A Parameterized Runtime Analysis of Randomized Local Search and Evolutionary Algorithm for Max l-Uncut*

Extended Abstract†

ABSTRACT

In the last few years, parameterized complexity has emerged as a new tool to analyze the running time of randomized local search algorithm. However, such analysis are few and far between. In this paper, we do a parameterized runtime analysis of a randomized local search algorithm and a (1 + 1) EA for a classical graph partitioning problem, namely, Max l-Uncut, and its balanced counterpart Max Balanced l-Uncut.

In Max l-Uncut, given an undirected graph $G = (V, E)$, the objective is to find a partition of $V(G)$ into l parts such that the number of uncut edges - edges within the parts - is maximized. In the last few years, Max l-Uncut and Max Balanced l-Uncut are studied extensively from the approximation point of view. In this paper, we analyze the parameterized runtime of a randomized local search algorithm (RLS) for Max Balanced l-Uncut where the parameter is the number of uncut edges. RLS generates a solution of specific fitness in polynomial time for this problem. Furthermore, we design a fixed parameter tractable randomized local search and a (1 + 1) EA for Max l-Uncut and prove that they perform equally well.

CCS CONCEPTS

• **Theory of computation** → *Design and analysis of algorithms*;

KEYWORDS

Max l-Uncut, Max Balanced l-Uncut, Running time Analysis, Randomized Local Search, (1 + 1) EA

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1 INTRODUCTION

Bioinspired computing is a widely used method to deal with NP-hard combinatorial optimization problems. It is always interesting to analyze the convergence of these techniques theoretically. In the past few years, parameterized complexity has emerged as a successful tool to analyze the convergence of randomized local search algorithms and evolutionary algorithms. This approach has been successfully used for Minimum Vertex Cover [7], Maximum Leaf Spanning Tree [6], Euclidean Travelling Salesperson Problem [11], Makespan Scheduling [8] and Weighted Vertex Cover Problem [9]. In this paper, we explore this technique for Max l-Uncut.

Max l-Uncut consists of partitioning of the vertex set of graph G into l parts such that number of uncut edges is maximized. An *uncut edge* is defined as an edge whose both the end points lie within the same part. It is defined mathematically as follows. Let $P = \{A_1, \cdots, A_l\}$ be a partition of $V(G)$. We define the weight of an edge $e \in E(G)$ corresponding to a partition P as follows.

$$w_e(P) = \begin{cases} 1 & \text{if } e \text{ is an uncut edge in } P \\ 0 & \text{otherwise} \end{cases}$$

Then the number of uncut edges for a partition P of $V(G)$ is

$$f_{\text{uncut}}(P) = \sum_{e \in E(G)} w_e(P)$$

Let $\mathcal{P}$ be the set of all partitions of $V(G)$ into l-parts. The objective of Max l-Uncut is to find a partition $P^\star \in \mathcal{P}$ such that $f_{\text{uncut}}(P^\star) \geq f_{\text{uncut}}(P)$, for all $P \in \mathcal{P}$. In Max Balanced l-Uncut, the vertex set of graph G must be partitioned into l almost equal parts, i.e., for a partition $P = \{A_1, \cdots, A_l\}$ of $V(G)$, $||A_i| - |A_j|| \leq 1$, for all $i, j \in [l]$. This problem was motivated from the study of the homophily law of large scale networks [15]. The Max l-Uncut problem is the complement of well studied Min l-Cut problem. It is well known that Min l-Cut is polynomial time solvable when l is fixed [4] and NP-complete when l is given as input [3], though the balanced version of the problem is known to be NP-complete even for fixed l, where $l \geq 2$. Since the problems Min l-Cut and Max l-Uncut (Balanced Min l-Cut and Max Balanced l-Uncut) are complements of each other, the above results hold for Max l-Uncut as well.

In the last few years, Max l-Uncut and Max Balanced l-Uncut have been studied extensively from the point of view of approximation algorithms. Ye and Zhang [14] developed a 0.602-approximation algorithm for Max Balanced 2-Uncut. Wu et al. [13] designed a 0.3456-approximation algorithm for

MAX BALANCED 3-UNCUT problem. Zhang et al. [15] proposed approximation algorithms for MAX l-UNCUT when l is given as input. They developed a randomized $\left(1-\frac{l}{n}\right)^2$-approximation algorithm and a greedy $\left(1-\frac{2(l-1)}{n}\right)$-approximation algorithm.

In this paper, we study MAX BALANCED l-UNCUT where l is fixed and also MAX MULTIPARTITE UNCUT which deals with finding a partition of $V(G)$ into l parts where l is given as part of input such that number of uncut edges is maximized. For MAX BALANCED l-UNCUT, we analyze the parameterized running time of a randomized local search algorithm (RLS), where the parameter is the number of uncut edges. We also study the fitness landscape for MAX BALANCED 2-UNCUT. This problem satisfies Grover's wave equation [5] under the neighborhood operator defined by our RLS strategy which gives that MAX BALANCED 2-UNCUT has elementary landscape. We also prove that RLS generates a solution of specific fitness in polynomial time for this problem, which is also an optimal solution for star graphs. Futhermore, we design a fixed parameter tractable randomized local search and a (1 + 1) EA for MAX MULTIPARTITE UNCUT. These algorithms also obtain a solution of specific fitness in polynomial time.

2 PRELIMINARIES

In parameterized complexity, each problem instance is associated with a parameter k. Formally, a parameterized problem is a language $L \subseteq \Sigma \times \mathbb{N}$, where Σ is fixed finite set of alphabet and $\mathbb{N}$ is a set of natural numbers. For an instance $(x, k) \in \Sigma \times \mathbb{N}$, k is called the parameter. A parameterized problem is called *fixed-parameter tractable* (FPT) if there exists an algorithm which decides whether $(x, k) \in L$ in time bounded by $g(k)|x|^c$. An evolutionary or a randomized local search algorithm is a FPT algorithm for a parameterized problem if its expected optimiztion time, $E(T)$, is upper bounded by $g(k)|x|^c$. For a detailed study of parameterized complexity, we refer the reader to [2].

Throughout the paper, we use following notations. For an undirected graph G, the vertex set of G is denoted by $V(G)$, and the edge set of G by $E(G)$. The number of vertices and edges in the graph G are denoted by n and m respectively.

