Visualization in Multiobjective Optimization

Bogdan Filipič Tea Tušar

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Department of Intelligent Systems
Jožef Stefan Institute
Ljubljana, Slovenia

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- A taxonomy of visualization methods
- Visualizing single approximation sets
- Visualizing repeated approximation sets

Visualizing problem landscapes

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Introduction

Multiobjective optimization problem

Minimize

$$\mathbf{f}\colon X \to F$$
$$\mathbf{f}\colon (x_1, \ldots, x_n) \mapsto (f_1(x_1, \ldots, x_n), \ldots, f_m(x_1, \ldots, x_n))$$

- X is an n-dimensional decision space
- $F \subseteq \mathbb{R}^m$ is an m-dimensional objective space $(m \geq 2)$

Conflicting objectives $\rightarrow$ a set of optimal solutions

- Pareto set in the decision space
- Pareto front in the objective space

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Introduction

Visualization in multiobjective optimization

Useful for different purposes [19]

- Analysis of solutions and solution sets
- Decision support in interactive optimization
- Analysis of algorithm performance

Visualizing solution sets in the decision space

- Problem-specific
- If $X \subseteq \mathbb{R}^m$, any method for visualizing multidimensional solutions can be used
- Not the focus of this tutorial

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Introduction

Visualizing solution sets in the objective space

- Interested in sets of mutually nondominated solutions called approximation sets
- Different from ordinary multidimensional solution sets
- The focus of this tutorial

Visualization of multiobjective problem landscapes

- Important for problem understanding, but few approaches exist
  - Multiobjective cost landscapes [16]
  - Cumulated gradient field landscapes [26]
- The focus of this tutorial

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Introduction

Challenges of visualizing solution sets in the objective space

- High dimension and large number of solutions
- Limitations of computing and displaying technologies
- Cognitive limitations

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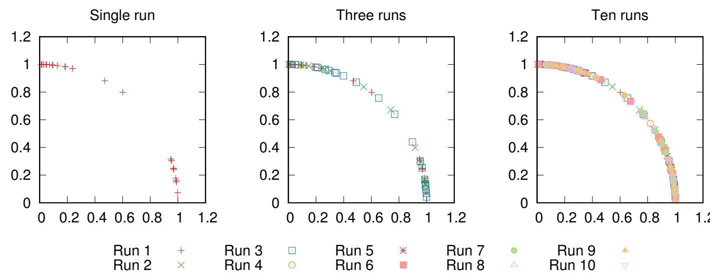
Introduction

Visualization can be hard even in 2-D

Stochastic optimization algorithms

- Single run → single approximation set
- Multiple runs → multiple approximation sets



The Empirical Attainment Function (EAF) [20] or the Average Runtime Attainment Function (aRTA) [4] can be used in such cases

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Introduction

This tutorial does not cover

- Visualization of a few solutions for decision making purposes (see [36])
- Visualization in the decision space
- General multidimensional visualization methods not previously used on approximation sets

This tutorial covers

- Visualization of entire sets in the objective space
  - Single approximation sets [2]
  - Repeated approximation sets [3, 4]
- Visualization of multiobjective landscapes

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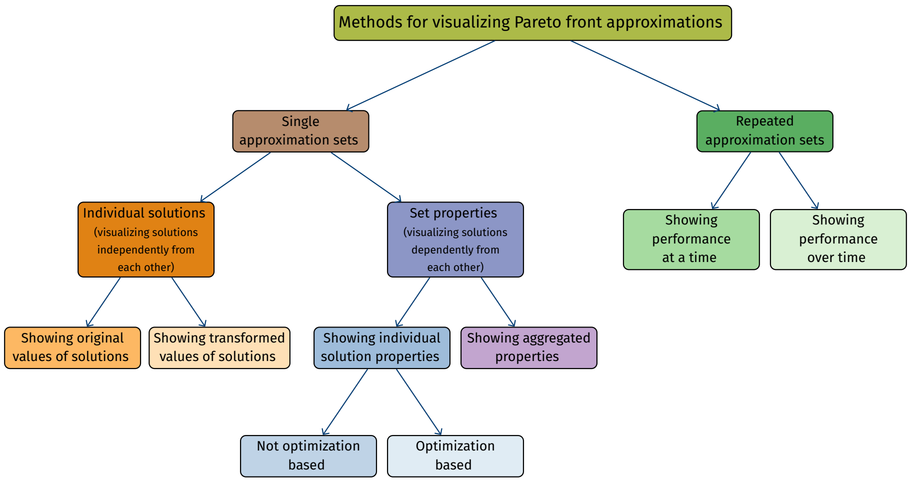
Visualizing approximation sets

Visualizing approximation sets

A taxonomy of visualization methods

A taxonomy of visualization methods [1]



Visualizing approximation sets

Visualizing single approximation sets

Methodology

Evaluating and comparing visualization methods

- No established methodology for evaluating or comparing visualization methods
- Propose benchmark approximation sets (analog to benchmark problems in multiobjective optimization)
- Visualize the sets using different methods
- Observe which set properties are distinguishable after visualization
- Only applicable to methods showing individual solutions or individual solution properties

