Fitness-Distance-Ratio Particle Swarm Optimization: Stability Analysis

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ABSTRACT

At present the fitness-distance-ratio particle swarm optimizer (FDR-PSO) has undergone no form of theoretical stability analysis. This paper theoretically derives the conditions necessary for order-1 and order-2 stability, under the well known stagnation assumption. Since it has been shown that particle stability has a meaningful impact on PSO's performance, it is important for PSO practitioners to know the actual criteria for particle stability. This paper validates its theoretical findings against an assumption free FDR-PSO algorithm. This empirical validation is necessary for a truly accurate representation of FDR-PSO's stability criteria.

CCS CONCEPTS

•Computing methodologies → Artificial intelligence; •Theory of computation → *Theory of randomized search heuristics;*

KEYWORDS

Particle swarm optimization, stability analysis, theory

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1 INTRODUCTION

Particle swarm optimization (PSO) is a well known stochastic population-based search algorithm originally developed by Kennedy and Eberhart [24]. Since PSO's inception it has be effectively applied to solve numerous real world optimization problems [30]. Given PSO's popularity there have been numerous alterations proposed to PSO algorithms in an attempt to improve performance. The proposed alterations vary from changing the construction of PSO's underlying social network structure to more fundamental changes to the PSO's update equations. Detailed discussion of some the numerous PSO variants can be found in the review articles [2, 3, 5, 20, 33].

While the original PSO has undergone thorough theoretical analysis [6, 7, 13, 14, 17, 18, 21, 22, 26–28, 31, 32, 35, 36], most PSO variants have never undergone theoretical analysis, which makes it difficult for PSO practitioners to make informed choices when utilizing the PSO variant.

In this paper, the early PSO variant, fitness-distance-ratio PSO (FDR-PSO) [29] is analyzed theoretically for the first time. Specifically, the criteria necessary for FDR-PSO's particles to converge to a point in expectation (order-1 stability), and a constant variance (order-2 stability) are derived. It has been shown that PSO particle stability (order-1 and order-2) has a substantial impact of performance [11]. Specifically, it was shown in [11] that parameter configurations that resulted in particle instability almost always caused PSO to perform worse than random search. Given the relationship between particle stability and performance it is important to understand the criteria that will ensure particle stability in PSO variants.

The theoretically derived region for particle stability of FDR-PSO is also empirically validated utilizing the assumption free methodology for convergent region validation, as presented in [8, 10], and used in [9, 12].

A brief description of FDR-PSO is given in section 2. The theoretical derivation of FDR-PSO's order-1 and order-2 stable regions are presented in section 3, along with a discussion on the influence of FDR-PSO's time dependent inertia coefficient on stability. The experimental setup and results validating the derived stable regions are presented in sections 4 and 5 respectively. Section 6 presents a summary of the findings of this paper.

2 FITNESS-DISTANCE-RATIO PARTICLE SWARM OPTIMIZER

The FDR-PSO was developed by Peram *et al* [29] as an extension of the inertia PSO as proposed by Shi and Eberhart [34]. The authors proposed the introduction of a third attractor, in addition to the usual social and cognitive attractors. The aim of the new attractor is to pull particles towards particles that are both nearby, and of higher fitness. This third attractor is constructed componentwise by selecting components from neighboring particles such that the relative fitness distance ratio is maximized.

The velocity and position update equations of FDR-PSO are defined as follows:

$$\begin{aligned}\boldsymbol{v}_i(t+1) = {} & w\boldsymbol{v}_i(t) + c_1\boldsymbol{r}_1 \otimes (\boldsymbol{p}_i(t) - \boldsymbol{x}_i(t)) \\ & + c_2\boldsymbol{r}_2 \otimes (\boldsymbol{n}_i(t) - \boldsymbol{x}_i(t)) \\ & + c_3\boldsymbol{r}_3 \otimes (\boldsymbol{\kappa}_i(t) - \boldsymbol{x}_i(t)) \qquad (1)\end{aligned}$$

$$\boldsymbol{x}_i(t+1) = \boldsymbol{x}_i(t) + \boldsymbol{v}_i(t+1), \qquad (2)$$

where $\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{r}_3 \sim U(0,1)^d$, and d is the dimension of the problem PSO is attempting to solve. The operator $\otimes$ is used to indicate

