a population P_0 of λ individuals uniformly at random.
2: **for** $t = 0, 1, 2, \dots$ until termination condition **do**
3: **evaluate** the individuals in population P_t.
4: **for** $i = 1$ to λ **do**
5: **select** two parents from population P_t.
6: **recombine** the two parents.
7: **mutate** the offspring and add it to population P_{t+1}.

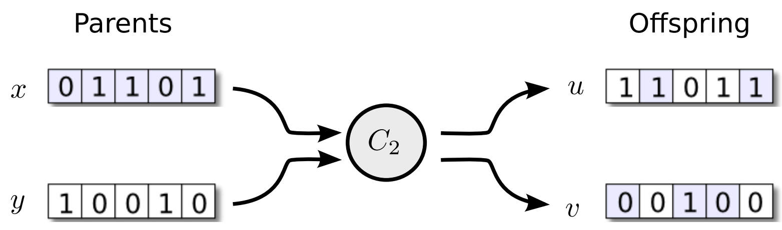
[2]Pseudo-code adapted from Eiben and Smith [2003].

Bitwise Mutation



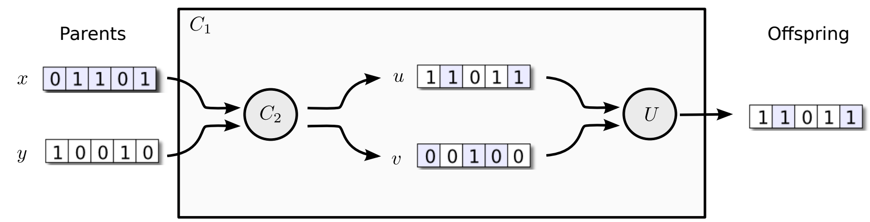
for $i = 1$ **to** n **do**
  with probability χ/n
    $x'_i := 1 - x_i$
  otherwise
    $x'_i := x_i$
return x'

Uniform Crossover - Two Offspring One Offspring



for $i = 1$ **to** n **do**
  with probability $1/2$
    $u_i := x_i$ and $v_i := y_i$
  otherwise
    $u_i := y_i$ and $v_i := x_i$
return u and v.

Uniform Crossover - Two Offspring One Offspring



for $i = 1$ **to** n **do**
  with probability $1/2$
    $u_i := x_i$ and $v_i := y_i$
  otherwise
    $u_i := y_i$ and $v_i := x_i$
return u or v with equal probability.

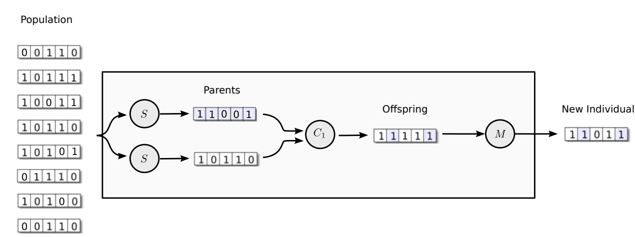
Tournament Selection

Tournament selection with tournament size k

1. Sample uniformly at random with replacement a subset $P' \subseteq P$ of k individuals from population P.
2. Select the individual in P' with highest fitness, with ties broken uniformly at random.

- Often, tournament size $k = 2$ is used.

A Model of Population-based EAs

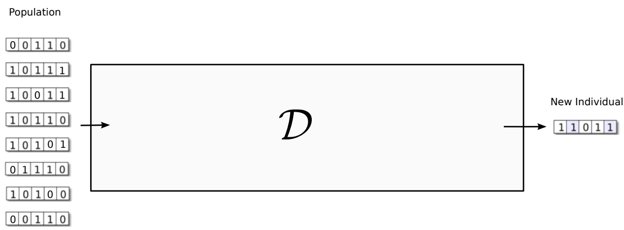


Wide range of evolutionary algorithms...

- selection mechanisms (ranking selection, (μ, λ)-selection, tournament selection, ...)
- fitness models (deterministic, stochastic, dynamic, partial, ...)
- variation operators
- search spaces (e.g. bitstrings, permutations, ...)

We will describe many of these with a general mathematical model.

A Model of Population-based EAs



Require: Search space $\mathcal{X}$ and random operator $\mathcal{D} : \mathcal{X}^\lambda \to \mathcal{X}$

$P_0 \sim \mathrm{Unif}(\mathcal{X}^\lambda)$

for $t = 0, 1, 2, \ldots$ until termination condition **do**

  for $i = 1$ to λ **do**

    $P_{t+1}(i) \sim \mathcal{D}(P_t)$

Outline

Aims and Goals of this Tutorial

- The scope of this tutorial is restricted to
  - population-based evolutionary algorithms, with finite parent– and offspring population sizes > 1,
  - using non-elitist selection mechanisms
- This tutorial will provide an overview of
  - the goals of runtime analysis of EAs
  - selected, generally applicable techniques
- You should attend if you wish to
  - theoretically understand the behaviour and performance of the EAs you design
  - familiarise yourself with some of the techniques used
  - pursue research in the area
- enable you or enhance your ability to
  1. understand theoretically population-dynamics of EAs on different problems
  2. perform time complexity analysis of population-based EAs on common toy problems
  3. have the basic skills to start independent research in the area

Evolutionary Algorithms are Algorithms

Criteria for evaluating algorithms

1. Correctness
   - Does the algorithm always give the correct output?
2. Computational Complexity
   - How much computational resources does the algorithm require to solve the problem?

Same criteria also applicable to evolutionary algorithms

1. Correctness.
   - Discover global optimum in finite time?
2. Computational Complexity.
   - Time (number of function evaluations) most relevant computational resource.

Runtime Analysis of Population-based EAs

Definition

Given any target subset $B(n) \subset \{0,1\}^n$ (e.g. optima), let

$$T_{B(n)} := \min_{t \in \mathbb{N}} \{t\lambda \mid P_t \cap B(n) \neq \emptyset\}$$

be the first time[3] the population contains an individual in $B(n)$.

Problem

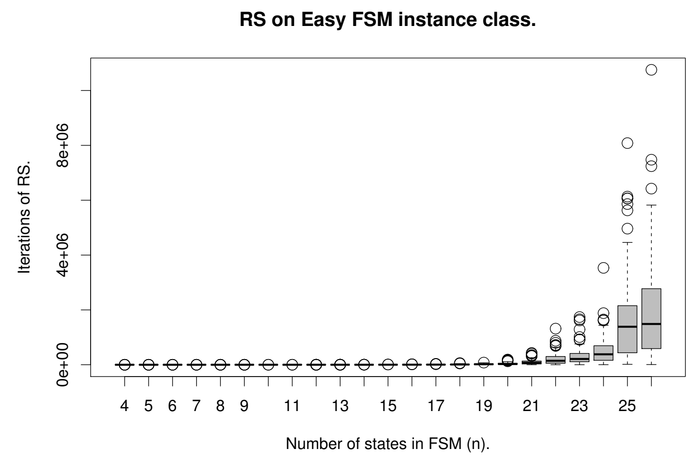
Show how

- $\mathbf{E}\left[T_{B(n)}\right]$ (the expected runtime)
- $\Pr\left(T_{B(n)} \leq t\right)$ (the "success" probability)

depend on the mapping $\mathcal{D}$.

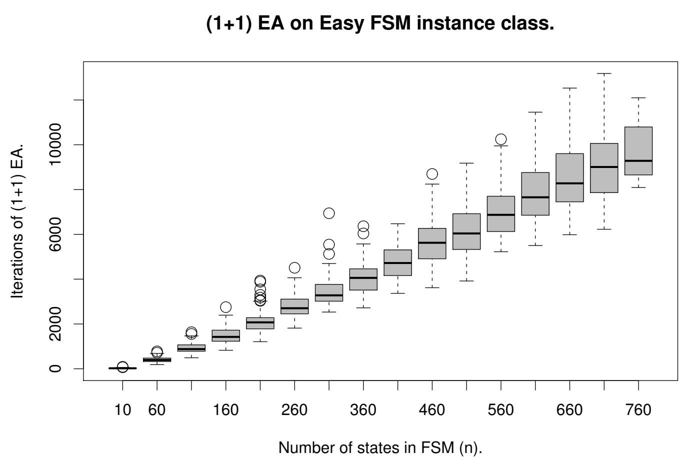
[3] We here count time as the number of search points that have been sampled since the start of the algorithm. For a typical $\mathcal{D}$ that models an EA, this corresponds to the number of times the fitness function is evaluated.