Benchmark approximation sets

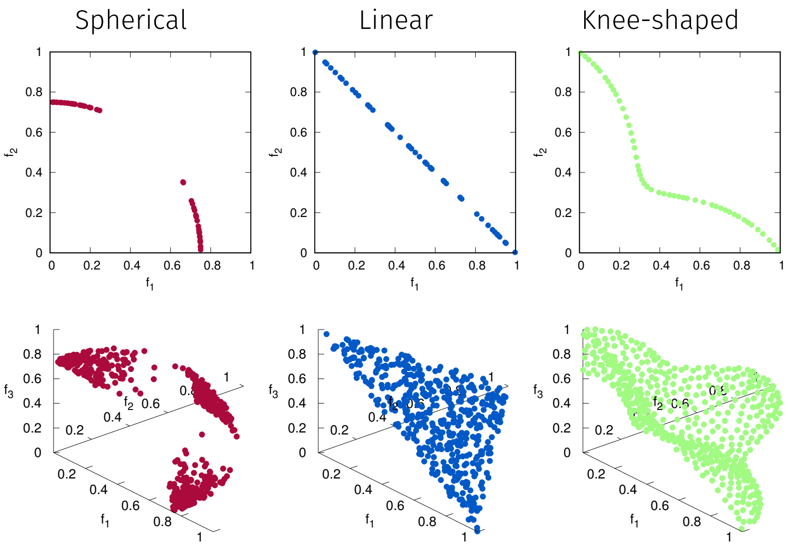
Three different sets that can be instantiated in any dimension

- Spherical with a clustered distribution of solutions (more at the corners and less at the center)
- Linear with a uniform distribution of solutions
- Knee-shaped with an even distribution of solutions

Size of each set

- 2-D: 50 solutions
- 3-D: 500 solutions
- 4-D: 500 solutions

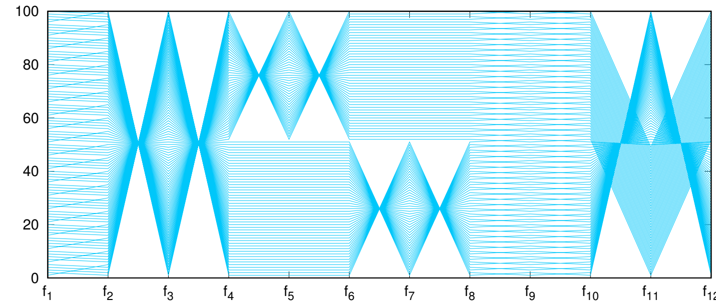
Benchmark approximation sets



Benchmark approximation sets

An additional set with redundant objectives

- Adapted from [18]
- 12 objectives
- Can be instantiated for any number of $10n$ solutions (here 100)



Desired properties of visualization methods

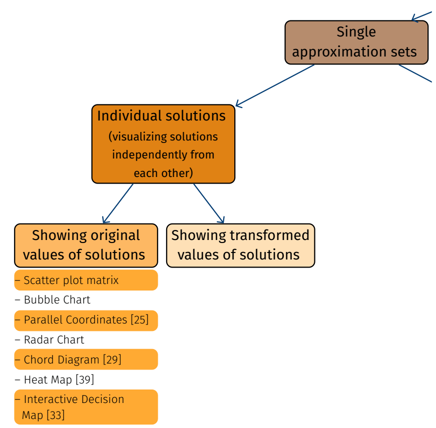
Demonstration on the 4-D spherical, linear and knee-shaped sets

- Preservation of the
  - Dominance relation between solutions
  - Front shape
  - Objective range
  - Distribution of solutions
- Robustness
- Handling of large sets
- Simultaneous visualization of multiple sets
- Scalability in number of objectives
- Simplicity

Demonstration on the 12-D approximation set

- Showing relations between objectives

Visualizing single approximation sets



Scatter plot matrix

Most often

- Scatter plot in a 2-D space
- Matrix of all possible combinations of objectives
- m objectives $\rightarrow \frac{m(m-1)}{2}$ different combinations

Alternatively

- Scatter plot in a 3-D space
- m objectives $\rightarrow \frac{m(m-1)(m-2)}{6}$ different combinations

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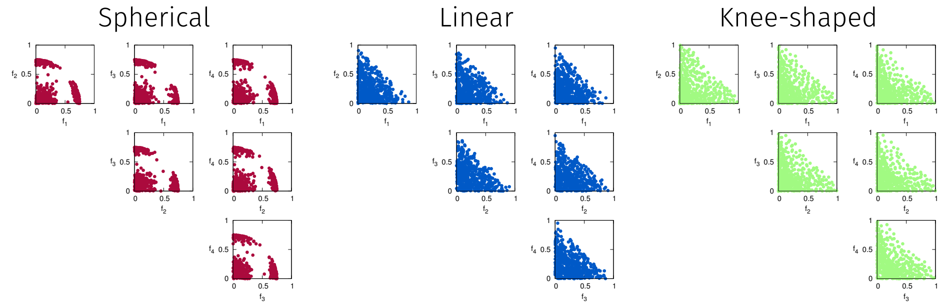
Scatter plot matrix