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component-wise multiplication of two vectors. The positions $\mathbf{p}_i$ and $\mathbf{n}_i$ are the "best" positions that particle i and particle i's neighborhood of particles have visited. In this paper "best" is defined as the location where a particle has obtained the lowest objective function evaluation. The coefficients c_1, c_2, and w are the cognitive, social, and inertia weights respectively. The new attractor $\boldsymbol{\kappa}_i$ is constructed as

$$\kappa_{i,j} = x_j^* \text{ such that } \max_{\boldsymbol{x}^* \in \mathcal{N}_i} \frac{\theta\left(f(\boldsymbol{x}^*) - f(\boldsymbol{x}_i)\right)}{|x_j^* - x_{i,j}|}, \tag{3}$$

where j indicates the vector component and $\mathcal{N}_i$ is the set of particle i's neighbor's personal best positions.

In the introduction of FDR-PSO the case of $x_j^* = x_{i,j}$ was not explicitly catered for, in this paper if $x_j^* = x_{i,j}$, then of the x^*'s with $x_j^* = x_{i,j}$ the one that maximizes $\theta\left(f(\boldsymbol{x}^*) - f(\boldsymbol{x}_i)\right)$ is selected. The term θ is set to 1 for maximizing problems and -1 for minimization problems. The new coefficient c_3 controls the influence of the attractor $\boldsymbol{\kappa}_i$, however, it was not named in the introduction of FDR-PSO.

The driving feature of PSO and its variants is social interaction, specifically the way in which knowledge about the search space is shared amongst the particles in the swarm. In general, the social topology of a swarm can be viewed as a graph, where the nodes represent particles, and the edges are the allowable direct communication routes. The social topology chosen has a direct impact on the behavior of the swarm as a whole [15, 23, 25]. Some of the most frequently used social topologies are discussed below:

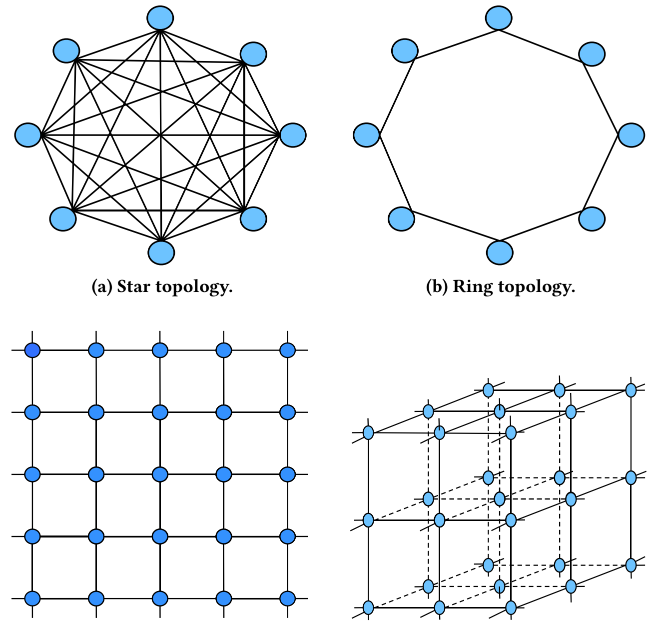
- **Star**: The star topology is one where all the particles in the swarm are interconnected as illustrated in figure 1a. The original implementation of the PSO algorithm utilized the star topology [24]. A PSO utilizing the star topology is commonly referred to as the Gbest PSO.
- **Ring**: The ring topology is one where each particle is in a neighborhood with only two other particles, with the resulting structure forming a ring as illustrated in figure 1b. The ring topology can be generalized to a network structure where larger neighborhoods are used. The resulting algorithm is referred to as the Lbest PSO.
- **Von Neumann**: The Von Neumann topology is one where the particles are arranged in a grid-like structure. The 2-D variant is illustrated in figure 1c, and the 3-D variant is illustrated in figure 1d.