Runtime as a function of problem size



- Exponential $\Longrightarrow$ Algorithm impractical on problem.
- Polynomial $\Longrightarrow$ Possibly efficient algorithm.

Runtime as a function of problem size



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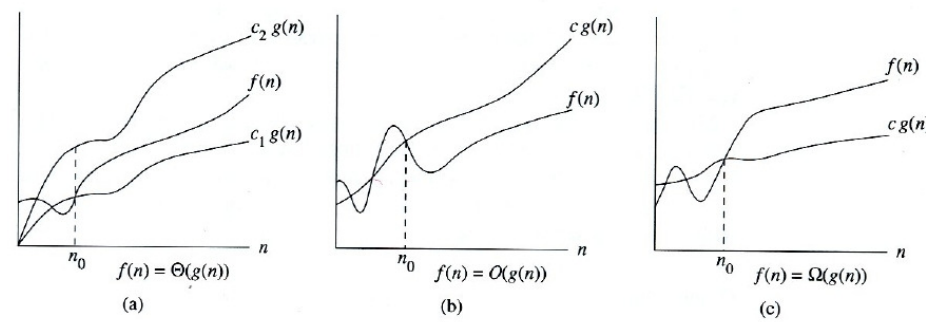
Asymptotic notation



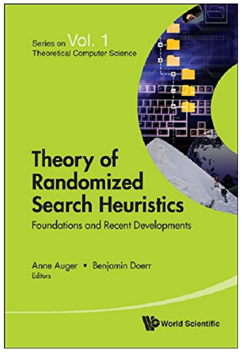
$$f(n) \in O(g(n)) \iff \exists \text{ constants } c, n_0 > 0 \text{ st. } 0 \leq f(n) \leq cg(n)$$

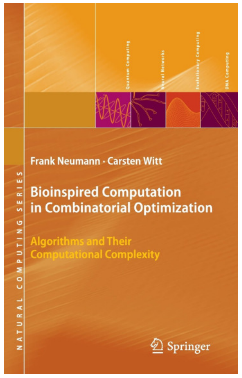
$$f(n) \in \Omega(g(n)) \iff \exists \text{ constants } c, n_0 > 0 \text{ st. } 0 \leq cg(n) \leq f(n)$$

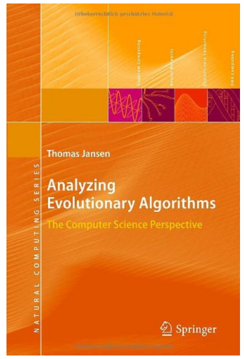
$$f(n) \in \Theta(g(n)) \iff f(n) \in O(g(n)) \text{ and } f(n) \in \Omega(g(n))$$

$$f(n) \in o(g(n)) \iff \lim_{n \to \infty} \frac{f(n)}{g(n)} = 0$$

Runtime Analysis of Evolutionary Algorithms







Level-based Theorem[4]

[4] Corus, Dang, Eremeev, and Lehre [2014] and arXiv:1407.7663

Approaches to Runtime Analysis of Populations

- Infinite population size
- Markov chain analysis He and Yao [2003]
- No parent population, or monomorphic populations
  - (1+1) EA
  - $(1+\lambda)$ EA Jansen, Jong, and Wegener [2005]
  - $(1,\lambda)$ EA Rowe and Sudholt [2012]
- Fitness-level techniques
  - $(1+\lambda)$ EA Witt [2006]
  - $(N+N)$ EAs Chen, He, Sun, Chen, and Yao [2009]
  - non-elitist EAs with unary variation operators Lehre [2011b], Dang and Lehre [2014]
- Classical drift analysis
  - Fitness proportionate selection Neumann, Oliveto, and Witt [2009], Oliveto and Witt [2014, 2015]
- Family trees
  - $(\mu+1)$ EA Witt [2006]
  - $(\mu+1)$ IA Zarges [2009]
- Multi-type branching processes Lehre and Yao [2012]
  - Negative drift theorem for populations Lehre [2011a]
- Level-based analysis Corus, Dang, Eremeev, and Lehre [2014]

Outline - Level-based Theorem[5]

1. Definition of levels of search space
2. Definition of "current level" of population
3. Statement of theorem and its conditions
4. Recommendations for how to apply the theorem
5. Some example applications
6. Derivation of special cases
   - Mutation-only EAs
   - Crossover
   - Mutation-only EAs with uncertain fitness (e.g. noise)

[5] It is out of scope of this tutorial to present the proof of this theorem. The proof uses drift analysis with a distance function that takes into account the current level, as well as the number of individuals above the current level.

Level Partitioning of Search Space $\mathcal{X}$

Definition

$(A_1, \ldots, A_m)$ is a level-partitioning of search space $\mathcal{X}$ if

- $\bigcup_{j=1}^{m} A_j = \mathcal{X}$ (together, the levels cover the search space)
- $A_i \cap A_j = \emptyset$ whenever $i \neq j$ (the levels are nonoverlapping)
- the last level A_m covers the optima for the problem

We will write $A_{\geq j}$ to denote everything in level j and higher, i.e.,

$$A_{\geq j} := \bigcup_{i=j}^{m} A_i.$$

Current level of a population P wrt $\gamma_0 \in (0, 1)$

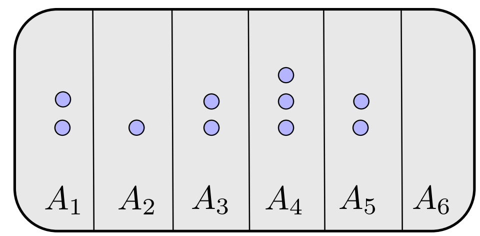
Definition

The unique integer $j \in [m-1]$ such that

$$|P \cap A_{\geq j}| \geq \gamma_0 \lambda > |P \cap A_{\geq j+1}|$$

Example

Current level wrt $\gamma_0 = \frac{1}{2}$ is ....4.



Level-based theorem (informal version)

If the following three conditions are satisfied

(G1) it is always possible to sample above the current level

(G2) the proportion of the population above the current level increases in expectation

(G3) the population size is large enough

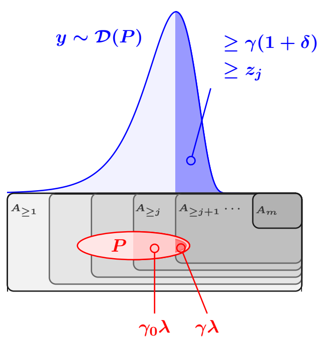
then the expected time to reach the last level cannot be too high.

Level-based Theorem[6] (1/2) (setup)

- Given a level-partitioning $(A_1, \ldots, A_m)$ of $\mathcal{X}$
- $m - 1$ *upgrade probabilities* $z_1, \ldots, z_{m-1} \in (0, 1]$ and $z_{\min} := \min_i z_i$
- a parameter $\delta \in (0, 1)$, and
- a constant $\gamma_0 \in (0, 1)$,

[6] This version of the theorem simplifies some of the conditions at the cost of a slightly less precise bound on the runtime.