We define *hamming distance* between two partitions P and P' of $V(G)$ as the minimum number of *moves* required to change one partition to the other. If a vertex is moved from one set to other then it is considered as one move. Let $P = \{A = \{1, 2, 3\}, B = \{4, 5, 6\}\}$ and $P' = \{A = \{1, 2\}, B = \{3, 4, 5, 6\}\}$, then P' can be generated from P by moving a vertex 3 from A to B. Hence, the hamming distance between P and P' is one. Suppose $P' = \{A = \{1, 2, 4\}, B = \{3, 5, 6\}\}$. Here, we move two vertices, 3 is moved from A to B and 4 is moved from B to A. Therefore, the hamming distance between P and P' is two. The partition P' is a hamming neighbor (or 2-hamming neighbor) of P if the hamming distance between them is 1 (or 2).

Let A and B be two sets. $A \setminus B$ is set of elements of A which are not in B. For a set A, $|A|$ denotes the number of elements in the set A. The set $\{1, 2, \cdots, l\}$ is denoted by $[l]$. For a real number $x \in \mathbb{R}$, $|x|$ denotes the absolute value of x. Let $n, l \in \mathbb{R}$. $x = n \bmod l$ denotes the remainder when n is divided by l.

A *landscape* is a triple $(X, \mathcal{N}, \phi)$, where X is called the solution space, $\mathcal{N} : X \to 2^X$ is called the neighborhood operator and $\phi : X \to \mathbb{R}$ is called the objective function. The pair $(X, \mathcal{N})$ defines the configuration space which can be represented in the form of a directed graph ζ, where the vertex set of ζ is defined as $V(\zeta) = X$ and the arc set $E(\zeta)$ consists of an arc (x, y) if $y \in \mathcal{N}(x)$. Let A denote the adjacency matrix of the graph ζ. The *degree matrix* corresponding to the neighborhood operator $\mathcal{N} : X \to 2^X$ is defined as follows. The rows and columns are indexed by vertices in X and

$$D_{x,y} = \begin{cases} |\mathcal{N}(x)| & \text{if } x = y \\ 0 & \text{otherwise} \end{cases}$$

The *Laplacian matrix*, Δ, associated with the neighborhood is defined as $A - D$. The landscape is said to be elementary if there exists a constant b and an eigenvalue λ of $-\Delta$ such that $-\Delta(\phi - b) = \lambda(\phi - b)$, i.e., ϕ is an eigenvector of the Laplacian matrix upto an additive constant. The objective function f is said to be *elementary* when the neighborhood is clear from the context [12]. For the details of the theory of fitness landscape we refer the reader to the survey by Reidy and Stadler [10].

3 RANDOMIZED LOCAL SEARCH FOR MAX BALANCED 2-UNCUT

In this section, we present a randomized local search algorithm, RLS1, for MAX BALANCED 2-UNCUT and analyse its parameterized running time. We also show that RLS1 has elementary landscape for MAX BALANCED 2-UNCUT. As a result it is able to obtain a solution of specific fitness in polynomial time. In RLS1, given an undirected graph G, we start with a uniform random partition of vertex set $V(G)$ into two sets A and B such that $||A| - |B|| \leq 1$. Now, given a partition (A, B) of vertex set $V(G)$, we select two vertices, one each from A and B, uniformly at random and swap them. Algorithm 1 outlines RLS1.

Algorithm 1: Randomized Local Search for MAX BALANCED 2-UNCUT (RLS1)

Input: An undirected graph G
1: choose a random partition $P = \{A, B\}$ of $V(G)$ such that $||A| - |B|| \leq 1$
2: choose $u \in A$ and $v \in B$ uniformly at random.
3: generate a new partition $P' = \{A', B'\}$, where $A' = \{A \setminus \{u\}\} \cup \{v\}$ and $B' = \{B \setminus \{v\}\} \cup \{u\}$
4: **if** $f_{\text{uncut}}(P) < f_{\text{uncut}}(P')$ **then**
    $P \leftarrow P'$
  end
5: Repeat Step 2 to 4

Given a partition $P = \{A, B\}$ of $V(G)$, let $N(P)$ denotes the set of all 2-hamming neighbors of P such that for all $P' = (A', B') \in N(P)$, $|A' \setminus A| = 1$ and $|B' \setminus B| = 1$. We will first show that the landscape induced by neighborhood defined by RLS1 and fitness function of Max Balanced 2-Uncut is elementary. Towards this, we first prove the following result.

Lemma 3.1. *For any partition P of vertex set of an undirected graph G, if n is even*

$$\sum_{P' \in N(P)} f_{\text{uncut}}(P') = (n-2)m + \left(\frac{n^2}{4} - 2n + 2\right) f_{\text{uncut}}(P)$$

otherwise,

$$\sum_{P' \in N(P)} f_{\text{uncut}}(P') \geq (n-2)m + \left(\frac{n^2}{4} - 2n + \frac{3}{4}\right) f_{\text{uncut}}(P)$$

Proof. Let $P = \{A, B\}$ be a partition of $V(G)$ such that $|A| = \lfloor \frac{n}{2} \rfloor$ and $|B| = \lceil \frac{n}{2} \rceil$. Suppose that there exists an edge $e = uv$ such that $u \in A$ and $v \in B$. Now, e will contribute to the set of uncut edges only if either u is moved to B or v is moved to A. Suppose in RLS1, u is moved to B, then we have $\lceil \frac{n}{2} \rceil - 1$ choices to select a vertex from B uniformly at random to move in A. Similarly, we can move v from B to A. In this manner, there are exactly $(n-2)$ 2-hamming neighbors $P' \in N(P)$ corresponding to which e is an uncut edge.

Now, suppose that $e = uv$ is an uncut edge corresponding to partition P. Since both the end points of edge belong to the same set, any of these vertices cannot be selected to move to the other set. Now we have two possibilities, either $u, v \in A$ or $u, v \in B$.