The FDR-PSO algorithm is summarized in algorithm 1.

3 THEORETICAL DERIVATION

This section presents the theoretical derivation of the criteria for order-1 and order-2 stability for the FDR-PSO algorithm, along with the point of particle convergence (in expectation). The impact of the time dependent inertia weight, as suggested for FDR-PSO in [29], on particle stability is also discussed.

Firstly, the stagnation assumption is used, specifically it is assumed that $\boldsymbol{g}_i(t) = \boldsymbol{g}$, $\boldsymbol{n}_i(t) = \boldsymbol{n}$ and $\boldsymbol{\kappa}_i(t) = \boldsymbol{\kappa}$ for all t. The accuracy of the subsequent theoretical derivation is tested without the stagnation assumption in sections 4 and 5. Given that there is no dependence between the vector components of the update equation, it is possible to without loss of generality focus on a one dimensional particle trajectory. The particle subscript i is also dropped for notational convenience.



(a) Star topology. (b) Ring topology.

(c) 2-D von Neumann topology. (d) 3-D von Neumann topology.

Figure 1: Common social topologies

The position update equation (2) can be rewritten into the following form:

$$x_i(t+1) + \alpha_1 x_i(t) + \alpha_2 x_i(t-1) = \alpha_3 \tag{4}$$

where

$$\begin{aligned}
\alpha_1 &= -(1+w) + c_1 r_1 + c_2 r_2 + c_3 r_3 \\
\alpha_2 &= w \\
\alpha_3 &= c_1 r_1 p + c_2 r_2 n + c_3 r_3 \kappa
\end{aligned} \tag{5}$$

The application of the expectation operator to equation (4) results in the following equation:

$$E[x_i(t+1)] + E[\alpha_1]E[x_i(t)] + E[\alpha_2]E[x_i(t-1)] = E[\alpha_3] \tag{6}$$

where:

$$\begin{aligned}
E[\alpha_1] &= -(1+w) + \frac{c_1}{2} + \frac{c_2}{2} + \frac{c_3}{2} \\
E[\alpha_2] &= w \\
E[\alpha_3] &= \frac{c_1 p}{2} + \frac{c_2 n}{2} + \frac{c_3 \kappa}{2}.
\end{aligned} \tag{7}$$

In order to obtain the order-1 region, equation (6) is rewritten into the following matrix form:

$$\begin{vmatrix} x_i(t+1) \\ x_i(t) \end{vmatrix} = M \begin{vmatrix} x_i(t) \\ x_i(t-1) \end{vmatrix} + \begin{vmatrix} E[\alpha_3] \\ 0 \end{vmatrix} \tag{8}$$

where

$$M = \begin{vmatrix} -E[\alpha_1] & -E[\alpha_2] \\ 1 & 0 \end{vmatrix} \tag{9}$$

Algorithm 1 FDR-PSO algorithm

Create and initialize a d-dimensional swarm, $\Omega(0)$, of N particles uniformly within a predefined hypercube.
Let f be the objective function.
Let $\boldsymbol{p}_i$ represent the personal best position of particle i, initialized to $\boldsymbol{x}_i(0)$.
Let $\boldsymbol{n}_i$ represent the neighborhood best position of particle i, initialized to $\boldsymbol{x}_i(0)$.
Initialize $\boldsymbol{v}_i(0)$ to $\mathbf{0}$.
repeat
  for all particles $i = 1, \cdots, N$ **do**
    if $f(\boldsymbol{x}_i) < f(\boldsymbol{y}_i)$ **then**
      $\boldsymbol{p}_i = \boldsymbol{x}_i$
    end if
    for all particles $\hat{i}$ with particle i in their NBD **do**
      if $f(\boldsymbol{p}_i) < f(\boldsymbol{n}_{\hat{i}})$ **then**
        $\boldsymbol{n}_{\hat{i}} = \boldsymbol{p}_i$
        if $f(\boldsymbol{n}_{\hat{i}}) < f(\boldsymbol{g})$ **then**
          $\boldsymbol{g} = \boldsymbol{n}_{\hat{i}}$
        end if
      end if
    end for
  end for
  for all particles $i = 1, \cdots, N$ **do**
    update velocity of particle i using equation (1)
    update position of particle i using equation (2)
  end for
until stopping condition is met