Level-based Theorem (2/2) [Corus, Dang, Eremeev, and Lehre, 2014]



If for all populations $P \in \mathcal{X}^\lambda$, an individual $y \sim \mathcal{D}(P)$ has

$$\Pr\left(y \in A_{\geq j+1}\right) \geq z_j, \qquad \text{(G1)}$$
$$\Pr\left(y \in A_{\geq j+1}\right) \geq \gamma(1+\delta), \qquad \text{(G2)}$$

where $j \in [m-1]$ is the current level of population P, i.e.,

$$|P \cap A_{\geq j}| \geq \gamma_0\lambda > |P \cap A_{\geq j+1}| = \gamma\lambda,$$

and the population size λ is bounded from below by

$$\lambda \geq \left(\frac{4}{\gamma_0\delta^2}\right)\ln\left(\frac{128m}{z_{\min}\delta^2}\right), \qquad \text{(G3)}$$

then the algorithm reaches the last level A_m in expected time

$$\mathbf{E}\left[T_{A_m}\right] \leq \left(\frac{8}{\delta^2}\right)\sum_{j=1}^{m-1}\left(\lambda\ln\left(\frac{6\delta\lambda}{4+z_j\delta\lambda}\right)+\frac{1}{z_j}\right).$$

Suggested recipe for application of level-based theorem

1. Find a partition $(A_1, \ldots, A_m)$ of $\mathcal{X}$ that reflects the state of the algorithm, and where A_m consists of all goal states.
2. Find parameters γ_0 and δ and a configuration of the algorithm (e.g., mutation rate, selective pressure) such that whenever $|P \cap A_{\geq j+1}| = \gamma\lambda > 0$, condition (G2) holds, i.e.,
$$\Pr\left(y \in A_{\geq j+1}\right) \geq \gamma(1+\delta)$$
3. For each level $j \in [m-1]$, estimate a lower bound $z_j \in (0,1)$ such that whenever $|P \cap A_{\geq j+1}| = 0$, condition (G1) holds, i.e.,
$$\Pr\left(y \in A_{\geq j+1}\right) \geq z_j$$
4. Calculate the sufficient population size λ from condition (G3).
5. Read off the bound on expected runtime.

Simple Example to Illustrate Theorem

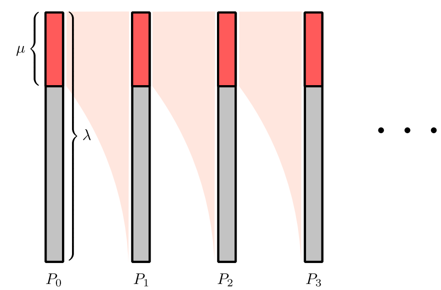
Problem

- search space $\mathcal{X} = \{1, \cdots, m\}$
- fitness function $f(x) = x$ (to be maximised)

Evolutionary Algorithm

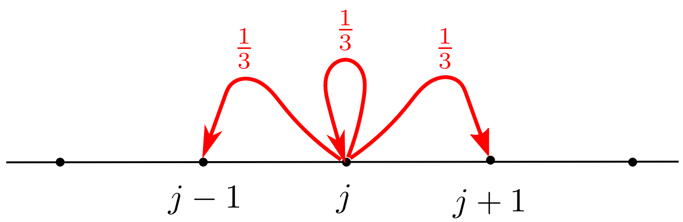
for $t = 0, 1, 2, \ldots$ until termination condition **do**
  for $i = 1$ to λ **do**
    Select a parent x from P_t using (μ, λ)-selection
    Obtain y by mutating x
    Set i-th offspring $P_{t+1}(i) = y$

(μ, λ)-selection mechanism



1. Sort the current population $P = (x_1, \ldots, x_\lambda)$ such that
$$f(x_1) \geq f(x_2) \geq \ldots \geq f(x_\lambda)$$
2. **return** $\text{Unif}(x_1, \ldots, x_\mu)$

A simple mutation operator...



$$\mathbf{Pr}\left(V(x) = y\right) = \begin{cases} \frac{1}{3} & \text{if } y \in \{x-1, x, x+1\} \\ 0 & \text{otherwise.} \end{cases}$$

Step 1: Level-partition

Problem

- search space $\mathcal{X} = \{1, \cdots, m\}$
- fitness function $f(x) = x$ (to be maximised)

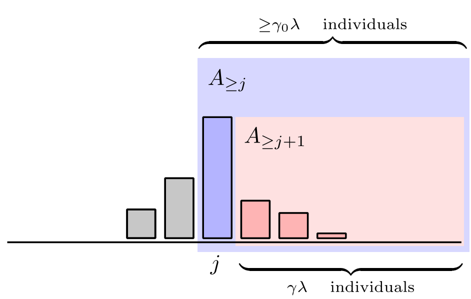
Level-partition of $\mathcal{X}$

$$A_j := \{j\}$$
$$A_{\geq j} = \{j, j+1, \ldots, m\}$$

Properties of a Population at Level j

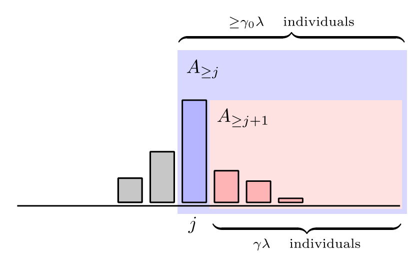
- Assume that the current level of the population P is j, i.e.,

$$\gamma\lambda = |P \cap A_{\geq j+1}| < \gamma_0\lambda \leq |P \cap A_{\geq j}| \qquad (1)$$



- (μ, λ) selects parent u.a.r. among best μ individuals
- by choosing parameter $\gamma_0 := \mu/\lambda$, assumption (1) implies
  - $\mathbf{Pr}\left(\text{select parent in } A_{\geq j}\right) = 1$
  - $\mathbf{Pr}\left(\text{select parent in } A_{\geq j+1}\right) = \frac{\gamma\lambda}{\mu}$

Condition (G2)

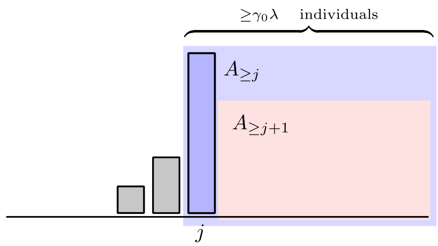


Assuming that $\frac{\lambda}{\mu} = \frac{9}{4} = \frac{1+\frac{1}{2}}{1-\frac{1}{3}}$

$$\begin{aligned} &\mathbf{Pr}\left(y \in A_{\geq j+1}\right) \\ &\quad \geq \mathbf{Pr}\left(\text{select parent in } A_{\geq j+1}\right) \cdot \mathbf{Pr}\left(\text{do not downgrade}\right) \\ &\quad \geq \gamma \cdot \frac{\lambda}{\mu} \cdot \left(1 - \frac{1}{3}\right) = \gamma\left(1 + \frac{1}{2}\right). \\ &\quad \geq \gamma(1+\delta) \end{aligned}$$

$\Longrightarrow$ Condition (G2) satisfied for $\delta = 1/2$.

Condition (G1)



$$\begin{aligned}
&\mathbf{Pr}\left(y \in A_{\geq j+1}\right) \\
&\quad\geq \mathbf{Pr}\left(\text{select parent in } A_j\right) \cdot \mathbf{Pr}\left(\text{upgrade offspring to } A_{\geq j+1}\right) \\
&\quad\geq 1 \cdot \frac{1}{3} \\
&\quad= z_j > 0
\end{aligned}$$

$\Longrightarrow$ Condition (G1) satisfied by choosing $z_j := \frac{1}{3}$ for all $j \in [m]$.

Condition (G3) - Sufficiently Large Population

Recall that $\gamma_0 = \mu/\lambda = 4/9$ and $\delta = 1/2$ and $z_{\text{min}} = 1/3$

$$\begin{aligned}
&\left(\frac{4}{\gamma_0\delta^2}\right)\ln\left(\frac{128m}{z_{\text{min}}\delta^2}\right) \\
&= 36\ln(1536m) \\
&< 36(\ln(m)+8) \leq \lambda
\end{aligned}$$

Hence, choosing $\lambda \geq 36(\ln(m)+8)$ sufficient to satisfy (G3).