(1) Suppose both the end points u and v of edge e belong to the set A. Then, we have $\lfloor \frac{n}{2} \rfloor - 2$ choices to select a vertex from set A and $\lceil \frac{n}{2} \rceil$ choices to select vertex from B. Hence,

$$\sum_{P' \in N(P)} w_e(P') = (n-2)(1 - w_e(P)) + \left(\left(\left\lfloor \frac{n}{2} \right\rfloor - 2\right)\left\lceil \frac{n}{2} \right\rceil\right) w_e(P)$$

If n is even, we get

$$\sum_{P' \in N(P)} w_e(P') = \left(\frac{n^2}{4} - 2n + 2\right) w_e(P) + n - 2$$

otherwise,

$$\sum_{P' \in N(P)} w_e(P') = \left(\frac{n^2}{4} - 2n + \frac{3}{4}\right) w_e(P) + n - 2$$

(2) If $u, v \in B$, then we have $\lceil \frac{n}{2} \rceil - 2$ choices to select a vertex from set B and $\lfloor \frac{n}{2} \rfloor$ choices to select vertex from A. Hence,

$$\sum_{P' \in N(P)} w_e(P') = (n-2)(1 - w_e(P)) + \left(\left(\left\lceil \frac{n}{2} \right\rceil - 2\right)\left\lfloor \frac{n}{2} \right\rfloor\right) w_e(P)$$

If n is even, we get

$$\sum_{P' \in N(P)} w_e(P') = \left(\frac{n^2}{4} - 2n + 2\right) w_e(P) + n - 2$$

otherwise,

$$\sum_{P' \in N(P)} w_e(P') = \left(\frac{n^2}{4} - 2n + \frac{11}{4}\right) w_e(P) + n - 2$$
$$\geq \left(\frac{n^2}{4} - 2n + \frac{3}{4}\right) w_e(P) + n - 2$$

Since, $f_{\text{uncut}}(P) = \sum_{e \in E(G)} w_e(P)$,

$$\sum_{P' \in N(P)} f_{\text{uncut}}(P') = \sum_{P' \in N(P)} \sum_{e \in E(G)} w_e(P')$$

If n is even,

$$\sum_{P' \in N(P)} f_{\text{uncut}}(P') = \sum_{e \in E(G)} \left(\frac{n^2}{4} - 2n + 2\right) w_e(P) + n - 2$$
$$= \left(\frac{n^2}{4} - 2n + 2\right) f_{\text{uncut}}(P) + (n-2)m$$

and if n is odd

$$\sum_{P' \in N(P)} f_{\text{uncut}}(P') \geq \left(\frac{n^2}{4} - 2n + \frac{3}{4}\right) f_{\text{uncut}}(P) + (n-2)m$$

This completes the proof. □

Now, we show that Max Balanced 2-Uncut has elementary landscape with respect to RLS1 when the number of vertices in the graph is even. Towards this, we show that it satisfies following linear difference equation which is also known as Grover's difference equation with respect to neighborhood defined by RLS1.

$$\nabla^2 \phi + \frac{K}{n} \phi = 0 \qquad (1)$$

where ϕ is the cost function, $K > 0$, n is the problem size and ∇^2 is the average difference operator over a specified neighborhood. We define normalized cost of Max Balanced 2-Uncut as $\bar{f}_{\text{uncut}} = f_{\text{uncut}} - f_{\text{uncut}_{AV}}$, where $f_{\text{uncut}_{AV}}$ is the average cost of all possible partitions. Here, the cost is referred as value of uncut.

Lemma 3.2. *Max Balanced 2-Uncut with the normalized cost $\bar{f}$uncut and the neighborhood defined by* RLS1 *satisfy the following difference equation for graph G with even number of vertices.*

$$\nabla^2 \bar{f}_{\text{uncut}} = -\frac{8(n-1)}{n^2} \bar{f}_{\text{uncut}}$$

Proof. The proof follows from Lemma 3.1 and the definition of average difference operator. Using Lemma 3.1, we have

$$\sum_{P' \in N(P)} (f_{\text{uncut}}(P') - f_{\text{uncut}}(P)) = m(n-2) - 2(n-1) f_{\text{uncut}}(P)$$

By the definition of average difference operator, we have

$$\nabla^2 f_{\text{uncut}} = \frac{1}{|N(P)|} \sum_{P' \in N(P)} (f_{\text{uncut}}(P') - f_{\text{uncut}}(P))$$

Since $|N(P)| = \frac{n^2}{4}$ when n is even,

$$\nabla^2 f_{\text{uncut}} = \frac{4}{n^2}\Big(m(n-2) - 2(n-1) f_{\text{uncut}}(P)\Big)$$

Observe that the fraction of the all partitions in which an edge $uv \in E(G)$ contributes to the cost is $\frac{n-2}{2(n-1)}$. This gives that the average cost, $f_{\text{uncut}_{AV}}$, over all partitions is $\frac{m(n-2)}{2(n-1)}$. Defining the normalized cost $\bar{f}_{\text{uncut}} = f_{\text{uncut}} - f_{\text{uncut}_{AV}}$, we get

$$\nabla^2 \bar{f}_{\text{uncut}} = -\frac{8(n-1)}{n^2} \bar{f}_{\text{uncut}}$$

This completes the proof. □

Grover [5] presented following result towards the significance of difference equation for local optima.

Theorem 3.3. (Theorem 6 [5]) *Let φ be the normalized cost. If φ satisfies equation 1, then all local optima have a cost better than or equal to the average cost of all configurations.*

Lemma 3.2 and Theorem 3.3 gives that the cost of the solution obtained by RLS1 for Max Balanced 2-Uncut is not arbitrarily poor. Equivalently, the landscape of Max Balanced 2-Uncut is elementary under RLS1 [1]. Now we prove that RLS1 obtains solution of certain quality in $O(mn^2)$ time.