Now, if $\rho(M) < 1$, order-1 stability is obtained [1], where ρ is the spectral radius of the matrix. The two eigenvalues of M are

$$\frac{-E[\alpha_1] \pm \sqrt{E[\alpha_1]^2 - 4E[\alpha_2]}}{2} \tag{10}$$

which means that the simplified conditions needed for $\rho(M) < 1$ are

$$|w| < 1 \text{ and } 0 < c_1 + c_2 + c_3 < 4(w+1) \tag{11}$$

In the presence of order-1 stability a fixed point γ for equation (6) exists. Specifically, γ can be calculated as

$$\gamma = \frac{c_1 p + c_2 n + c_3 \kappa}{c_1 + c_2 + c_3} \tag{12}$$

by setting

$$E[x_i(t+1)] = E[x_i(t)] = E[x_i(t-1)] = \gamma$$

in equation (6) and solving for γ.

In order to obtain the conditions necessary for order-2 stability, the following theorem from Blackwell [4] is use,

Theorem 3.1. *For all PSO algorithms that can be rearranged into the following form:*

$$x_i(t+1) + a x_i(t) + b x_i(t-1) = c(\mathcal{N}_i), \tag{13}$$

were a and b are random variables, and $c(\mathcal{N}_i)$ is a random variable that also depends on stagnant neighborhood position information. If the sequence $(x_i(t))$ is order-1 stable and the following conditions hold,

$$1 + E[a] + E[b] \neq 0 \tag{14}$$

$$1 - E\left[a^2\right] - E\left[b^2\right] + \left(\frac{2E[ab]\,E[a]}{1 + E[b]}\right) > 0 \tag{15}$$

then the sequence $(x_i(t)))$ is also order-2 stable.

Equation (4) is already in the correct form to use theorem 3.1 with $a = \alpha_1$, $b = \alpha_2$, and $c = \alpha_3$. In order to utilize theorem 3.1 a number of expected value calculations are needed, which now follow:

$$E[\alpha_1]^2 = (1+w)^2 - (1+w)(c_1 + c_2 + c_3) + \frac{(c_1 + c_2 + c_3)^2}{4},$$

$$E[\alpha_2] = w, \quad E[\alpha_2]^2 = w^2,$$

$$E[\alpha_1 \alpha_2] = E[\alpha_1 w] = -w(1+w) + \frac{c_1 w}{2} + \frac{c_2 w}{2} + \frac{c_3 w}{2}.$$

In order to calculate $E[\alpha_1^2]$, α_1^2 is first calculated as

$$\begin{aligned}
&\alpha_1^2 \\
&= (-(1+w) + c_1 r_1 + c_2 r_2 + c_3 r_3)^2 \\
&= (1+w)^2 - (1+w)c_1 r_1 - (1+w)c_2 r_2 - (1+w)c_3 r_3 + \\
&- (1+w)c_1 r_1 + c_1^2 r_1^2 + c_1 r_1 c_2 r_2 + c_1 r_1 c_3 r_3 \\
&- (1+w)c_2 r_2 + c_1 r_1 c_2 r_2 + c_2^2 r_2^2 + c_2 r_2 c_3 r_3 \\
&- (1+w)c_3 r_3 + c_1 r_1 c_3 r_3 + c_2 r_2 c_3 r_3 + c_3^2 r_3^2 \\
&= (1+w)^2 - 2(1+w)(c_1 r_1 + c_2 r_2 + c_3 r_3) \\
&+ 2c_1 r_1 c_2 r_2 + 2c_1 r_1 c_3 r_3 + 2c_2 r_2 c_3 r_3 \\
&+ c_1^2 r_1^2 + c_2^2 r_2^2 + c_3^2 r_3^2
\end{aligned}$$