Example: Summary

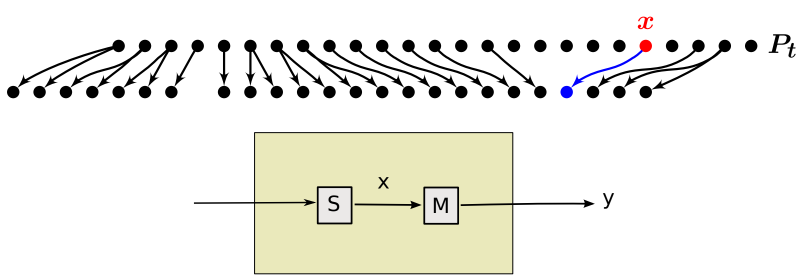
We have shown that if $\lambda \geq 36(\ln(m)+8)$ and $\mu = 4\lambda/9$

- (G1) is satisfied for $z_j = 1/3$ for all $j \in [m-1]$
- (G2) is satisfied for $\delta = 1/2$, and
- (G3) is satisfied

hence, by the level-based theorem, the expected running time of the EA is no more than

$$\begin{aligned}
&\left(\frac{8}{\delta^2}\right)\sum_{j=1}^{m-1}\left(\lambda\ln\left(\frac{6\delta\lambda}{4+z_j\delta\lambda}\right)+\frac{1}{z_j}\right) \\
&\quad< \left(\frac{8}{\delta^2}\right)\sum_{j=1}^{m-1}\left(\lambda\ln\left(\frac{6}{z_j}\right)+\frac{1}{z_j}\right) \\
&\quad= 32\sum_{j=1}^{m-1}(\lambda\ln(18)+3) < 100m\lambda.
\end{aligned}$$

Population-Selection Variation Algorithm (PSVA)



for $t = 0$ to ∞ **do**
  for $i = 1$ to λ **do**
    Sample i-th parent x according to select(P_t)
    Sample i-th offspring $P_{t+1}(i)$ according to mutate(x)

Measuring Selective Pressure

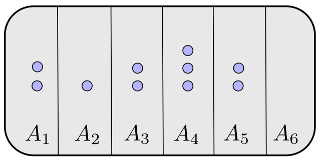
Definition (Cumulative selection probability)

For any population P of λ individuals, where the levels of the individuals are in decreasing order $\ell_0 \geq \ell_1 \geq \cdots \geq \ell_{\lambda-1}$, define for all $\gamma \in (0, \gamma_0)$

$$\zeta(\gamma, P) := \Pr\left(\text{select}(P) \in A_{\geq \ell_{\lceil \gamma\lambda \rceil}}\right),$$

(i.e., prob. of not selecting a worse individual than the $\gamma\lambda$-ranked).

Example



ℓ_0	ℓ_1	ℓ_2	ℓ_3	ℓ_4	ℓ_5	ℓ_6	ℓ_7	ℓ_8	ℓ_9
5	5	4	4	4	3	3	2	1	1

$$\zeta(1/10, P) = \Pr\left(\text{select}(P) \in A_{\geq \ell_1}\right) = \Pr\left(\text{select}(P) \in A_{\geq 5}\right)$$
$$\zeta(3/10, P) = \Pr\left(\text{select}(P) \in A_{\geq \ell_3}\right) = \Pr\left(\text{select}(P) \in A_{\geq 4}\right)$$

Corollary for PSVA

If for any level $j \in [m-1]$ and all search points $x \in A_{\geq j}$,

(C1) $\Pr\left(\text{mutate}(x) \in A_{\geq j+1}\right) \geq s_j \geq s_{\text{min}}$

(C2) $\Pr\left(\text{mutate}(x) \in A_{\geq j}\right) \geq p_0$

and for all non-optimal populations $P \in (\mathcal{X} \setminus A_m)^\lambda$ and $\gamma \in (0, \gamma_0]$

(C3) $\zeta(\gamma, P) \geq \dfrac{(1+\delta)\gamma}{p_0}$

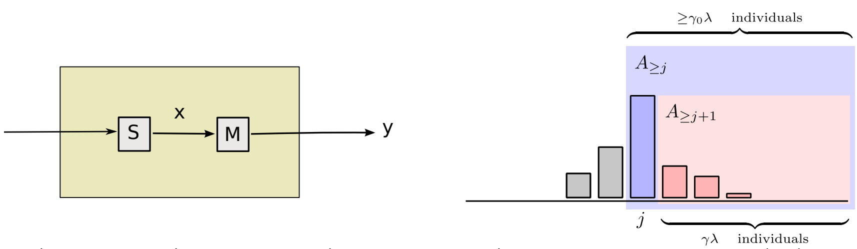
and the population size λ satisfies

(C4) $\lambda \geq \left(\dfrac{4}{\gamma_0 \delta^2}\right) \ln\left(\dfrac{128m}{\gamma_0 s_{\text{min}} \delta^2}\right)$

then the expected time to reach the last level A_m is less than

$$\left(\frac{8}{\delta^2}\right) \sum_{j=1}^{m-1} \left(\lambda \ln\left(\frac{6\delta\lambda}{4 + \gamma_0 s_j \delta\lambda}\right) + \frac{1}{\gamma_0 s_j}\right).$$

Proof of Corollary: (C2) & (C3) $\implies$ (G2)



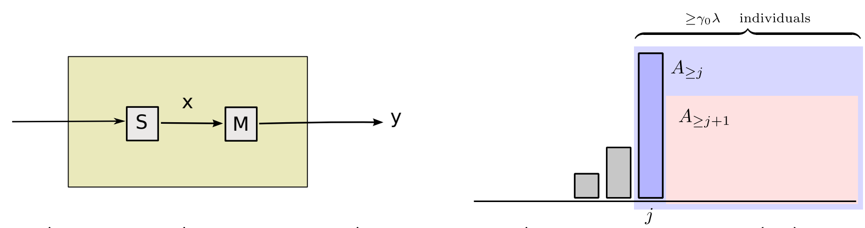
If $|P \cap A_{\geq j}| \geq \gamma_0\lambda > |P \cap A_{\geq j+1}| =: \gamma\lambda$ and $y \sim \mathcal{D}(P)$ then

$$\begin{aligned}
&\Pr\left(y \in A_{\geq j+1}\right) \\
&\quad\geq \Pr\left(x \in A_{\geq j+1}\right) \Pr\left(y \in A_{\geq j+1} \mid x \in A_{\geq j+1}\right)
\end{aligned}$$

(i.e., select x from level $j+1$ and do not downgrade it)

$$\begin{aligned}
&\quad\geq \zeta(\gamma, P) p_0 \\
&\quad\geq \gamma(1+\delta).
\end{aligned}$$

Proof of Corollary: (C1) & (C3) $\implies$ (G1)



If $|P \cap A_{\geq j}| \geq \gamma_0\lambda$ and $|P \cap A_{\geq j+1}| = 0$ and $y \sim \mathcal{D}(P)$

$$\Pr\left(y \in A_{\geq j+1}\right) \geq \Pr\left(x \in A_j\right) \Pr\left(y \in A_{\geq j+1} \mid x \in A_j\right)$$

(i.e., select x from level j and upgrade it)

$$\begin{aligned}
&\geq \zeta(\gamma_0, P) s_j \\
&\geq \gamma_0(1+\delta) s_j / p_0 \\
&= z_j > 0
\end{aligned}$$

Example Application

$$\textsc{LeadingOnes}(x) = \sum_{i=1}^{n} \prod_{j=1}^{i} x_j$$

Partition into $n+1$ levels

$$A_j := \{x \in \{0,1\}^n \mid x_1 = \cdots = x_{j-1} = 1 \land x_j = 0\}$$

Example Application

(μ,λ) EA with bit-wise mutation rate χ/n on LeadingOnes.
For any const. $\delta \in (0,1)$ and large n, no bits mutated with prob.

$$\left(1 - \frac{\chi}{n}\right)^n > \frac{1-\delta}{e^\chi}.$$

If $x \in A_{\geq j}$, $\lambda/\mu > e^\chi \left(\frac{1+\delta}{1-\delta}\right)$ and $\lambda > c'' \ln(n)$ then

(C1) $\mathbf{Pr}\left(\text{mutate}(x) \in A_{\geq j+1}\right) \geq \frac{\chi(1-\delta)}{ne^\chi} \quad =: \quad s_j =: s_{\text{min}}$

(C2) $\mathbf{Pr}\left(\text{mutate}(x) \in A_{\geq j}\right) \geq \frac{1-\delta}{e^\chi} \quad =: \quad p_0$