Lemma 3.4. RLS1 *finds a solution P' after an expected $O(mn^2)$ iterations such that*

$$f_{\text{uncut}}(P') \geq \begin{cases} \frac{m(n-2)}{2(n-1)} & \text{if } n \text{ is even} \\ \frac{m(n-2)}{2n-1} & \text{if } n \text{ is odd} \end{cases}$$

Proof. We first prove that after polynomial number of steps, RLS1 reaches to a state of local optima. Formally, RLS1 obtains a solution P' such that for all partitions $Z \in N(P')$, $f_{\text{uncut}}(Z) \leq f_{\text{uncut}}(P')$. Let P be the current solution generated by RLS1. If there is no partition Z in $N(P)$ such that $f_{\text{uncut}}(Z) > f_{\text{uncut}}(P)$, then $P' = P$. Otherwise, suppose that there exists at least one partition $Z \in N(P)$ such that $f_{\text{uncut}}(Z) > f_{\text{uncut}}(P)$. The probability of generating a specific hamming neighbor in $N(P)$ is at least $\frac{4}{n^2}$. Hence, the expected waiting time to make an improvement from such a solution is at most $O(n^2)$. Since there are m number of edges, RLS1 can obtain a solution P' such that there is no improving solution in $N(P')$ in expected time bounded by $O(mn^2)$. Hence,

$$\frac{1}{|N(P')|} \sum_{Z \in N(P')} f_{\text{uncut}}(Z) \leq f_{\text{uncut}}(P')$$

Now if n is even, using Lemma 3.1 and the fact that $|N(P')| = \frac{n^2}{4}$, we get

$$f_{\text{uncut}}(P') \geq \frac{m(n-2)}{2(n-1)}$$

and if n is odd, $|N(P')| = \frac{n^2-1}{4}$ which gives that

$$f_{\text{uncut}}(P') \geq \frac{m(n-2)}{2n-1}$$

This completes the proof. □

We can observe that starting with any random partition, RLS1 returns optimal partition for star graph in $O(mn^2)$ time.

Now we show that RLS1 is a fixed parameter algorithm with respect to standard parameterization of Max Balanced 2-Uncut. The parameterized version of Max Balanced 2-Uncut is defined as follows.

Max Balanced 2-Uncut
Input: An undirected graph G and a non-negative integer k
Parameter: k
Goal: Find a partition $P = \{A, B\}$ of $V(G)$ where $\|\|A\| - \|B\|\| \leq 1$ such that number of uncut edges in P is at least k

Theorem 3.5. RLS1 *solves a parameterized instance (G, k) of* Max Balanced 2-Uncut *in $O(\max(mn^2, k^{O(k)}))$ expected time. Futhermore, after $O(\max(mn^3, nk^{O(k)}))$ iterations,* RLS1 *solves parameterized instance of* Max Balanced 2-Uncut *with constant probability.*

Proof. Let T denote the random variable that corresponds to the first time RLS1 solves the standard parameterization of Max Balanced 2-Uncut. If $k \leq \frac{m(n-2)}{2n-1}$, then using Lemma 3.4, RLS1 finds a partition P of $V(G)$ such that $f_{\text{uncut}}(P) \geq \frac{m(n-2)}{2n-1} \geq k$ after an expected $O(mn^2)$ steps. Hence, $E(T) = O(mn^2)$ if $k \leq \frac{m(n-2)}{2n-1}$. Now suppose that $k > \frac{m(n-2)}{2n-1}$. Without loss of generality, we can assume that the graph G is connected. Hence, $m \geq n-1$. This implies that $k > \frac{n-4}{2}$. Since, number of 2-hamming neighbors is $\frac{n^2-1}{4}$ when n is odd and $\frac{n^2}{4}$ when n is even, the probability that RLS1 perform an optimal swap in each iteration is at least $\frac{4}{n^2}$. Since in each iteration, we move two vertices in the partitions to which they belong in the optimal partition, these vertices will not be moved in the further iterations to generate an optimal partition. So, we need to perform $\lfloor \frac{n}{2} \rfloor$ successful swap operations. Hence, the probability that RLS1 transforms an arbitrary partition into an optimal partition is at least $\left(\frac{4}{n^2}\right)^{\lfloor \frac{n}{2} \rfloor}$. We can assume that $k \geq 5$, otherwise $k \leq 4$ is a trivial case. Since, $n < 2(k+2)$,

$$\left(\frac{4}{n^2}\right)^{\frac{n-1}{2}} \geq (2k)^{-(2k+3)}$$

Hence, $E(T) = O(k^{O(k)})$, when $k > \frac{m(n-2)}{2n-1}$.

Now we use Markov inequality to bound the time T. Markov inequality states that $Pr(T \geq \lambda E(T)) \leq \lambda^{-1}$, for all $\lambda \geq 1$. Let $c \geq 1$ be an arbitrary constant. Suppose that $k \leq \frac{m(n-2)}{2n-1}$. Using Markov inequality, we get

$$Pr(T \geq cmn^3) \leq \frac{1}{cn}$$

and when $k > \frac{m(n-2)}{2n-1}$,

$$Pr(T \geq cnk^{2k+3}) \leq \frac{1}{cn}$$

Hence, RLS1 solves standard parameterization of Max Balanced 2-Uncut in $O(\max(mn^3, nk^{O(k)}))$ time almost surely. □

3.1 Modified FPT for Max Balanced 2-Uncut

When the number of vertices are bounded by $g(k)$, where $g(k)$ is a linear function of k, then the algorithm can perform better without the swap operator. For Max Balanced 2-Uncut, if $n \geq 2(k+2)$, RLS1 generates a solution such that $f_{\text{uncut}} \geq k$ in polynomial time. When $n < 2(k+2)$, we can choose a balanced partition P uniformly at random. Since, $n < 2(k+2)$, total number of possible balanced partitions is $\binom{2k+4}{k+2} \leq 4^{k+2}$. Therefore, the probability that P is an optimal partition is at least $\frac{1}{4^{k+2}}$. Hence, after $O(4^k)$ trials, we choose an optimal partition with constant probability.