Applying the expectation operator to equation (16) leads to

$$\begin{aligned}
&E[\alpha_1^2] \\
&= (1+w)^2 - 2(1+w)(c_1 E[r_1] + c_2 E[r_2] + c_3 E[r_3]) \\
&+ 2c_1 c_2 E[r_1]E[r_2] + 2c_1 c_3 E[r_1]E[r_3] + 2c_2 c_3 E[r_2]E[r_3] \\
&+ c_1^2 E[r_1^2] + c_2^2 E[r_2^2] + c_3^2 E[r_3^2] \\
&= (1+w)^2 - (1+w)(c_1 + c_2 + c_3) + \frac{c_1 c_2}{2} + \frac{c_1 c_3}{2} + \frac{c_2 c_3}{2} \\
&+ \frac{c_1^2}{3} + \frac{c_2^2}{3} + \frac{c_3^2}{3}
\end{aligned}$$

Using the calculated expected values it is now possible to obtain the conditions necessary for order-2 stability. The first condition obtained from equation (14) leads to

$$\begin{gathered}
1 - (1+w) + \frac{c_1}{2} + \frac{c_2}{2} + \frac{c_3}{2} + w \neq 0 \\
\Longrightarrow \\
c_1 + c_2 + c_3 \neq 0
\end{gathered} \tag{16}$$

The second condition obtained from equation (15) lead to

$$1 - \left((1+w)^2 - 3c(1+w) + \frac{5}{2}c^2 \right) - w^2 + \frac{2w\left(\frac{3}{2}c - (1+w)\right)^2}{1+w} > 0 \tag{17}$$

where $c = c_1 = c_2 = c_3$ for simplicity and based on the recommendation in [29] to use equal coefficients. Using $|w| < 1$ and $c_1 + c_2 + c_3 > 0$, from the order-1 stable region of equation (11), equation (17) can be simplified, by completing the square and rearranging the equation, to

$$c < \frac{6(1-w^2)}{5-4w} \tag{18}$$

In order for a particle's movement to be seen as convergent it should exhibit order-1 and order-2 stability. As such the criteria for particle convergence is merged as

$$c_1 + c_2 + c_3 > 0 \tag{19}$$

$$|w| < 1 \tag{20}$$

$$c_1 + c_2 + c_3 < \frac{18(1-w^2)}{5-4w} \tag{21}$$

when $c_1 = c_2 = c_3$. This merger is possible because the region defined by equation (18) is a subset of the region defined by $0 < c_1 + c_2 + c_3 < 4(w+1)$. The stability criteria are illustrated in figure 2.

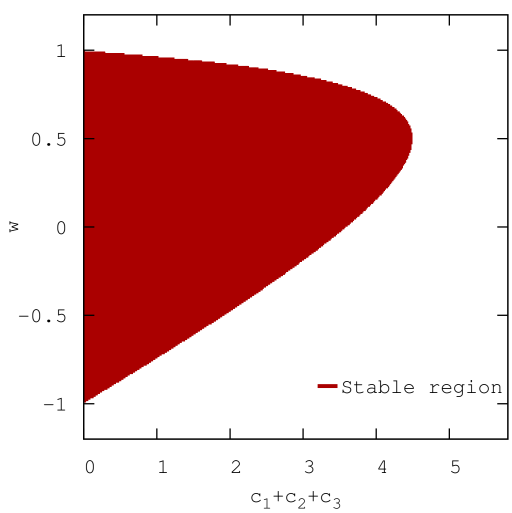


Figure 2: FDR-PSO region of stability

In the proposal of FDR-PSO [29] it was stated that the following time dependent inertia coefficient could be used

$$w(t+1) = \frac{(w(t) - 0.4)(T - t)}{T - 0.4} \tag{22}$$

where T is the maximum number of iterations the PSO will be run for. The immediate question is what the effect of this time dependent inertia coefficient is on particle stability. The relationship between equation (22) and particle stability can be understood by checking how often during a run the criteria of equations (19), (20), and (21) are satisfied. Ideally an explicit version of the recurrence relation in equation (22), would be used. However, despite the simple appearance of equation (22), the type of non-linearity makes finding an usable explicit version nontrivial. Therefore, an empirical approach is used to assess how often the stability criteria are satisfied. In the original work of FDR-PSO it was recommend that $c_1 = c_2 = c_3 = 1$ should be used, but an initial inertia weight, $w(0)$ was not recommended. In table 1, for $T = 5000$, the percentage of a run that FDR-PSO is stable is shown, for initial inertia weights $-1 \leq w(0) \leq 1$, in step sizes of 0.2 and for $0.2 \leq c = c_1 = c_2 = c_3 \leq 1.4$, in step sizes of 0.2.