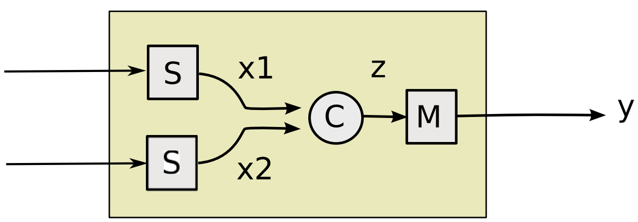
(C3) $\zeta(\gamma, P) \geq \gamma\lambda/\mu > \gamma e^\chi \left(\frac{1+\delta}{1-\delta}\right) \quad = \quad \gamma(1+\delta)/p_0$

(C4) $\lambda > c'' \ln(n) \quad > \quad c \ln(m/s_{\text{min}})$

then

$$\mathbf{E}[T] = \mathcal{O}\left(\sum_{j=1}^{m-1} \lambda \ln\left(\frac{\lambda}{1+s_j\lambda}\right) + \frac{1}{s_j}\right) = \mathcal{O}(n\lambda\ln(\lambda) + n^2)$$

Genetic Algorithms with Crossover



Definition (Genetic Algorithm)

for $t = 0, 1, 2, \ldots$ until termination condition **do**
  for $i = 1$ to λ **do**
    Select parents x_1 and x_2 from population P_t acc. to p_{sel}
    Create z by applying a crossover operator to x_1 and x_2.
    Create y by applying a mutation operator to y.

Corollary for Genetic Algorithms

If for any level $j \in [m-1]$ and all search points $x \in A_{\geq j}$

(C1) $\mathbf{Pr}\left(\text{mutate}(x) \in A_{\geq j+1}\right) \geq s_j \geq s_{\text{min}}$

(C2) $\mathbf{Pr}\left(\text{mutate}(x) \in A_{\geq j}\right) \geq p_0$

and for all $u \in A_{\geq j}$ and $v \in A_{\geq j+1}$

(C3) $\mathbf{Pr}\left(\text{crossover}(u, v) \in A_{\geq j+1}\right) \geq \varepsilon_1$

and for all non-optimal populations $P \in (\mathcal{X} \setminus A_m)^\lambda$ and $\gamma \in (0, \gamma_0]$

(C4) $\zeta(\gamma, P) \geq \gamma\sqrt{\frac{1+\delta}{p_0\varepsilon_1\gamma_0}}$

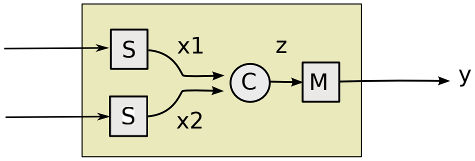
and the population size λ satisfies

(C5) $\lambda \geq \left(\frac{4}{\gamma_0\delta^2}\right) \ln\left(\frac{128m}{\gamma_0\delta^2 s_{\text{min}}}\right)$

then the expected time to reach the last level A_m is less than

$$\left(\frac{8}{\delta^2}\right) \sum_{j=1}^{m-1} \left(\lambda \ln\left(\frac{6\delta\lambda}{4 + \gamma_0 s_j \delta\lambda}\right) + \frac{1}{\gamma_0 s_j}\right).$$

Example application – (μ,λ) GA on Onemax



$$\textsc{OneMax}(x) := \sum_{i=1}^{n} x_i.$$

(μ,λ) Genetic Algorithm (GA)

for $t = 0, 1, 2, \dots$ until termination condition **do**
  for $i = 1$ to λ **do**
    Select a parent x from population P_t acc. to (μ,λ)-selection
    Select a parent y from population P_t acc. to (μ,λ)-selection
    Apply uniform crossover to x and y, i.e. $z :=$ crossover(x, y)
    Create $P_{t+1}(i)$ by flipping each bit in z with probability χ/n.

Theorem
If $\lambda > c\ln(n)$ for a sufficiently large constant $c > 0$, and $\frac{\lambda}{\mu} > 2e^{\chi}(1+\delta)$ for any constant $\delta > 0$, then the expected runtime of (μ,λ) GA on $\textsc{OneMax}$ *is $O(n\lambda)$.*

Partition of Search Space into Levels

Partition into $m := n + 1$ levels $A_0, \dots, A_n$

$$A_j := \{x \in \{0,1\}^n \mid \textsc{OneMax}(x) = j\}$$

Condition (C1) and (C2)

Given any search point $x \in A_{\geq j}$,

- to remain at the same level, it is sufficient to not flip any bits

$$\Pr\left(\text{mutate}(x) \in A_{\geq j}\right) \geq \left(1 - \frac{\chi}{n}\right)^n \geq \frac{1-\delta}{e^{\chi}} =: p_0.$$

- to reach a higher level, it suffices to flip a zero-bit into a one-bit and leave the other bits unchanged, i.e.,

$$\Pr\left(\text{mutate}(x) \in A_{\geq j+1}\right) \geq (n-j)\frac{\chi}{n}\left(1 - \frac{\chi}{n}\right)^{n-1} \geq \frac{\chi(n-j)(1-\delta)}{ne^{\chi}} =: s_j.$$

Example application – (μ,λ) GA on Onemax

If $\lambda/\mu > \dots$ and $\lambda > c\ln(n)$ and $x \in A_{\geq j}$ then

(C1) $\Pr\left(\text{mutate}(x) \in A_{\geq j+1}\right) \geq \frac{\chi(n-j)(1-\delta)}{ne^{\chi}} =: s_j$ ✓

(C2) $\Pr\left(\text{mutate}(x) \in A_{\geq j}\right) \geq \frac{1-\delta}{e^{\chi}} =: p_0$ ✓

and for all $u \in A_{\geq j}$ and $v \in A_{\geq j+1}$

(C3) $\Pr\left(\text{crossover}(u, v) \in A_{\geq j+1}\right) \geq \varepsilon_1 > 0$

and for all non-optimal populations $P \in (\mathcal{X} \setminus A_m)^\lambda$ and $\gamma \in (0, \gamma_0]$

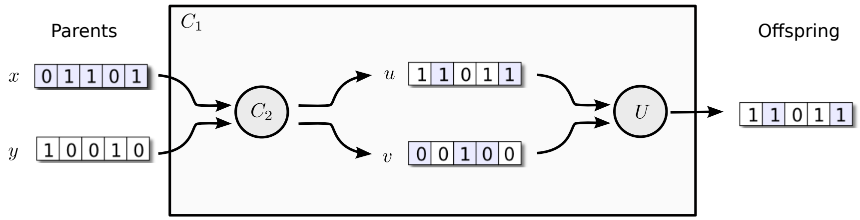
(C4) $\zeta(\gamma, P) \geq \frac{\gamma\lambda}{\mu} \geq \gamma\sqrt{\frac{1+\delta}{p_0\varepsilon_1\gamma_0}}$

and the population size λ satisfies

(C5) $\lambda > c\ln(n) \geq \left(\frac{4}{\gamma_0\delta^2}\right)\ln\left(\frac{128m}{\gamma_0\delta^2 s_{\min}}\right)$ ✓

- (C5) holds if the constant $c > 0$ is large enough $(m = n + 1)$
- Remains to show that (C3) and (C4) can be satisfied
  - Need to determine the parameter ε_1.
  - Need to determine a lower bound for the ratio λ/μ.

