Unlimited Budget Analysis

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ABSTRACT

Performance analysis of randomised search heuristics is a rapidly growing and developing field. We contribute to its further development by introducing a novel analytical perspective that we call unlimited budget analysis. It has its roots in the very recently introduced approximation error analysis and bears some similarity to fixed budget analysis. The focus is on the progress an optimisation heuristic makes towards a set goal, not on the time it takes to reach this goal, setting it far apart from runtime analysis. We present the framework, apply it to simple mutation-based algorithms, covering both, local and global search. We provide analytical results for a number of simple example functions for unlimited budget analysis and compare them to results derived within the fixed budget framework for the same algorithms and functions.

KEYWORDS

Performance measures, Theory, Working principles of evolutionary computing

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Performance analysis of randomised search heuristics (RSHs) is a rapidly growing and developing field. The current mainstream method is runtime analysis whose purpose is to estimate the number of function evaluations for obtaining an optimal solution. Recently an alternative perspective appeared which is to consider solution quality that an algorithm achieves. There are two ways to measure solution quality: (1) the expected function value that can be achieved with a pre-defined number of computational steps, called the fixed budget setting [4]; (2) the approximation error between the achieved objective function value and the optimal value [1, 2].

In this paper we present a technique following an idea rooted from both fixed budget analysis and approximation error: analysing the distance of the achieved function value to the optimal value depending on the number of computational steps. Since the approach does not consider a computational budget that is fixed in advance and explicitly works for any number of computational steps, we call it unlimited budget analysis.

Consider a maximisation problem:

$$\max f(x), \quad \text{subject to } x \in \mathcal{D}, \tag{1}$$

where $f(x)$ is a fitness function such that $f(x) < +\infty$ and $\mathcal{D}$ is a finite state set or a closed set in $\mathbb{R}^n$. Let $f^* := \max\{f(x); x \in \mathcal{D}\}$ and $\mathcal{D}^* := \{x \mid f(x) = f^*\}$.

RSHs for solving the above optimisation problem can be described by a sequence of random variables $\{X^{[t]}; t = 0, 1, \dots\}$ where $X^{[t]}$ represents the solution (or a population of solutions for population-based RSHs) at the tth step. The fitness of $X^{[t]}$ is

$$f(X^{[t]}) := \max\{f(x); x \in X^{[t]}\}$$

and its expected value $f^{[t]} := \mathbb{E}[f(X^{[t]})$. The approximation error of $X^{[t]}$ is $e(X^{[t]}) := |f(X^{[t]}) - f^*|$ and its expected value $e^{[t]} := \mathbb{E}[e(X^{[t]})]$.

We assume that the sequence $\{f^{[t]}; t = 0, 1, \dots\}$ converges to f^*. Unlimited budget analysis is to find a lower (or upper) bound $b(t)$ on the fitness value $f^{[t]}$ satisfying two conditions: (1) the lower (or upper) bound holds for any $t \in [0, +\infty)$; (2) the bound error $|f^* - b(t)|$ converges to 0 as $t \longrightarrow +\infty$.

From the approximation error $e^{[t]} = |f^* - f^{[t]}|$, it is obvious that a bound on $e^{[t]}$ will lead to a bound on $f^{[t]}$. Following the work on the estimation of the approximation error of EAs [1, 2], this paper focuses on drawing a bound on $f^{[t]}$ through a bound on $e^{[t]}$.

The convergence rate of the error sequence $\{e^{[t]}; t = 0, 1, \dots\}$ at the t-th generation [3] is $r^{[t]} = e^{[t+1]}/e^{[t]}$ if $e^{[t]} \neq 0$. If $e^{[t]} = 0$, $X^{[t]}$ is an optimal solution. Based on this rate, we estimate upper and lower bounds on $f^{[t]}$ as follows:

Theorem 1. *Given an error sequence $\{e^{[t]}; t = 0, 1, \dots\}$,*

(1) *if there exists some $\lambda > 0$, $e^{[t+1]}/e^{[t]} \leq \lambda$ for any t, then $f^{[t]} \geq f^* - e^{[0]}\lambda^t$.*
(2) *If there exists some $\lambda > 0$, $e^{[t+1]}/e^{[t]} \geq \lambda$ for any t, then $f^{[t]} \leq f^* - e^{[0]}\lambda^t$.*

Proof. It is sufficient to prove the first claim. From the condition $e^{[t+1]}/e^{[t]} \leq \lambda$, we get $e^{[t+1]} \leq e^{[t]}\lambda$ and then $e^{[t]} \leq e^{[0]}\lambda^t$. Equivalently $f^{[t]} \geq f^* - e^{[0]}\lambda^t$. □

Unlimited budget analysis can do a similar job as fixed budget analysis. We show this similarity through an example. Consider random local search (RLS) [4] on maximising the OneMax function,

$$\max f(x) = |x|, \quad x \in \{0, 1\}^n, \tag{2}$$

where $x = (x_1, \cdots, x_n) \in \{0, 1\}^n$ and $|x| = x_1 + \cdots + x_n$. The approximation error of x is $e(i) = n - i$.

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Without loss of generality, we assume that $X^{[t]} = x$ such that $|x| = i$ where $i < n$. Then $f(x) = i$ and $e(x) = n - i$. The fitness of $X^{[t+1]}$ increases if and only if one of the $n - i$ 0-valued bits in x is flipped. The probability of this event happening is $\frac{n-i}{n}$. Denote the conditional expectation of the fitness change by

$$\Delta(x) := \mathbb{E}[f(X^{[t+1]}) - f(X^{[t]}) \mid X^{(t)} = x].$$

Then $\frac{e^{[t+1]}}{e^{[t]}} = 1 - \frac{\Delta(x)}{e^{[t]}}$. We have

$$\Delta(x) = \frac{n-i}{n} \cdot 1, \tag{3}$$

$$\frac{e^{[t+1]}}{e^{[t]}} = 1 - \frac{\Delta(x)}{e^{[t]}} = 1 - \frac{1}{n}, \tag{4}$$

$$e^{[t]} = \left(1 - \frac{1}{n}\right)^t e^{[0]}, \tag{5}$$

$$f^{[t]} = n - \left(1 - \frac{1}{n}\right)^t e^{[0]}. \tag{6}$$

Jansen and Zarges [4] also studied RLS on the OneMax function but use a different method. We arrive at the the same result here.