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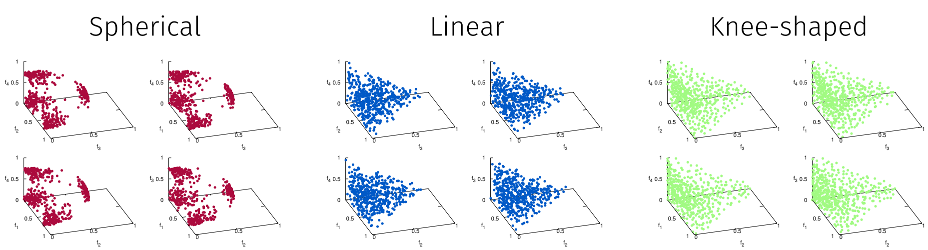
Scatter plot matrix



Preservation of the dominance relation	Preservation of the front shape	Preservation of the objective range	Preservation of the distribution of solutions	Robustness	Handling of large sets	Simultaneous visualization	Scalability	Simplicity
×	≈	✓	≈	✓	≈	✓	×	✓

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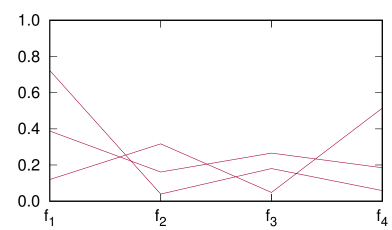
Parallel coordinates

- m objectives $\rightarrow$ m parallel axes
- Solution represented as a polyline with vertices on the axes
- Position of each vertex corresponds to that objective value
- No loss of information



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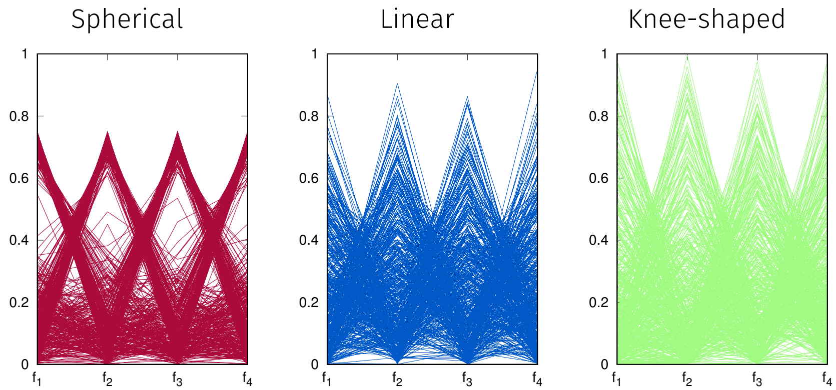
Parallel coordinates



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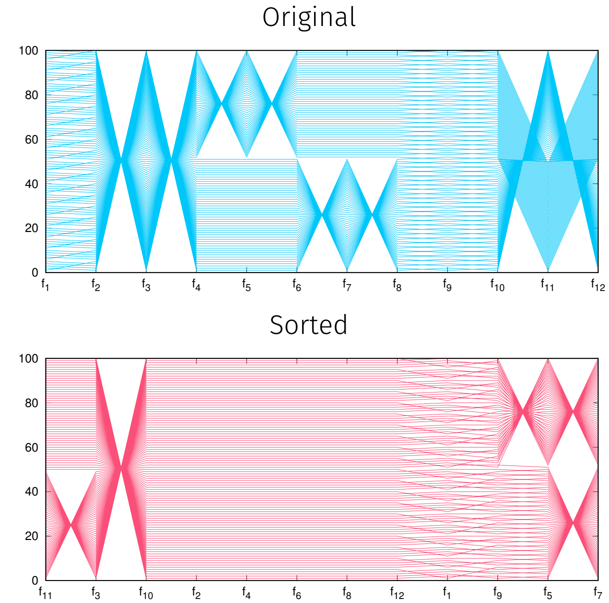
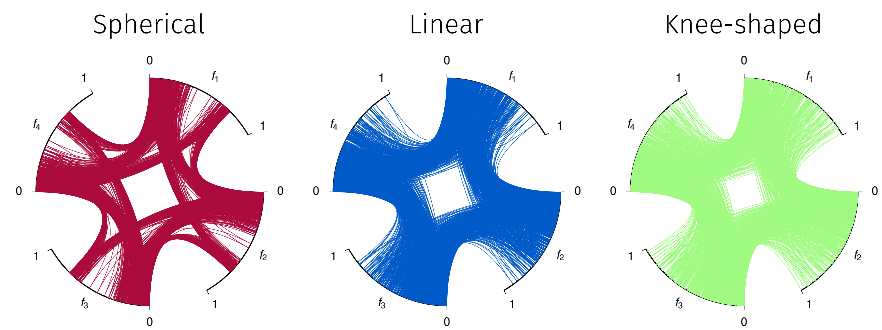
Parallel coordinates



Chord diagram

- Similar to parallel coordinates
- m objectives $\rightarrow$ m arcs

Chord diagram



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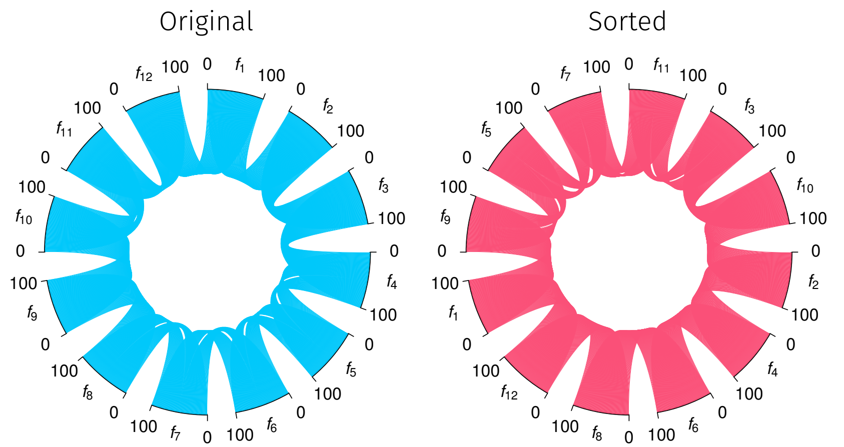
×	×	✓	≈	✓	×	×	✓	≈