Condition (C3) – (μ,λ) GA on OneMax



Proof.
Assume that $x \in A_{\geq j+1}$ and $y \in A_{\geq j}$, and w.l.o.g. that $|u| \geq |v|$

$$\begin{aligned} 2j+1 &\leq |x| + |y| \\ &= |u| + |v| \\ &\leq 2|u|. \end{aligned}$$

Therefore $\Pr\left(u \in A_{\geq j+1}\right) = 1$ and

$$\Pr\left(\text{crossover}(x,y) \in A_{\geq j+1} \mid x \in A_{\geq j+1} \text{ and } y \in A_{\geq j}\right) \geq \frac{1}{2} =: \varepsilon.$$

Example application – (μ,λ) GA on Onemax

If $\lambda/\mu > 2e^{\chi}\left(\frac{1+\delta}{1-\delta}\right)$ for any const. $\delta > 0$, and $\lambda > c\ln(n)$

(C1) $\Pr\left(\text{mutate}(x) \in A_{\geq j+1}\right) \geq \frac{\chi(n-j)(1-\delta)}{ne^{\chi}} =: s_j$ ✓

(C2) $\Pr\left(\text{mutate}(x) \in A_{\geq j}\right) \geq \frac{1-\delta}{e^{\chi}} =: p_0$ ✓

(C3) $\Pr\left(\text{crossover}(u,v) \in A_{\geq j+1}\right) > 1/2 =: \varepsilon_1 > 0$ ✓

(C4) $\beta(\gamma) \geq \frac{\gamma\lambda}{\mu} \geq \gamma\sqrt{\frac{1+\delta}{p_0\varepsilon_1\gamma_0}}$ ✓

(C5) $\lambda > c\ln(n) \geq \left(\frac{4}{\gamma_0\delta^2}\right)\ln\left(\frac{128m}{\gamma_0\delta^2 s_{\text{min}}}\right)$ ✓

We have all the necessary parameters, and would like to find a simple expression for the expected runtime

$$\left(\frac{8}{\delta^2}\right)\left(\lambda\sum_{j=0}^{n-1}\ln\left(\frac{6\delta\lambda}{4+\gamma_0 s_j\delta\lambda}\right) + \sum_{j=0}^{n-1}\frac{1}{\gamma_0 s_j}\right).$$

Bounding the first term (first attempt, imprecise)

$$\sum_{j=0}^{n-1}\ln\left(\frac{6\delta\lambda}{4+\gamma_0 s_j\delta\lambda}\right) < \sum_{j=0}^{n-1}\ln\left(\frac{6\delta\lambda}{4}\right) = \mathcal{O}(n\ln(\lambda)).$$

- This upper bound is imprecise because it does not exploit that the upgrade probabilities s_j are large for small j.

Bounding the first term (second attempt, more precise)

$$\sum_{j=0}^{n-1}\ln\left(\frac{6\delta\lambda}{4+\gamma_0 s_j\delta\lambda}\right) < \sum_{j=0}^{n-1}\ln\left(\frac{6}{\gamma_0 s_j}\right)$$

using $\ln(a)+\ln(b) = \ln(ab)$ and defining $c := \frac{6e^{\chi}}{\gamma_0(1-\delta)\chi}$

$$= \ln\left(\prod_{j=0}^{n-1}\frac{cn}{n-j}\right) = \ln\left(\frac{(cn)^n}{n!}\right)$$

and using the lower bound $n! > (n/e)^n$

$$< \ln\left(\frac{(cn)^n e^n}{n^n}\right) = n\ln(ec) = \mathcal{O}(n).$$

Bounding the second term

Recall the definition of the n-th Harmonic number

$$H_n := \sum_{i=1}^{n} \frac{1}{i} = \mathcal{O}(\ln(n)).$$

The second term can therefore be bounded as

$$\sum_{j=0}^{n-1} \frac{1}{\gamma_0 s_j} = \mathcal{O}\left(\sum_{j=0}^{n-1} \frac{n}{n-j}\right) = \mathcal{O}(n \ln(n))$$

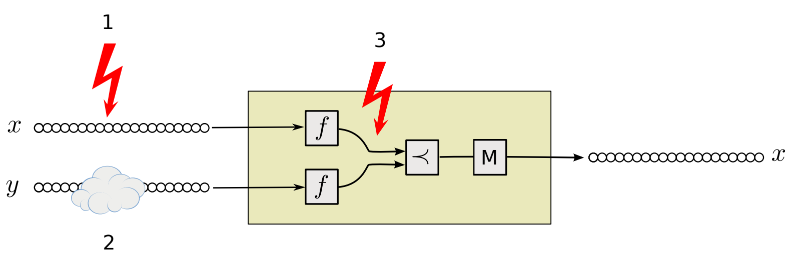
Final result

Theorem

If $\lambda > c\ln(n)$ for a sufficiently large constant $c > 0$, and $\frac{\lambda}{\mu} > 2e^{\chi}(1+\delta)$ for any constant $\delta > 0$, then the expected runtime of (μ,λ) GA on OneMax *is*

$$\left(\frac{8}{\delta^2}\right)\left(\lambda \sum_{j=0}^{n-1} \ln\left(\frac{6\delta\lambda}{4+\gamma_0 s_j \delta\lambda}\right) + \sum_{j=0}^{n-1} \frac{1}{\gamma_0 s_j}\right)$$
$$= \mathcal{O}(n\lambda) + \mathcal{O}(n \ln n) = \mathcal{O}(n\lambda).$$

Uncertainty in Comparison-based PSVAs



Sources of uncertainty

1. Droste noise model (Droste, 2004)
2. Partial evaluation
3. Noisy fitness (Prügel-Bennet, Rowe, Shapiro, 2015)

Sufficient with mutation rate $\delta/(3n)$ and

$$\Pr(x \text{ choosen} \mid f(x) > f(y)) \geq \frac{1}{2} + \delta \quad \text{with } 1/\delta \in \text{poly}(n)$$

(Dang & Lehre, GECCO 2014 & FOGA 2015)

Lower Bounds

Lower Bounds

Problem

Consider a sequence of populations $P_1, \ldots$ over a search space $\mathcal{X}$, and a target region $A \subset \mathcal{X}$ (e.g., the set of optimal solutions), let

$$T_A := \min\{ \lambda t \mid P_t \cap A \neq \emptyset \}$$

We would like to prove statements on the form

$$\Pr(T_A \leq t(n)) \leq e^{-\Omega(n)}. \qquad (2)$$

- i.e., with overwhelmingly high probability, the target region A has not been found in $t(n)$ evaluations
- lower bounds often harder to prove than upper bounds
- will present an easy to use method that is applicable in many situations

Reproductive rate[7]

Definition

For any population $P = (x_1, \ldots, x_\lambda)$ let $p_{\text{sel}}(x_i)$ be the probability that individual x_i is selected from the population P

- The reproductive rate of individual x_i is $\lambda \cdot p_{\text{sel}}(x_i)$.
- The reproductive rate of a selection mechanism is bounded from above by α_0 if

$$\forall P \in \mathcal{X}^\lambda, \ \forall x \in P \quad \lambda \cdot p_{\text{sel}}(x) \leq \alpha_0$$

(i.e., no individual gets more than α_0 offspring in expectation)

[7] The reproductive rate of an individual as defined here corresponds to the notion of "fitness" as used in the field of population genetics, i.e., the expected number of offspring.

Algorithms considered for lower bounds

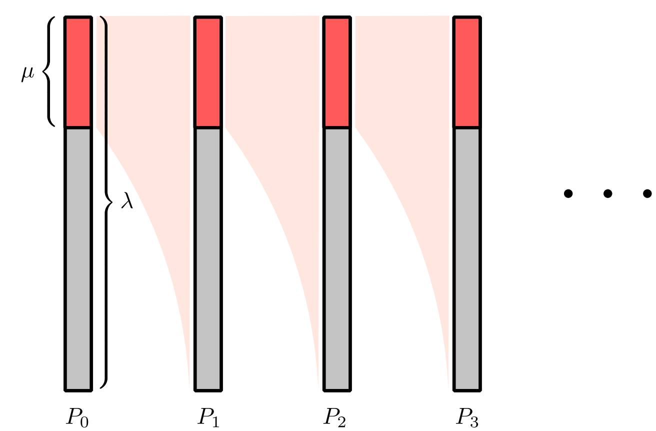
Definition (Non-elitist EA with selection and mutation)

for $t = 0, 1, 2, \ldots$ until termination condition **do**
 for $i = 1$ to λ **do**
  Select parent x from population P_t according to p_{sel}
  Flip each position in x independently with probability χ/n.
  Let the i-th offspring be $P_{t+1}(i) := x$.
  (i.e., create offspring by mutating the parent)

Assumptions

- population size $\lambda \in \text{poly}(n)$, i.e. not exponentially large
- bitwise mutation with probability χ/n, but no crossover.
- results hold for any non-elitist selection scheme p_{sel} that satisfy some mild conditions to be described later.