It is apparent from table 1 that the initial inertia, $w(0)$, has a negligible effect on stability of FDR-PSO during the course of a run. Specifically, for the same value of c, different values of $w(0)$ resulted in approximately the same percentage of time the FDR-PSO algorithm was stable. More interestingly, it is clear from table 1 that the use of the recommend parameters $c_1 = c_2 = c_3 = 1$ in conjunction with equation (22) to control the inertia weight is not actually advisable, as particle behavior is unstable for over 67% of the run. Increasing c to 1.2 is catastrophic for stability, as FDR-PSO is unstable for 98% of the run. A detailed empirical justification for why particle instability is bad for PSO performance can be found in [11]. Similar observations have been made for a collection of self adaptive PSO variants [19]. The amount of time that FDR-PSO is stable increased as the parameter c decreased, which is in line with what would be expected, because the acceptable range of w clearly increases as c decreases as illustrated in figure 2. However, even for low values of c over 31% of the run is unstable. The data from table 1 indicates that the use of equation (22) is not well suited if particle stability is a concern.

Table 1: Percentage of Time FDR-PSO is stable

$w(0)\backslash c$	0.2	0.4	0.6	0.8	1.0	1.2
−1	67.94	63.42	57.20	48.00	32.64	0.02
−0.80	67.94	63.42	57.20	48.00	32.64	0.02
−0.60	67.94	63.42	57.20	48.00	32.64	0.02
−0.40	67.96	63.42	57.20	48.00	32.64	0.02
−0.20	67.96	63.44	57.20	48.00	32.64	0.02
0.00	67.98	63.44	57.22	48.00	32.64	0.02
0.20	67.98	63.46	57.22	48.02	32.64	0.02
0.40	68.00	63.46	57.24	48.02	32.66	0.04
0.60	68.00	63.48	57.24	48.04	32.66	0.04
0.80	68.02	63.48	57.26	48.04	32.68	0.04
1.00	68.02	63.50	57.26	48.06	32.68	0.06

4 EXPERIMENTAL SETUP

This section utilizes a method for empirically investigating the convergence region of PSO variants as proposed by Cleghorn and Engelbrecht [8, 10].

The experiment utilizes a population size of 64, and 5000 iterations. Particle positions were initialized within

$(-100, 100)$ and velocities were initialized to $\mathbf{0}$ [16]. The analysis is done in 5, 10, 20, 30, 40, and 50 dimensions, with the maximum possible distance between particles in the initial search space being 447.214, 632.456, 894.427, 1095.445, 1264.911, and 1414.214 respectively. This maximum distance is referred to as $\Delta_{max}(d)$ from this point forward, where d is the search space dimension. Reported results were bounded at the respective Δ_{max}s to prevent highly divergent parameter configurations from obscuring the data, and to form a classification boundary between stable and unstable parameter configurations.

The empirical measure of convergence used in this paper is:

$$\Delta(t+1) = \frac{1}{k}\sum_{i=1}^{k} \|\boldsymbol{x}_i(t+1) - \boldsymbol{x}_i(t)\|_2. \tag{23}$$

The objective function used is:

$$CF(\boldsymbol{x}) \sim U(-1000, 1000), \tag{24}$$

which was shown to be an effective objective function for stability analysis in [8]. The value of CF, for each $\boldsymbol{x}$ in the domain of CF, is calculated and stored the first time it is required in the execution of the PSO algorithm. The calculated value for each $\boldsymbol{x}$ in the domain of CF remains static after its initial computation. Objective function values are generated anew for each independent run of the PSO algorithm.