Chord diagram



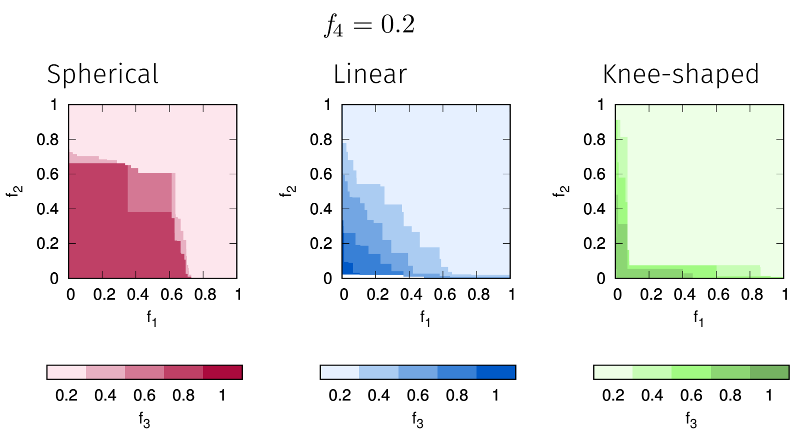
Interactive decision maps

The Edgeworth-Pareto hull (EPH) of an approximation set A contains all points in the objective space that are weakly dominated by any solution in A.

Interactive decision maps

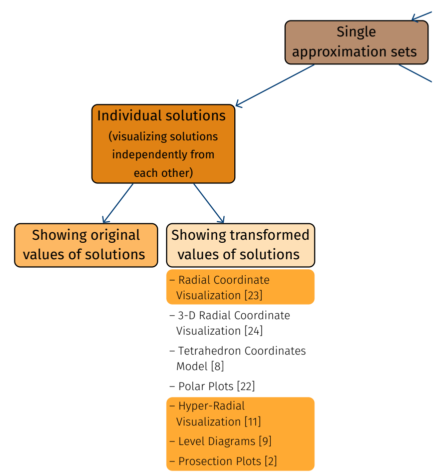
- Visualize the surface of the EPH, not the actual approximation set
- Plot a number of axis-aligned sampling surfaces of the EPH
- Color used to denote third objective
- Fixed value of the forth objective

Interactive decision maps



Preservation of the				Robustness	Handling of large sets	Simultaneous visualization	Scalability	Simplicity
dominance relation	front shape	objective range	distribution of solutions					
×	≈	✓	≈	✓	✓	×	×	≈

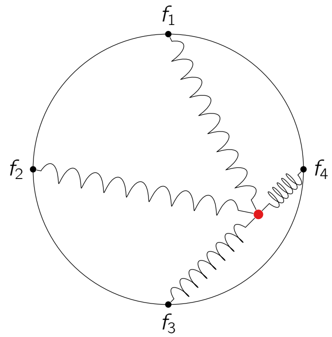
Visualizing single approximation sets



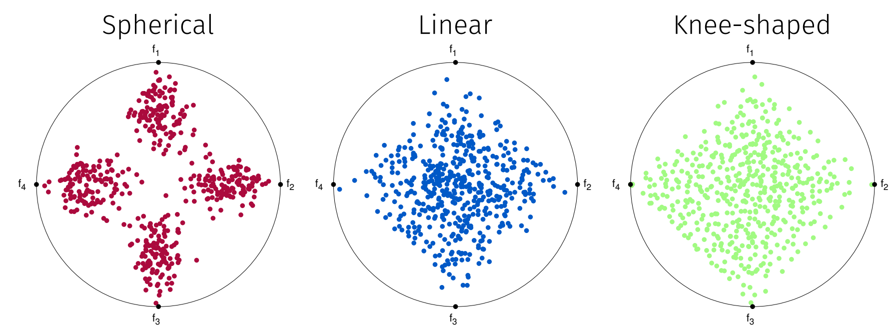
Radial coordinate visualization

Also called RadViz

- Inspired from physics
- Objectives treated as anchors, equally spaced around the circumference of a unit circle
- Solutions attached to anchors with ‘springs’
- Spring stiffness proportional to the objective value
- Solution placed where the spring forces are in equilibrium



Radial coordinate visualization

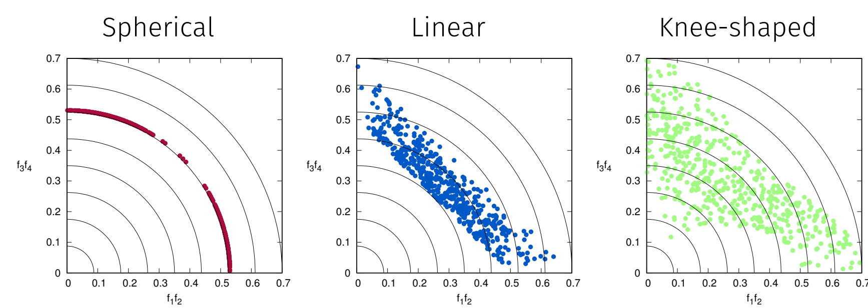


Preservation of the				Robustness	Handling of large sets	Simultaneous visualization	Scalability	Simplicity
dominance relation	front shape	objective range	distribution of solutions					
×	×	×	≈	✓	≈	✓	✓	✓