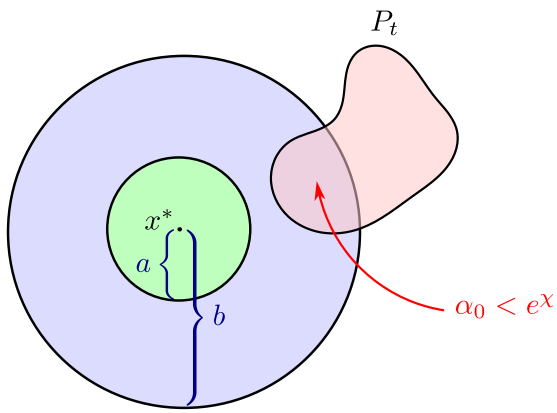
(μ, λ)-selection mechanism



Probability of selecting i-th individual is $p_i \in \{0, \frac{1}{\mu}\}$.

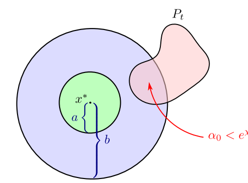
- reproductive rate bounded by $\alpha_0 = \lambda/\mu$

Negative Drift Theorem for Populations (informal)



If individuals closer than b of target has reproductive rate $\alpha_0 < e^{\chi}$,
then it takes exponential time $e^{c(b-a)}$ to reach within a of target.

Negative Drift Thm. for Populations [Lehre, 2011a]



Consider the non-elitist EA with

- population size $\lambda = \text{poly}(n)$
- bitwise mutation rate χ/n for $0 < \chi < n$

let $T := \min\{t \mid H(P_t, x^*) \leq a\}$ for any $x^* \in \{0,1\}^n$.

If there are constants $\alpha_0 \geq 1$, $\delta > 0$ and integers $a(n)$ and $b(n) < \frac{n}{\chi}$ where $b(n) - a(n) = \omega(\ln n)$, st.

(C1) If $a(n) < H(x, x^*) < b(n)$ then $\lambda \cdot p_{\text{sel}}(x) \leq \alpha_0$.

(C2) $\psi := \ln(\alpha_0)/\chi + \delta < 1$

(C3) $b(n) < \min\left\{\frac{n}{5}, \frac{n}{2}\left(1 - \sqrt{\psi(2-\psi)}\right)\right\}$

then there exist constants $c, c' > 0$ such that

$$\Pr\left(T \leq e^{c(b(n)-a(n))}\right) \leq e^{-c'(b(n)-a(n))}.$$

The worst individuals have low reproductive rate

Lemma

Consider any selection mechanism which for $x, y \in P$ satisfies

(a) *If $f(x) > f(y)$, then $p_{sel}(x) > p_{sel}(y)$.*
(selection probabilities are monotone wrt fitness)

(b) *If $f(x) = f(y)$, then $p_{sel}(x) = p_{sel}(y)$.*
(ties are drawn randomly)

If $f(x) = \min_{y \in P} f(y)$, then $p_{sel}(x) \leq 1/\lambda$.
(individuals with lowest fitness have reproductive rate ≤ 1)

Proof.

- By (a) and (b), $p_{\text{sel}}(x) = \min_{y \in P} p_{\text{sel}}(y)$.
- $1 = \sum_{x \in P} p_{\text{sel}}(x) \geq \lambda \cdot \min_{y \in P} p_{\text{sel}}(y) = \lambda \cdot p_{\text{sel}}(x)$.

□

Example 1: Needle in the haystack

Definition

$$\text{NEEDLE}(x) = \begin{cases} 1 & \text{if } x = 1^n \\ 0 & \text{otherwise.} \end{cases}$$

Theorem

The optimisation time of the non-elitist EA with any selection mechanism satisfying the properties above[8] on NEEDLE *is at least e^{cn} with probability $1 - e^{-\Omega(n)}$ for some constant $c > 0$.*

[8] From black-box complexity theory, it is known that NEEDLE is hard for all search heuristics (Droste et al 2006).

Example 1: Needle in the haystack (proof[9])

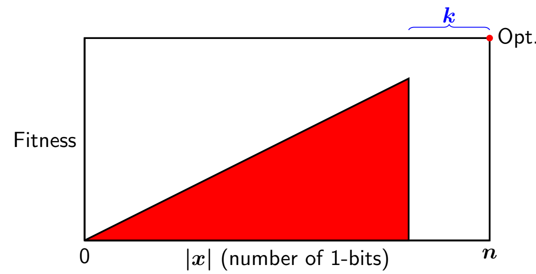
- Apply negative drift theorem with $a(n) := 1$.
- By previous lemma, can choose $\alpha_0 = 1$ for any $b(n)$, hence $\psi = \ln(\alpha)/\chi + \delta = \delta < 1$ for all χ and $\delta < 1$.
- Choosing the parameters $\delta := 1/10$ and $b(n) := n/6$ give

$$\min\left\{\frac{n}{5}, \frac{n}{2}\left(1 - \sqrt{\psi(2-\psi)}\right)\right\} = \frac{n}{5} < b(n).$$

- It follows that $\mathbf{Pr}\left(T \leq e^{c(b(n)-a(n))}\right) \leq e^{-\Omega(n)}$.

[9]For simplicity, we assume that $b(n) \leq n/\chi$.

Exercise: Optimisation time on Jump_k



Recipe

- $a(n) = 1$
- $b(n) = k$
- $\alpha_0 = 1$ as before
- small δ

$$\text{JUMP}_k(x) := \begin{cases} 0 & \text{if } n-k \leq |x| < n, \\ |x| & \text{otherwise.} \end{cases}$$

When the best individuals have low reproductive rate

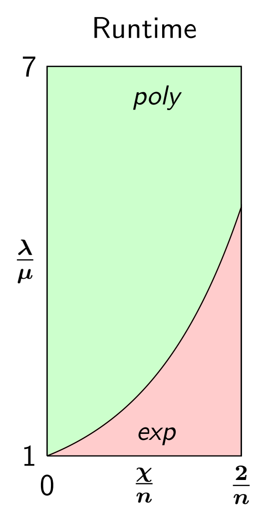
Remark

- The negative drift conditions hold trivially if $\alpha_0 < e^{\chi}$ holds for all individuals.

Example (Insufficient selective pressure)

Selection mechanism	Parameter settings
Linear ranking selection	$\eta < e^{\chi}$
k-tournament selection	$k < e^{\chi}$
(μ,λ)-selection	$\lambda < \mu e^{\chi}$
Any in cellular EAs	$\Delta(G) < e^{\chi}$

Mutation-selection balance



Example

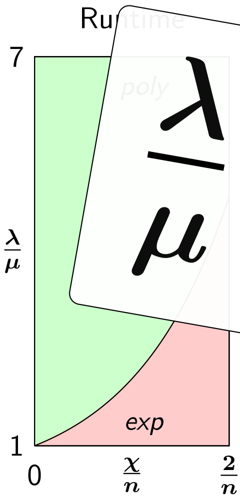
The runtime T of a non-elitist EA with

- (μ, λ)-selection
- bit-wise mutation rate χ/n
- population size $\lambda > c\log(n)$

on LEADINGONES has expected value

$$\mathbf{E}[T] = \begin{cases} e^{\Omega(n)} & \text{if } \lambda < \mu e^{\chi} \\ O(n\lambda\ln(\lambda) + n^2) & \text{if } \lambda > \mu e^{\chi} \end{cases}$$

Mutation-selection balance



$$\frac{\lambda}{\mu} > e^{\chi}$$

Example

The runtime T of a non-elitist EA with

- (μ, λ)-selection
- bit-wise mutation rate χ/n
- population size $\lambda > c \log(n)$

on LeadingOnes has expected value

$$\mathbf{E}[T] = \begin{cases} e^{\Omega(n)} & \text{if } \lambda < \mu e^{\chi} \\ O(n\lambda \ln(\lambda) + n^2) & \text{if } \lambda > \mu e^{\chi} \end{cases}$$