However, in many cases, there exists a significant difference between fixed budget analysis and unlimited budget analysis. We show this difference through another example. Consider the (1+1) EA [4] on maximising the LeadingOnes function.

$$f(x) = \textstyle\sum_{i=1}^{n} \prod_{j=1}^{i} x_j, \qquad x \in \{0,1\}^n \tag{7}$$

We assume that $X^{[t]} = x$ which satisfies $x_1 = 1, \cdots, x_i = 1, x_{i+1} = 0$ for $i < n$. The fitness $f(x) = i$ and the error $e(x) = n - i$. The fitness of $X^{[t+1]}$ increases if the leftmost 0-valued bit in x is flipped into 1 and other bits are unchanged. The probability of this event happening is $1/n\left(1 - \frac{1}{n}\right)^{n-1}$. Thus, the probability of $f(X^{[t+1]}) > f(X^{[t]})$ is at least $\frac{1}{n}\left(1 - \frac{1}{n}\right)^{n-1}$.

Then we have for any t

$$\Delta(x) \geq \frac{1}{n}\left(1 - \frac{1}{n}\right)^{n-1} \cdot 1.$$

$$\frac{e^{[t+1]}}{e^{[t]}} \leq 1 - \frac{1}{n}\left(1 - \frac{1}{n}\right)^{n-1} \frac{1}{n-i} \leq 1 - \frac{1}{n^2}\left(1 - \frac{1}{n}\right)^{n-1}, \tag{8}$$

$$e^{[t]} \leq \left(1 - \frac{1}{n^2}\left(1 - \frac{1}{n}\right)^{n-1}\right)^t e^{[0]}, \tag{9}$$

$$f^{[t]} \geq n - \left(1 - \frac{1}{n^2}\left(1 - \frac{1}{n}\right)^{n-1}\right)^t e^{[0]}, \tag{10}$$

$$\text{and } \lim_{t \to +\infty} \left| f^* - n + \left(1 - \frac{1}{n^2}\left(1 - \frac{1}{n}\right)^{n-1}\right)^t e^{[0]} \right| = 0. \tag{11}$$

We compare the result (10) with the result by fixed budget analysis [4]. According to Theorem [4, Theorem 13], for any t under a threshold,

$$f^{[t]} = 1 + \frac{2t}{n} - o(\frac{t}{n}), \tag{12}$$

$$\text{but } \lim_{t \to +\infty} \left| f^* - 1 - \frac{2t}{n} + o(\frac{t}{n}) \right| = +\infty \neq 0. \tag{13}$$

For LeadingOnes, fixed budget analysis provides a linear approximation of $f^{[t]}$ for t within a threshold. But for $t \in [0, +\infty)$, the relationship between $f^{[t]}$ and t is nonlinear. Thus, unlimited budget analysis finds an exponential approximation of $f^{[t]}$ for any t. We may regard unlimited budget analysis as fixed budget analysis with unlimited computational budget.

Different from runtime analysis, a bound on $f(X^{[t]})$ is related to the function $f(x)$. Therefore, scaling a function may change the representation of the bound significantly. We show this change via an example. Consider RLS on the square function which scales the OneMax function,

$$\max f(x) = |x|^2, \quad x \in \{0,1\}^n. \tag{14}$$

We assume that $X^{[t]} = x$ with $|x| = i$ where $i < n$. Then $f(x) = i^2$ and $e(x) = n^2 - i^2$. x includes $n - i$ 0-valued bits. The fitness of $X^{[t+1]}$ increases if and only if one of the $n - i$ 0-valued bits in x is flipped. The probability of this event happening is $\frac{n-i}{n}$. Then we have

$$\Delta(x) = \frac{n-i}{n} \cdot [(i+1)^2 - i^2], \tag{15}$$

$$\frac{e^{[t+1]}}{e^{[t]}} = 1 - \frac{n-i}{n} \cdot \frac{(i+1)^2 - i^2}{n^2 - i^2} \leq 1 - \frac{1}{n^2}, \tag{16}$$

$$e^{[t]} \leq \left(1 - \frac{1}{n^2}\right)^t e^{[0]}, \tag{17}$$

$$f^{[t]} \geq n^2 - \left(1 - \frac{1}{n^2}\right)^t e^{[0]}. \tag{18}$$

Notice the lower bound (18) on the square function is different from the bound (6) on the OneMax function. From this example, we see that it is easy to apply unlimited budget analysis to scaled versions of a function. However, scaling might bring a trouble in fixed budget framework because the existing analysis of OneMax relies on the linearity of expectation [4].

In summary, we have presented unlimited budget analysis, an analytical framework to derive results about the expected performance, measured by means of function values, of a randomised search heuristics after an arbitrary number of computational steps. We have demonstrated the applicability of our method by considering random local search and the (1+1) EA through three examples. We observe that for OneMax we obtain the same result as with fixed budget analysis and that for LeadingOnes the result we obtain is different from the bound obtained with fixed budget analysis, where the former is an exponential approximation for $t \in [0, +\infty)$ but the later is a linear approximation for t within a threshold. We also demonstrate that unlimited budget analysis could be easy to apply to scaled functions.

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