4 MAX BALANCED l-UNCUT

In this section, we present a randomized local search for Max Balanced l-Uncut when l is fixed and analyse its parameterized running time. Before going into the details of algorithm, we prove a lower bound on the number of uncut edges. We first define the notion of 2-hamming neighbor in l-partition of vertices. Given a partition $P = \{A_1, \cdots, A_l\}$ where $|A_i| = \lfloor \frac{n}{l} \rfloor$ $\left(\text{or } \lceil \frac{n}{l} \rceil\right)$, for all $i \in [l]$, $P' = \{A'_1, \cdots, A'_l\}$ is a 2-hamming neighbor of P if there exists $i, j \in [l]$ such that $|A'_i \setminus A_i| = 1$, $|A'_j \setminus A_j| = 1$ and $A'_q = A_q$, for all $q \in [l] \setminus \{i, j\}$. Let $N(P)$ denote the set of all 2-hamming neighbors of P. Let $S_1 = \left\{A_i \colon |A_i| = \lfloor \frac{n}{l} \rfloor, i \in [l]\right\}$ and $S_2 = \left\{A_i \colon |A_i| = \lceil \frac{n}{l} \rceil, i \in [l]\right\}$. Let $x = n \bmod l$. Clearly, $|S_2| = x$. Suppose that $P' = \{A'_1, \cdots, A'_l\} \in N(P)$, where $|A'_i \setminus A_i| = 1$, $|A'_j \setminus A_j| = 1$, then either $A_i, A_j \in S_1$ or $A_i, A_j \in S_2$ or $A_i \in S_1$ and $A_j \in S_2$. It can be observed that

$$
\begin{aligned}
|N(P)| &= \binom{x}{2}\left(\left\lceil \frac{n}{l} \right\rceil\right)^2 + \binom{l-x}{2}\left(\left\lfloor \frac{n}{l} \right\rfloor\right)^2 + x(l-x)\left\lceil \frac{n}{l} \right\rceil \left\lfloor \frac{n}{l} \right\rfloor \\
&= \frac{n^2(l-1) - x(l-x)}{2l}
\end{aligned}
$$

Now, we prove the following lower bound for Max Balanced l-Uncut.

Lemma 4.1. *Let P be a partition of $V(G)$ into l parts, where l is fixed. If there does not exist a 2-hamming neighbor P' of P such that $f_{\text{uncut}}(P') > f_{\text{uncut}}(P)$, then $f_{\text{uncut}}(P) \geq \frac{m(n-x-l)}{(n-1)l}$.*

Proof. Let $P = \{A_1, \cdots, A_l\}$ be a partition of $V(G)$. We first compute the number of 2-hamming neighbors of P in which uv is an uncut edge. Let $N(P)$ denote the set of these neighbors.

(1) Suppose uv is an uncut edge in P. Let $u, v \in A_i$, for some $i \in [l]$. It is clear that we cannot move u or v from A_i to some other set.

(a) If $A_i \in S_1$, then

$$
|N(P)| = \frac{n^2(l-1) - x(l-x)}{2l} - 2\left(n - \left\lfloor \frac{n}{l} \right\rfloor\right)
$$

(b) If $A_i \in S_2$, then

$$
\begin{aligned}
|N(P)| &= \frac{n^2(l-1) - x(l-x)}{2l} - 2\left(n - \left\lceil \frac{n}{l} \right\rceil\right) \\
&\geq \frac{n^2(l-1) - x(l-x)}{2l} - 2\left(n - \left\lfloor \frac{n}{l} \right\rfloor\right)
\end{aligned}
$$

(2) Suppose uv is not an uncut edge in P. Let $u \in A_i$ and $v \in A_j$.

(a) If $A_i, A_j \in S_1$, then

$$
|N(P)| = 2\left(\left\lfloor \frac{n}{l} \right\rfloor - 1\right) = 2\left(\frac{n-x}{l} - 1\right)
$$

(b) If $A_i, A_j \in S_2$, then

$$
|N(P)| = 2\left(\left\lceil \frac{n}{l} \right\rceil - 1\right) \geq 2\left(\left\lfloor \frac{n}{l} \right\rfloor - 1\right)
$$

(c) $A_i \in S_1$ and $A_j \in S_2$

$$
|N(P)| = \left\lceil \frac{n}{l} \right\rceil + \left\lfloor \frac{n}{l} \right\rfloor - 2 \geq 2\left(\left\lfloor \frac{n}{l} \right\rfloor - 1\right)
$$

Now, the number of partitions in $N(P)$ corresponding to which uv is an uncut edge is

$$
\begin{aligned}
\sum_{P' \in N(P)} w_e(P) \quad &\geq \quad \left(\frac{n^2(l-1) - x(l-x)}{2l} - 2\left(n - \left\lfloor \frac{n}{l} \right\rfloor\right)\right) w_e(P) \\
&\quad + \left(2\left(\left\lfloor \frac{n}{l} \right\rfloor - 1\right)\right)(1 - w_e(P)) \\
&\geq \quad 2\left(\frac{n-x}{l} - 1\right) \\
&\quad + \left(\frac{n^2(l-1) - x(l-x)}{2l} - 2n + 2\right) w_e(P)
\end{aligned}
$$

Since, $f_{\text{uncut}}(P) = \sum_{e \in E(G)} w_e(P)$

$$
\begin{aligned}
\sum_{P' \in N(P)} f_{\text{uncut}}(P') \quad &= \quad \sum_{P' \in N(P)} \sum_{e \in E(G)} w_e(P) \\
&\geq \quad 2\left(\frac{n-x}{l} - 1\right) m \\
&\quad + \left(\frac{n^2(l-1) - x(l-x)}{2l} - 2n + 2\right) f_{\text{uncut}}(P)
\end{aligned}
$$

Let P be a partition such that there is no 2-hamming neighbor P' of P for which $f_{\text{uncut}}(P') > f_{\text{uncut}}(P)$. Hence,

$$
\frac{1}{|N(P)|} \sum_{P' \in N(P)} f_{\text{uncut}}(P') \leq f_{\text{uncut}}(P)
$$

Since, $|N(P)| = \frac{n^2(l-1) - x(l-x)}{2l}$, we get

$$
f_{\text{uncut}}(P) \geq \frac{m(n-x-l)}{(n-1)l}
$$

□

Now we give a randomized local search algorithm, RLS2, for Max Balanced l-Uncut. Algorithm 2 outlines this algorithm. Now, we prove that Algorithm 2 generates a neighbor of specific fitness in polynomial time.