Hyper-radial visualization

- Solutions preserve distance (hyper-radius) to the ideal point
- Distances are computed separately for two subsets of objectives
- Indifference curves denote points with the same preference

Hyper-radial visualization



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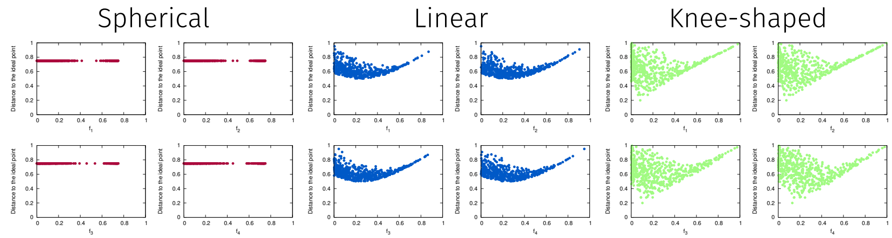
×	≈	×	×	✓	≈	✓	✓	✓

Level diagrams

- m objectives $\rightarrow$ m diagrams
- Plot solutions with objective f_i on the x axis and distance to the ideal point on the y axis

Level diagrams



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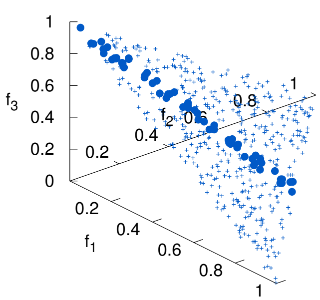
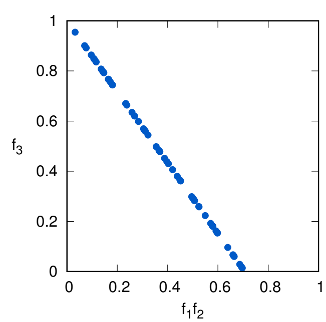
×	≈	✓	×	✓	≈	✓	✓	✓

Prosections

- Visualize only part of the objective space
- Dimensionality reduction by projection of solutions in a section
- Need to choose prosection plane, angle and section width

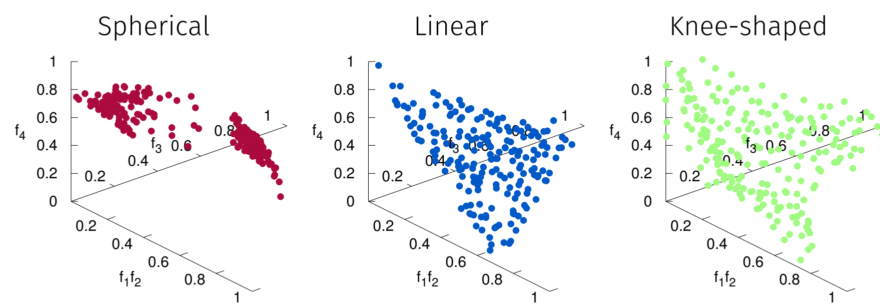


Before prosection



After prosection

Prosections



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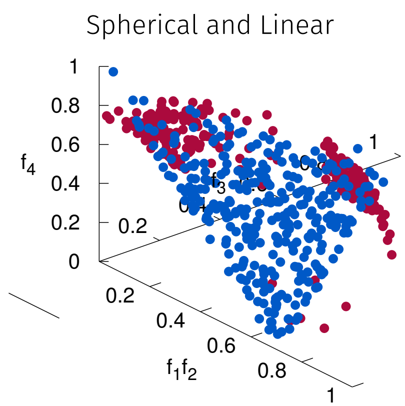
Prosections



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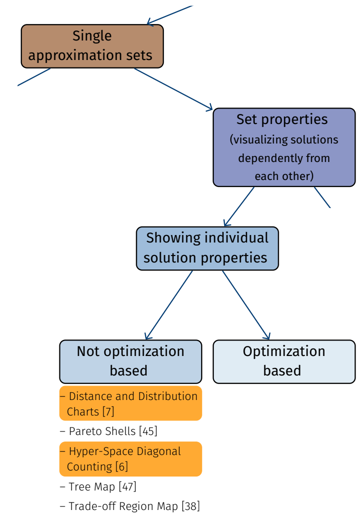
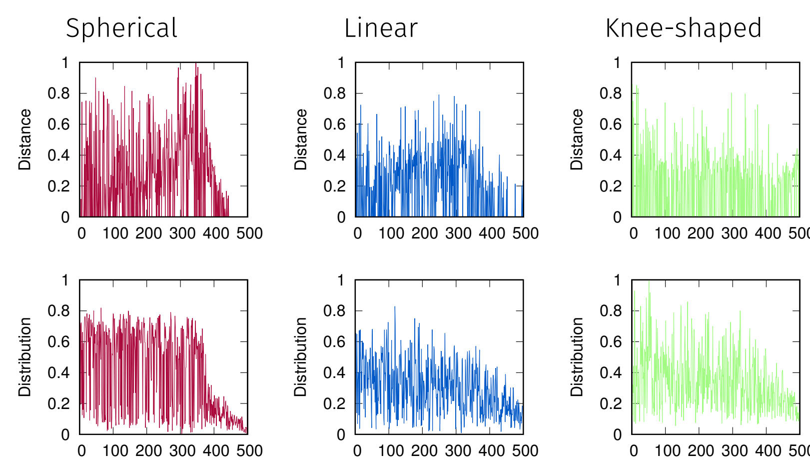
Visualizing single approximation sets