Other Example Applications

Expected runtime of EA with bit-wise mutation rate χ/n

Selection Mechanism	High Selective Pressure	Low Selective Pressure
Fitness Proportionate	$\nu > f_{\max} \ln(2e^{\chi})$	$\nu < \chi/\ln 2$ and $\lambda \geq n^3$
Linear Ranking	$\eta > e^{\chi}$	$\eta < e^{\chi}$
k-Tournament	$k > e^{\chi}$	$k < e^{\chi}$
(μ, λ)	$\lambda > \mu e^{\chi}$	$\lambda < \mu e^{\chi}$
Cellular EAs		$\Delta(G) < e^{\chi}$
OneMax	$O(n\lambda)$	$e^{\Omega(n)}$
LeadingOnes	$O(n\lambda \ln(\lambda) + n^2)$	$e^{\Omega(n)}$
Linear Functions	$O(n\lambda \ln(\lambda) + n^2)$	$e^{\Omega(n)}$
r-Unimodal	$O(r\lambda \ln(\lambda) + nr)$	$e^{\Omega(n)}$
Jump_r	$O(n\lambda + (n/\chi)^r)$	$e^{\Omega(n)}$

Fitness proportional selection + crossover Oliveto and Witt [2014, 2015]

Definition (Simple Genetic Algorithm (SGA) (Goldberg 1989))

for $t = 0, 1, 2, \ldots$ until termination condition **do**
  for $i = 1$ to λ **do**
    Select two parents x and y from P_t proportionally to fitness
    Obtain z by applying uniform crossover to x and y with $p = 1/2$
    Flip each position in z independently with $p = 1/n$.
    Let the i-th offspring be $P_{t+1}(i) := x$.
    (i.e., create offspring by crossover followed by mutation)

Application to OneMax

Expected Behaviour

- Backward drift due to mutation close to the optimum
- no positive drift due to crossover
- selection too weak to keep positive fluctuations

Difficulties When Introducing Crossover:

- Variance of offspring distribution
- # flipping bits due to mutation Poisson-distributed → variance $O(1)$
- # of one-bits created by crossover binomially distributed according to Hamming distance of parents and $1/2$ → deviation $\Omega(\sqrt{n})$ possible

Negative Drift Theorem With Scaling

Let X_t, $t \geq 0$, random variable describing a stochastic process over finite state space $S \subseteq \mathbb{R}$;

If there $\exists$ interval $[a, b]$ and, possibly depending on $\ell := b - a$, bound $\epsilon(\ell) > 0$ and scaling factor $r(\ell)$ st.

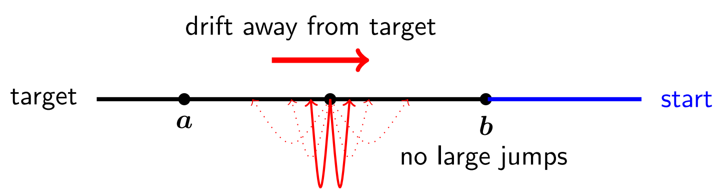
(C1) $E(X_{t+1} - X_t \mid X_0, \dots, X_t \;\wedge\; a < X_t < b) \;\geq\; \epsilon$,

(C2) $\mathrm{Prob}(|X_{t+1} - X_t| \geq jr \mid X_0, \dots, X_t \;\wedge\; a < X_t) \;\leq\; e^{-j}$ for $j \in \mathbb{N}_0$,

(C3) $1 \leq r \leq \min\{\epsilon^2 \ell, \sqrt{\epsilon\ell/(132\log(\epsilon\ell))}\}$.

then

$$\Pr\left(T \leq e^{\epsilon\ell/(132r^2)}\right) = O(e^{-\epsilon\ell/(132r^2)}).$$



Diversity

X_t: # individuals with 1 in some fixed position at time t

Assume uniform selection (and no mutation). Then:

- The probability crossover produces an individual with 1 in the fixed position is $(X_t = k)$:
- $\frac{k}{\mu} \cdot \frac{k}{\mu} + 2 \cdot \frac{1}{2} \cdot \frac{k(\mu-k)}{\mu^2} = \frac{k}{\mu}$
- $\{X_t\} \approx B(\mu, k/\mu) \rightsquigarrow E(X_t \mid X_{t-1} = k) = k$ (martingale)
- But random fluctuations $\rightsquigarrow$ absorbing state 0 or μ due to the variance ($E(T_{0 \vee \mu}) = O(\mu \log \mu)$ [drift analysis]).
- Progress by crossover is at most $n^{1/2+\epsilon}$ w.o.p. (Chernoff Bounds when ones are $n/2$).
- If $\mu \leq n^{1/2-\epsilon}$ a bit has converged to 0 before optimum is found w.o.p.

Diversity

X_t: # individuals with 1 in some fixed position at time t

Assume uniform selection (and no mutation). Then:

- The probability crossover produces an individual with 1 in the fixed position is $(X_t = k)$:
- $\frac{k}{\mu} \cdot \frac{k}{\mu} + 2 \cdot \frac{1}{2} \cdot \frac{k(\mu-k)}{\mu^2} = \frac{k}{\mu}$
- $\{X_t\} \approx B(\mu, k/\mu) \rightsquigarrow E(X_t \mid X_{t-1} = k) = k$ (martingale)
- But random fluctuations $\rightsquigarrow$ absorbing state 0 or μ due to the variance

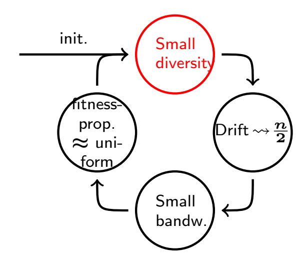
Compare fitness-prop. and uniform selection:

- Basically no difference for small population *bandwidth* (difference of best and worst OneMax-value in pop.)
- $E(X_t \mid X_{t-1} = k) = k \pm 1/(7\mu)$

Result

Let $\mu \leq n^{1/8-\epsilon}$ for an arbitrarily small constant $\epsilon > 0$. Then with probability $1 - 2^{-\Omega(n^{\epsilon/9})}$, the SGA on OneMax does not create individuals with more than $(1+c)\frac{n}{2}$ or less than $(1-c)\frac{n}{2}$ one-bits, for arbitrarily small constant $c > 0$, within the first $2^{n^{\epsilon/10}}$ generations. In particular, it does not reach the optimum then.

Overall Proof Structure



Not a loop, but in each step only exponentially small failure prob.

Steady-state (μ+1) GA

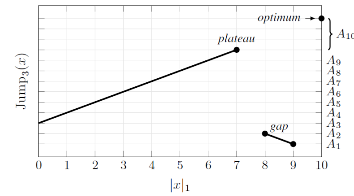
Definition ((μ+1) GA)

$P_0 \leftarrow \mu$ individuals, uniformly at random from $\{0,1\}^n$
for $t = 1, 2, \ldots$ until termination condition **do**
  Select x and y from P_t unif. at random with replacement
  Obtain z by applying uniform crossover to x and y with $p = 1/2$
  Mutate each position in z independently with $p = c/n$
  Select one element from P with lowest fitness and remove it.

Summary

- Runtime analysis of evolutionary algorithms
  - mathematically rigorous statements about EA performance
  - most previous results on simple EAs, such as (1+1) EA
  - special techniques developed for population-based EAs
- Level-based method Corus et al. [2014]
  - EAs analysed from the perspective of EDAs
  - Upper bounds on expected optimisation time
  - Example applications include crossover and noise
- Negative drift theorem Lehre [2011a]
  - reproductive rate vs selective pressure
  - exponential lower bounds
  - mutation-selection balance
- Diversity + Bandwidth analysis for fitness proportional selection
  - analysis of crossover
  - low selection pressure
  - exponential lower bounds
- Speed-up via crossover for steady state GAs to escape local optima

Crossover allows faster escape from local optima Dang, Friedrich, Kötzing, Krejca, Lehre, Oliveto, Sudholt, and Sutton [2016]



Expected Runtimes ($k > 2$)

- (μ+1) EA with $p_m = 1/n$: $\Theta(n^k)$ (i.e., no crossover);
- (μ+1) GA with $p_m = 1/n$: $O(n^{k-1} \log n)$ [$\mu = \Theta(n)$];
- (μ+1) GA with $p_m = (1+\delta)/n$ is $O(n^{k-1})$ [$\mu = \Theta(\log n)$].

The interplay between mutation and crossover can create diversity on the top of the plateau; Then crossover + mutation can take advantage of the diversity to jump more quickly.

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