Lemma 4.2. *Given any random partition of vertices of G into l parts where l is fixed,* RLS2 *generates a solution P' such that there is no neighbor $Z \in N(P')$ such that $f_{\text{uncut}}(P') < f_{\text{uncut}}(Z)$ in $O(mn^2)$ time.*

Algorithm 2: Randomized Local Search for Max Balanced l-Uncut (RLS2)

Input: An undirected graph G

1: choose a random partition $P = (A_1, \cdots, A_l)$ of $V(G)$ such that $||A_i| - |A_j|| \leq 1$ for any $i, j \in [l]$
2: choose $i, j \in [l]$ randomly
3: choose $u \in A_i$ and $v \in A_j$ uniformly at random.
4: generate a new partition $P' = (A'_1, \cdots, A'_l)$, where $A'_i = A_i \setminus \{u\} \cup \{v\}$ and $A'_j = A_j \setminus \{v\} \cup \{u\}$
5: **if** $f_{\text{uncut}}(P) < f_{\text{uncut}}(P')$ **then**
    $P \leftarrow P'$
  end
6: Repeat Step 3 to 5

Proof. Let $P = (A_1, \cdots, A_l)$ be a partition of $V(G)$ into l parts. Clearly, neighbor of P generated by RLS2 is a 2-hamming neighbor of P. If there is no neighbor $Z \in N(P)$ such that $f_{\text{uncut}}(Z) > f_{\text{uncut}}(P)$, then $P' = P$, otherwise let Z be a partition in $N(P)$ such that $f_{\text{uncut}}(Z) > f_{\text{uncut}}(P)$. The probability of generating this particular neighbor is $\frac{2l}{n^2(l-1)-x(l-x)} \geq \frac{2}{n^2}$. Hence, the expected waiting time to generate such a neighbor is $O(n^2)$. Since, there are m number of edges, the waiting time to generate P' is $O(mn^2)$. □

Now, we analyse the parameterized running time of RLS2 for Max Balanced l-Uncut. We define the parameterized version of Max Balanced l-Uncut as follows.

> Max Balanced l-Uncut
> **Input:** An undirected graph G and a non-negative integer k
> **Parameter:** k
> **Goal:** Find a partition $P = \{A_1, \cdots, A_l\}$ of $V(G)$, where $||A_i| - |A_j|| \leq 1$ for all $i, j \in [l]$ such that number of uncut edges in P is at least k

Theorem 4.3. *RLS2 solves a parameterized instance (G, k) of Max Balanced l-Uncut in $O(\max(mn^2, k^{O(k)}))$ expected time. Futhermore, after $O(\max(mn^3, nk^{O(k)}))$ iterations, RLS2 solves parameterized instance of Max Balanced l-Uncut with constant probability.*

Proof. Let T be a random variable which denotes the first time RLS2 solves the parameterized instance of Max Balanced l-Uncut. If $k \leq \frac{m(n-x-l)}{(n-1)l}$, then using Lemma 4.1 and 4.2, RLS2 solves the instance (G, k) of Max Balanced l-Uncut in $O(mn^2)$ expected time. Suppose that $k > \frac{m(n-x-l)}{(n-1)l}$. Without loss of generality, we can assume that the graph is connected. Hence, $m \geq n - 1$. This gives, $n < l(k+2)$. Since, number of 2-hamming neighbors of a partition P is $\frac{n^2(l-1)-x(l-x)}{2l}$, the probability of performing an optimal swap in each iteration is at least $\frac{2l}{n^2(l-1)-x(l-x)}$. Since in each iteration, two vertices are moved in the partitions to which they belong in the optimal partition, these vertices will not be moved in the further iterations to generate an optimal partition. So, we need to perform $\frac{n}{2}$ successful swap operations. Hence, the probability that RLS2 transforms an arbitrary assignment into an optimal assignment is at least $\left(\frac{2l}{n^2(l-1)-x(l-x)}\right)^{\frac{n}{2}} \geq \left(\frac{2}{n^2}\right)^{\frac{n}{2}}$. Since, $n < l(k+2)$, the waiting expected time to generate an optimal assignment is $O(k^{O(k)})$.

Let $c \geq 1$ be an arbitrary constant. Using Markov inequality, if $k \leq \frac{m(n-x-l)}{(n-1)l}$,

$$Pr(T \geq cmn^3) \leq \frac{1}{cn}$$

and when $k > \frac{m(n-x-l)}{(n-1)l}$,

$$Pr(T \geq cnk^{2l(k+2)}) \leq \frac{1}{cn}$$

Hence, RLS2 solves standard parameterization of Max Balanced l-Uncut in $O(\max(mn^3, nk^{O(k)}))$ expected time almost surely. □

Theorem 4.4. *There exists a randomized* FPT *algorithm for* Max Balanced l-Uncut *with running time $O(l^{lk})$ when the number of vertices in the graph G is less than $l(k+2)$.*

Proof. The algorithm and running time analysis is similar to the one presented in section 3.1. □

5 MAX MULTIPARTITE UNCUT

Max MultiPartite Uncut deals with partitioning the vertex set of graph G into l parts where l is given as input. In this section, we present a randomized local search and a simple evolutionary algorithm for Max MultiPartite Uncut and analyse their parameterized running time. We first prove the following lower bound for Max MultiPartite Uncut. Further, we prove that such a lower bound can be obtained in polynomial time.

Lemma 5.1. *Let P be a partition of $V(G)$ into l parts. If there does not exist a hamming neighbor P' of P such that $f_{\text{uncut}}(P') > f_{\text{uncut}}(P)$, then $f_{\text{uncut}}(P) \geq \frac{m}{l}$, where m is the number of edges in the graph G.*

Proof. Suppose that $P = (A_1, \cdots, A_l)$ be a partition of $V(G)$. Let $\mathcal{H}$ be the set of hamming neighbors of P, i.e., $P' = \{A'_1, \cdots, A'_l\} \in \mathcal{H}$ if $|A'_i \setminus A_i| = 1$ for some $i \in [l]$ and $A'_j = A_j$ for all $j \in [l] \setminus \{i\}$. If an edge $uv \in E(G)$ is an uncut edge, then there are $(n-2)(l-1)$ partitions in $\mathcal{H}$ corresponding to which uv is an uncut edge because we cannot move u or v to any other partition. Now, suppose that uv is not an uncut edge. Let $u \in A_i$ and $v \in A_j$ for some $i, j \in [l]$, $i \neq j$. Now uv is an uncut edge only if either u is moved to A_j or v is moved to A_i. Hence, there are only two partitions in $\mathcal{H}$ corresponding to which uv is an uncut edge. Therefore,

$$\sum_{P' \in P} w_e(P') = (n-2)(l-1)w_e(P) + 2(1 - w_e(P))$$
$$= 2 + (nl - n - 2l)w_e(P)$$