Distance and distribution charts

- Plot solutions against their distance to the Pareto front and distance to other solutions
- Distance chart
  - Plot distance to the nearest non-dominated solution
- Distribution chart
  - Sort solutions w.r.t. first objective
  - Plot distances between consecutive solutions
  - For the first/last solution, compute distance to first/last non-dominated solution
  - k solutions $\rightarrow k+1$ distances
- All distances normalized to $[0, 1]$

Distance and distribution charts



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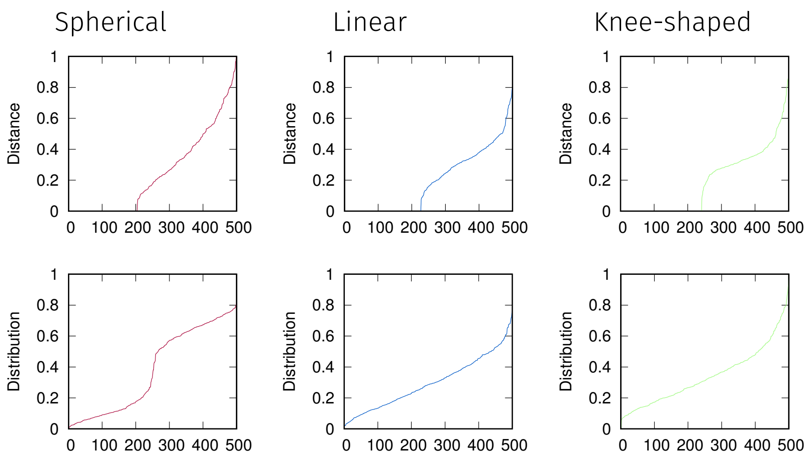
Distance and distribution charts



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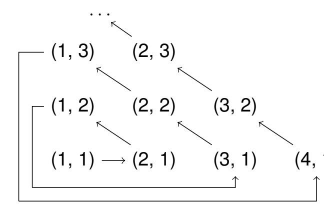
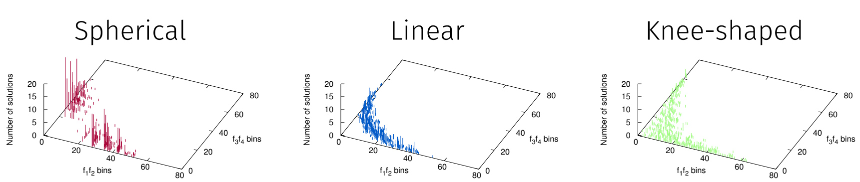
Hyper-space diagonal counting

- Inspired by Cantor's proof that shows $|\mathbb{N}| = |\mathbb{N}^2| = |\mathbb{N}^3| \ldots$



- Discretize each objective (choose a number of bins)
- In the 4-D case
  - Enumerate the bins for objectives f_1 and f_2
  - Enumerate the bins for objectives f_3 and f_4
  - Plot the number of solutions in each pair of bins

Hyper-space diagonal counting



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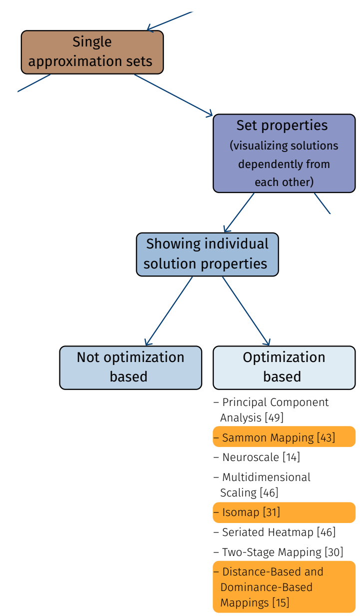
Visualizing single approximation sets



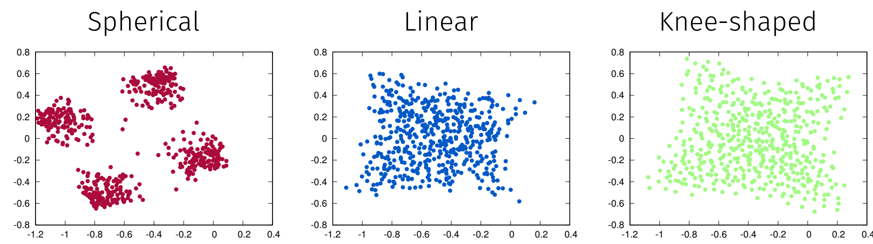
Sammon mapping

- A non-linear mapping
- Aims to preserve distances between solutions
  - d^*_{ij} distance between solutions $\mathbf{x}_i$ and $\mathbf{x}_j$ in the objective space
  - d_{ij} distance between solutions $\mathbf{x}_i$ and $\mathbf{x}_j$ in the visualized space
- Stress function to be minimized