The experiment was conducted over the following parameter region:

$$w \in [-1.1, 1.1] \text{ and } c_1 + c_2 + c_3 \in (0, 5.6], \tag{25}$$

where $c_1 = c_2 = c_3$, with a sample point every 0.1 along w and $c_1 + c_2 + c_3$. A total of 1288 sample points from the region defined in equation (25) were used. The results reported in section 5 are derived from 50 independent runs for each sample point.

5 EXPERIMENTAL RESULTS AND DISCUSSION

This section presents the results of the experiments described in section 4.

A snapshot of all parameter configurations' resulting convergence measure values are presented in figures 3, 4, and 5 for the 5000th iteration for FDR-PSO in 5, 20, and 50 dimensions respectively. The reported convergence measures are the maximum recorded over the 50 independent runs. The maximum is used because unstable particle behavior occurring in any run for a given parameter configuration indicates that the parameter configuration is not truly stable.

The number of parameter configurations that empirically agree or disagree with the stable/unstable behavior predicted by the theoretically derived stability region of equations (19), (20), and (21) is presented in table 2. Eight measurements are given in table 2: the number of parameter configurations that are theoretically stable (TS) and unstable (TUS), the number of parameter configurations that where empirically stable (ES) and unstable (EUS), the number of parameter configurations that were found to be empirically stable despite the theory predicting unstable behavior (ES despite TUS), the number of parameter configurations that were found to be empirically unstable despite the theory predicting stable behavior (EUS despite TS), and lastly the percentage error and agreement between

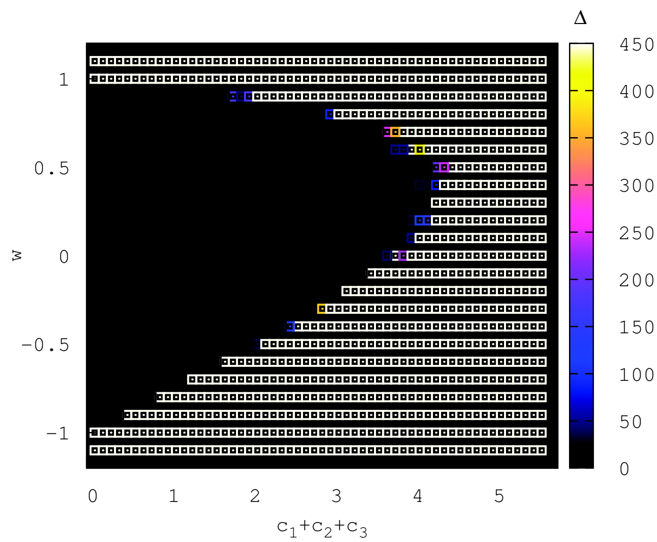


Figure 3: FDR-PSO stability results for 5 dimensions

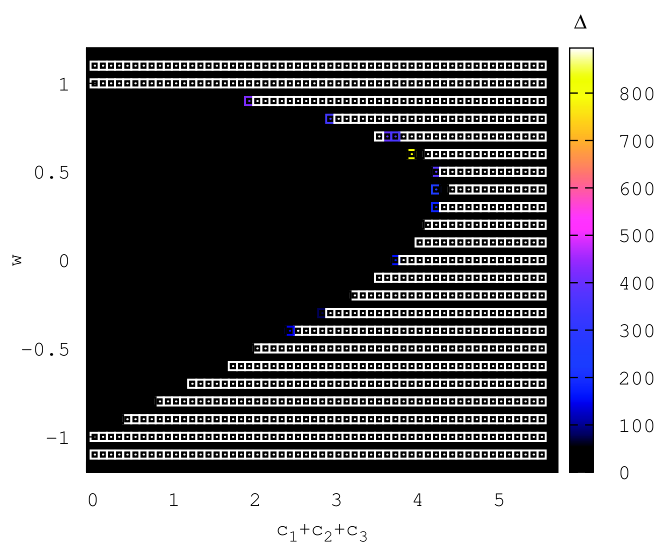


Figure 4: FDR-PSO stability results for 20 dimensions

the theoretical derivation and the empirical findings. A parameter configuration is classified to be stable if the value of the recorded convergence measure of equation (23) is less than $\Delta_{max}(d)$, and unstable if greater than or equal to $\Delta_{max}(d)$, in accordance with the approach of Cleghorn and Engelbrecht [10].