Since,

$$f_{\text{uncut}}(P) = \sum_{e \in E(G)} w_e(P)$$

we have

$$\sum_{P' \in N(P)} f_{\text{uncut}}(P) = 2m + (nl - n - 2l) f_{\text{uncut}}(P)$$

Let P be a partition such that there is no hamming neighbor P' of P for which $f_{\text{uncut}}(P') > f_{\text{uncut}}(P)$. Hence,

$$\frac{1}{|N(P)|} \sum_{P' \in N(P)} f_{\text{uncut}}(P') \leq f_{\text{uncut}}(P)$$

Since, $|N(P)| = n(l-1)$,

$$\frac{1}{n(l-1)}(2m + (nl - n - 2l) f_{\text{uncut}}(P)) \leq f_{\text{uncut}}(P)$$

Hence, we get following bound on $f_{\text{uncut}}(P)$

$$f_{\text{uncut}}(P) \geq \frac{m}{l}$$

□

5.1 Randomized Local Search for Max MultiPartite Uncut

In this section, we present a randomized local search algorithm, RLS3, for Max MultiPartite Uncut and analyse its parameterized running time. Towards this, we first select a vertex v uniformly at random and then choose a partition A_i to move v. Algorithm 3 outlines this algorithm. Now, we prove

Algorithm 3: Randomized Local Search for Max MultiPartite Uncut (RLS3)

Input: An undirected graph G and number of partitions l

1: choose a random partition $P = (A_1, \cdots, A_l)$ of $V(G)$.
2: choose a vertex v uniformly at random from V(G)
3: Let $v \in A_j$. choose a partition $A_i \in P$ randomly, where $i \in [l]$ and $i \neq j$
4: generate a new partition $P' = (A'_1, \cdots, A'_l)$ such that $A'_j = A_j \setminus \{v\}$, $A'_i = A_i \cup \{v\}$ and $A'_k = A_k$, $\forall k \in [l] \setminus \{i, j\}$
5: **if** $f_{\text{uncut}}(P) < f_{\text{uncut}}(P')$ **then**
  $P \leftarrow P'$
end
6: Repeat Step 2 to 5

that RLS3 reaches to a solution of certain quality in polynomial time.

Lemma 5.2. *Given any random partition P, RLS3 generates a partition P' for which there is no partition $Z \in N(P')$ such that $f_{\text{uncut}}(Z) > f_{\text{uncut}}(P')$ in $O(nml)$ time.*

Proof. Let P be a current partition. Clearly, RLS3 generates a hamming neighbor of a partiotion in each iteration. If there does not exist any hamming neighbor Z of P such that $f_{\text{uncut}}(Z) > f_{\text{uncut}}(P)$, then $P' = P$. Suppose that there exists a partition $Z \in N(P)$ such that $f_{\text{uncut}}(Z) > f_{\text{uncut}}(P)$. The probability of generating this particular neighbor is $\frac{1}{n(l-1)}$. Hence, the expected waiting time to generate this partition that improvs the solution is $O(nl)$. Since, there are m edges in the graph G, the waiting time to generate a partition P' is $O(nml)$. This completes the proof.

□

Theorem 5.3. RLS3 *generates a partition P in $O(nml)$ time such that $f_{\text{uncut}}(P) \geq \frac{m}{l}$.*

Proof. The proof follows from Lemma 5.1 and 5.2. □

Now, we analyse the parameterized running time of RLS3 for Max MultiPartite Uncut. We define the parameterized version of Max MultiPartite Uncut as follows.

> Max MultiPartite Uncut
> **Input:** An undirected graph $G = (V, E)$, number of partitions l and a non-negative integer k
> **Parameter:** l, k
> **Goal:** Find a partition P of $V(G)$ into l parts such that number of uncut edges in P is at least k

Theorem 5.4. RLS3 *is an* FPT *algorithm which solves an instance (G, l, k) of* Max MultiPartite Uncut *in $O(\max(mnl, (kl^2)^{O(kl)}))$ expected time. Futhermore,* RLS3 *solves parameterized instance of* Max MultiPartite Uncut *with constant probability after $O(\max(mn^2l, n(kl^2)^{O(kl)}))$ iterations.*

Proof. Let T be a random variable which denotes the first time when RLS3 solves the instance (G, l, k) of Max MultiPartite Uncut. If $k \leq \frac{m}{l}$, then using Lemma 5.1 and 5.2, parameterized instance of Max MultiPartite Uncut can be solved in $O(nml)$ expected stime. Now, suppose that $k > \frac{m}{l}$. Since, number of vertices in any graph G is at most $2m$, we have $n < 2kl$. In each iteration of RLS3, a vertex v which needs to move to some other partition to generate an optimal partition is selected with probability at least $\frac{1}{n}$ and the probability that v is moved to the partition to which it belongs in the optimal partition is at least $\frac{1}{l-1}$. Hence, in each iteration a vertex moves to its correct partition with probability at least $\frac{1}{n(l-1)}$. Since, at most n successful moves are required to generate an optimal partition, the probability that RLS3 generates an optimal assignment starting with any random assignment is at least $\left(\frac{1}{n(l-1)}\right)^n \geq (2kl^2)^{-2kl}$. Hence, $E(T) = O(kl^2)^{O(kl)}$.

Now, using Markov inequalty, if $k \leq \frac{m}{l}$ then the probability that $T \geq cn^2ml$ is at most $\frac{1}{cn}$ and if $k > \frac{m}{l}$, $Pr(T \geq (ckl^2)^{2kl}) \leq \frac{1}{cn}$ for some constant $c > 1$. Hence, RLS3 solves the parameterized instance of Max MultiPartite Uncut in $O(\max(mn^2l, n(kl^2)^{O(kl)}))$ time almost surely.

□

5.2 (1 + 1) EA for Max MultiPartite Uncut

In this section, we present a (1 + 1) EA for Max MultiPartite Uncut. In this algorithm, every vertex has equal probability to move in some other partition. We show that performance of RLS3 and (1 + 1) EA is same for Max MultiPartite Uncut. Algrithm 4 outlines this procedure.