$$S = \sum_i \sum_{j>i} (d^*_{ij} - d_{ij})^2$$

- Minimization by gradient descent or other (iterative) methods

Sammon mapping



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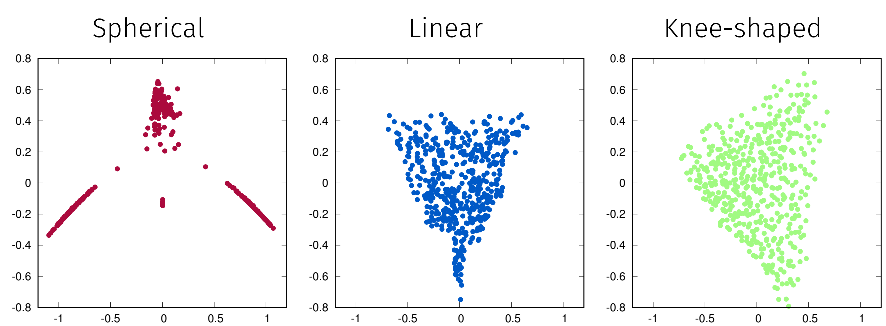
×	×	×	✓	≈	≈	✓	✓	×

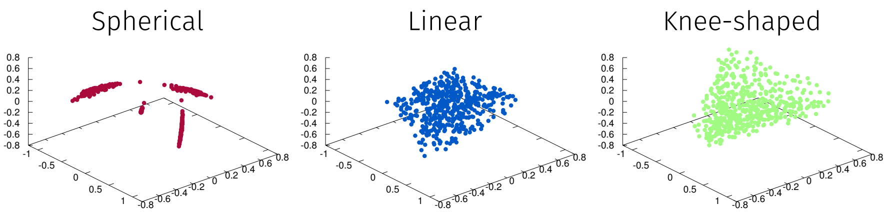
Isomap

- Assumes solutions lie on some low-dimensional manifold and the distances along this manifold should be preserved
- Creates a graph of solutions, where only the neighboring solutions are linked
- The geodesic distance between any two solutions is calculated as the sum of Euclidean distances on the shortest path between the two solutions
- Uses multidimensional scaling to perform the mapping based on these distances

Isomap



Isomap



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×	×	×	≈	≈	≈	✓	✓	×

Distance- and dominance-based mappings

Both mappings

- Use nondominated sorting to split solutions to fronts
- Project solutions onto the circumference of circles (with circle radius proportional to front number)

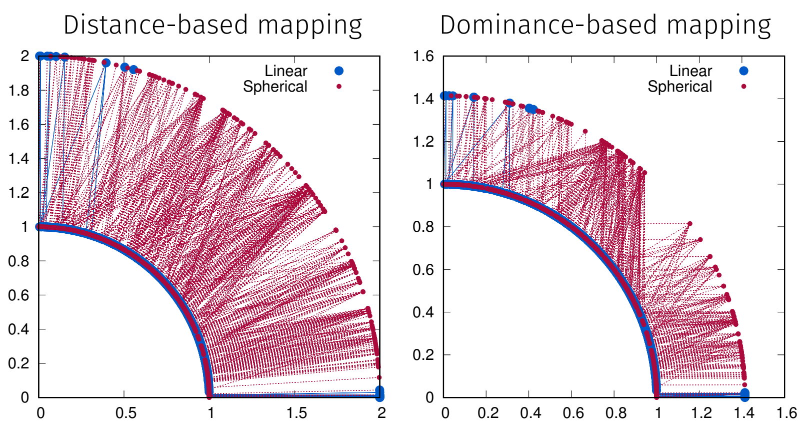
Distance-based mapping

- Tries to preserve closeness of solutions
- Two solutions are very close if their relations to other solutions are mostly equal

Dominance-based mapping

- Aims at preserving dominance relations among solutions
- All $\mathbf{x} \prec \mathbf{y}$ can be shown correctly
- Tries to minimize cases where $\mathbf{x} \not\prec \mathbf{y}$ is not shown correctly

Distance- and dominance-based mappings



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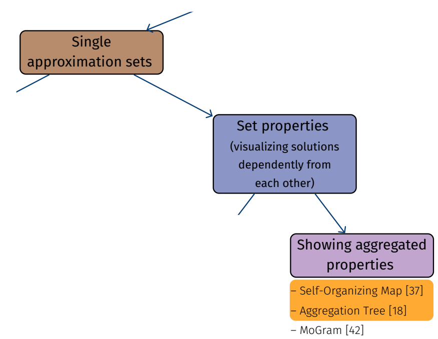
× / ✓	×	×	× / ≈	≈	×	✓	✓	×