It is clear that the actual stable region remains constant as the dimensionality increases, as there is only a negligible amount of difference between the stable regions for 5, 20 and 50 dimensions as seen in figures 3, 4, and 5 respectively. In all cases both the shape and size of the empirically obtained region in figures 3, 4, and 5 are

Table 2: Theoretical prediction versus empirical findings

Dimension	TS	TUS	ES	EUS	ES despite TUS	EUS despite TS	Error	Agreement
5	553	735	557	731	22	18	3.11%	96.89%
10	553	735	558	730	24	19	3.34%	96.66%
20	553	735	559	729	23	17	3.11%	96.89%
30	553	735	563	725	25	15	3.11%	96.89%
40	553	735	560	728	25	18	3.34%	96.66%
50	553	735	558	730	24	19	3.34%	96.66%

The measurements presented are the number of parameter configurations that are theoretically stable (TS) and unstable (TUS), the number of parameter configurations that where empirically stable (ES) and unstable (EUS), the number of parameter configurations that were found to be empirically unstable despite the theory predicting unstable behavior (ES despite TUS), the number of parameter configurations that were found to be empirically unstable despite the theory predicting stable behavior (EUS despite TS), and lastly the percentage error and agreement between the theoretically derivation and the empirical finding.

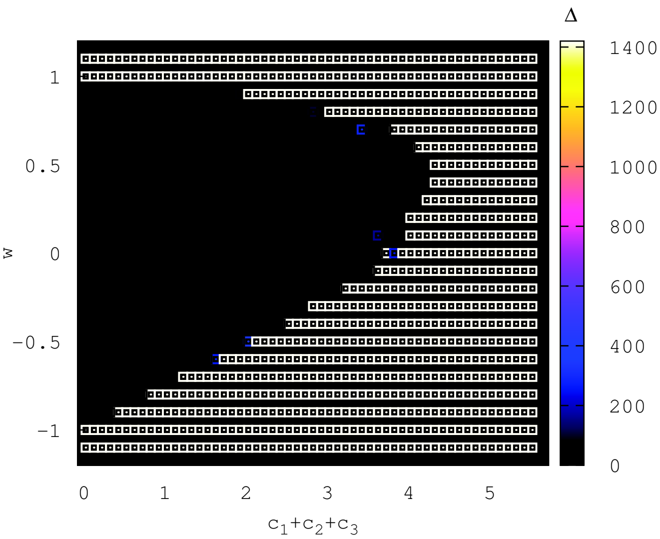


Figure 5: FDR-PSO stability results for 50 dimensions

in agreement with the theoretically predicted region as illustrated in figure 2.

As can be seen in table 2 the empirically obtained region of particle stability is in strong agreement with the theoretically derived region, as defined using equations (19), (20), and (21), with the highest and lowest level of agreement across all test case being 96.89% and 96.66% respectively. There is a very small variance of 0.126% in the percentage of agreement across dimensions, implying that the accuracy of the theoretically derived stability region is consistent across all dimensions.

It is clear that the theoretically derived region for particle stability as described by equations (19), (20), and (21) is an accurate representation of the real world parameter configurations necessary to ensure particle stability.

6 CONCLUSION

This paper theoretically derived the order-1 and order-2 stable regions for FDR-PSO, along with fixed point of particle convergence, in expectation. The derived order-1 and order-2 stable regions can be utilized by PSO practitioners to make an informed choice when selecting control parameters of FDR-PSO. The derived criteria for stability were validated empirically utilizing the method verified by Cleghorn and Engelbrecht [8], where no simplifying assumptions were made on the FDR-PSO algorithm. Given the empirical validation, the theoretical derivation is an accurate representation of FDR-PSO's stability criteria despite the criteria being derived under the stagnation assumption.

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