Algorithm 4: 1 + 1 EA for Max MultiPartite Uncut

```
   Input: An undirected graph G and number of partitions l
1: choose a random partition P = (A_1, ··· , A_l) of V(G).
2: move every vertex v ∈ V(G) to some other partition with
   probability 1/n to generate a new partition P'
3: if f_uncut(P) < f_uncut(P') then
       P ← P'
   end
4: Repeat Step 2 to 3
```

Now, we prove that (1 + 1) EA also generates a solution of specific fitness in $O(nml)$ time.

Theorem 5.5. *(1 + 1) EA generates a partition P' of $V(G)$ such that $f_{\text{uncut}}(P') \geq \frac{m}{l}$ after an expected $O(nml)$ iterations.*

Proof. Given a partition P, (1 + 1) EA generates a specific hamming neighbor of P with probability at least $\frac{1}{n(l-1)}\left(1 - \frac{1}{n}\right)^{n-1} \geq \frac{1}{en(l-1)} = \Omega(\frac{1}{n(l-1)})$. The rest of the proof is similar to Lemma 5.2. □

Hence, RLS3 and (1 + 1) EA both generates a partition P such that $f_{\text{uncut}}(P) \geq \frac{m}{l}$ in $O(nml)$ expected time. Now, we can show that (1+1) EA is also FPT for Max MultiPartite Uncut.

Theorem 5.6. *(1+1) EA solves a parameterized instance (G, l, k) of Max MultiPartite Uncut in $O(\max(mnl, (kl^2)^{O(kl)}))$ expected time. Futhermore, (1+1) EA solves parameterized instance of Max MultiPartite Uncut with constant probability after $O(\max(mn^2 l, n(kl^2)^{O(kl)}))$ iterations.*

Proof. The proof is similar to Theorem 5.4. □

6 CONCLUSION

In this paper, the running time analysis of randomized local search for Max Balanced l-Uncut and Max MultiPartite Uncut has been presented. We showed that the fitness landscape of Max Balanced 2-Uncut is elementary under the neighborhood operator defined by RLS2. The lower bounds for these problems have been proved and showed that there exists algorithms that are able to achieve these bounds in polynomial time. We also showed that these algorithms are fixed parameter tractable when parameterized by the solution size, i.e., the number of uncut edges. Furthermore, the parameterized running time analysis of (1+1) EA for Max MultiPartite Uncut has been carried out and obtained that the performance of randomized local search and (1 + 1) EA is same for Max MultiPartite Uncut.

REFERENCES

[1] J Wesley Barnes, Boryana Dimova, Steftcho P Dokov, and Andrew Solomon. 2003. The theory of elementary landscapes. *Applied Mathematics Letters* 16, 3 (2003), 337–343.

[2] Marek Cygan, Fedor V. Fomin, Lukasz Kowalik, Daniel Lokshtanov, Dániel Marx, Marcin Pilipczuk, Michal Pilipczuk, and Saket Saurabh. 2015. *Parameterized Algorithms.* Springer.

[3] M. R. Garey and David S. Johnson. 1979. *Computers and Intractability: A Guide to the Theory of NP-Completeness.* W. H. Freeman.

[4] Olivier Goldschmidt and Dorit S. Hochbaum. 1988. Polynomial Algorithm for the k-Cut Problem. In *29th Annual Symposium on Foundations of Computer Science, White Plains, New York, USA, 24-26 October 1988.* 444–451.

[5] Lov K Grover. 1992. Local search and the local structure of NP-complete problems. *Operations Research Letters* 12, 4 (1992), 235–243.

[6] Stefan Kratsch, Per Kristian Lehre, Frank Neumann, and Pietro Simone Oliveto. 2010. Fixed Parameter Evolutionary Algorithms and Maximum Leaf Spanning Trees: A Matter of Mutation. In *Parallel Problem Solving from Nature - PPSN XI, 11th International Conference, Kraków, Poland, September 11-15, 2010, Proceedings, Part I.* 204–213.

[7] Stefan Kratsch and Frank Neumann. 2013. Fixed-Parameter Evolutionary Algorithms and the Vertex Cover Problem. *Algorithmica* 65, 4 (2013), 754–771.

[8] Frank Neumann and Carsten Witt. 2015. On the Runtime of Randomized Local Search and Simple Evolutionary Algorithms for Dynamic Makespan Scheduling. In *Proceedings of the Twenty-Fourth International Joint Conference on Artificial Intelligence, IJCAI 2015, Buenos Aires, Argentina, July 25-31, 2015.* 3742–3748.

[9] Mojgan Pourhassan, Feng Shi, and Frank Neumann. 2016. Parameterized Analysis of Multi-objective Evolutionary Algorithms and the Weighted Vertex Cover Problem. In *Parallel Problem Solving from Nature - PPSN XIV - 14th International Conference, Edinburgh, UK, September 17-21, 2016, Proceedings.* 729–739.

[10] Christian M Reidys and Peter F Stadler. 2002. Combinatorial landscapes. *SIAM review* 44, 1 (2002), 3–54.

[11] Andrew M. Sutton, Frank Neumann, and Samadhi Nallaperuma. 2014. Parameterized Runtime Analyses of Evolutionary Algorithms for the Planar Euclidean Traveling Salesperson Problem. *Evolutionary Computation* 22, 4 (2014), 595–628.

[12] Darrell Whitley and Francisco Chicano. 2014. Elementary Landscape Decomposition of the Hamiltonian Path Optimization Problem. In *European Conference on Evolutionary Computation in Combinatorial Optimization.* 121–132.

[13] Chenchen Wu, Dachuan Xu, Donglei Du, and Wenqing Xu. 2016. An approximation algorithm for the balanced Max-3-Uncut problem using complex semidefinite programming rounding. *Journal of Combinatorial Optimization* 32, 4 (2016), 1017–1035.

[14] Yinyu Ye and Jiawei Zhang. 2003. Approximation of dense-n/2-subgraph and the complement of min-bisection. *Journal of Global Optimization* 25, 1 (2003), 55–73.

[15] Peng Zhang, Chenchen Wu, Dachuan Xu, and Xinghe Zhang. 2016. Approximation and Hardness Results for the Max k-Uncut Problem. In *Combinatorial Optimization and Applications - 10th International Conference, COCOA 2016, Hong Kong, China, December 16-18, 2016, Proceedings.* 49–61.