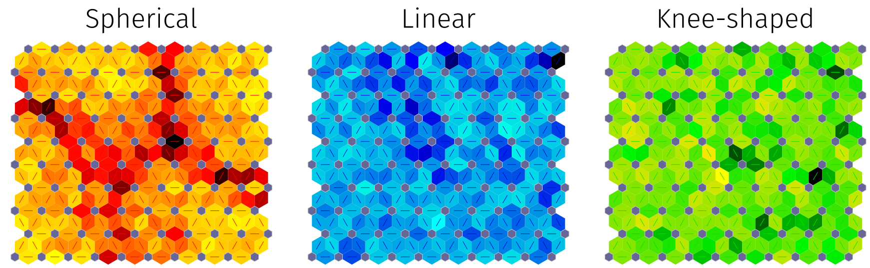
Visualizing single approximation sets



Self-organizing maps

- Self-organizing maps (SOMs) are neural networks
- Nearby solutions are mapped to nearby neurons in the SOM
- A SOM can be visualized using the unified distance matrix
- Distance between adjacent neurons is denoted with color
  - Similar neurons → light color
  - Different neurons (cluster boundaries) → dark color

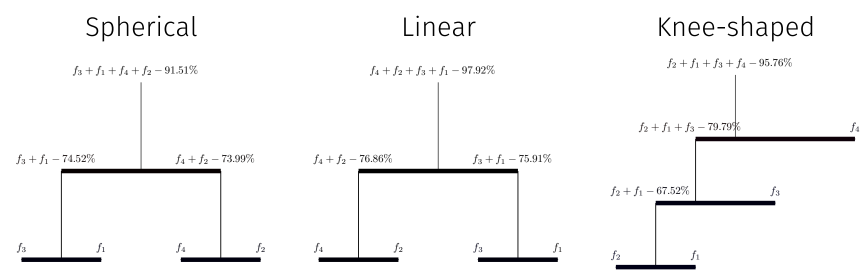
Self-organizing maps



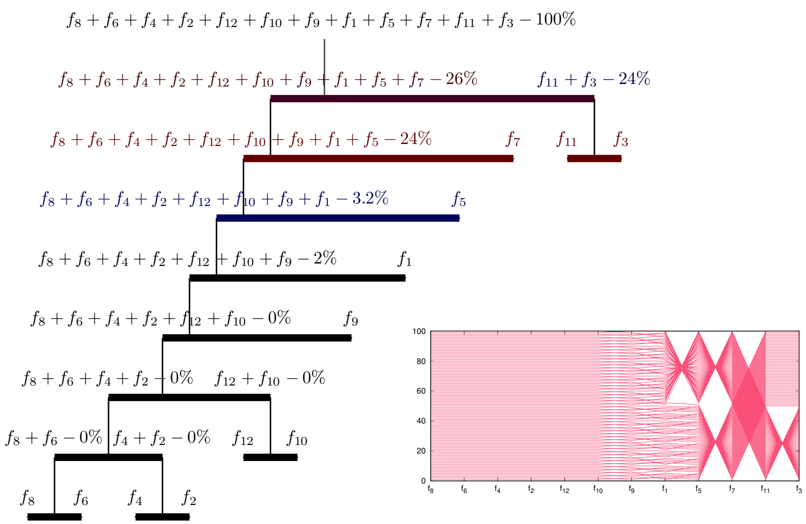
Aggregation trees

- Binary trees that show relationships between objectives
- Iterative clustering of objectives based on their harmony
- Computation of different types of conflict
- Percentages quantify the conflict between objectives
- Colors used to show type of conflict
  - global conflict (black)
  - local conflict on 'good' values (red)
  - local conflict on 'bad' values (blue)
- Can be used to sort objectives in other representations (parallel coordinates, radial charts, heat maps)

Aggregation trees



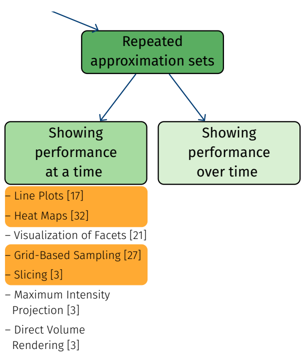
Aggregation trees



Visualizing approximation sets

Visualizing repeated approximation sets

Visualizing repeated approximation sets

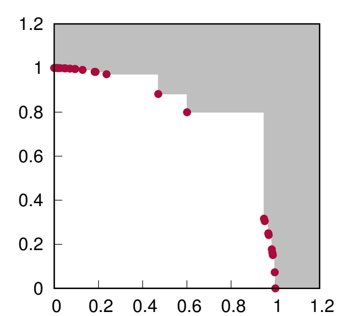


- Showing performance at a time with the Empirical Attainment Function (EAF) [20]
- Showing performance over time with the Average Runtime Attainment Function (aRTA) [4]

Empirical attainment function

Goal-attainment

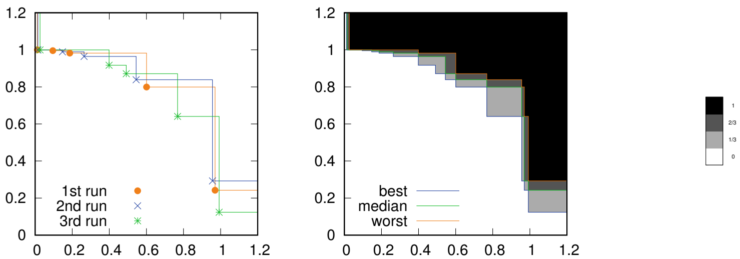
- Approximation set A
- A point in the objective space $\mathbf{z}$ is attained by A when $\mathbf{z}$ is weakly dominated by at least one solution from A



Empirical attainment function

EAF values [20]

- Algorithm $\mathcal{A}$, approximation sets $A_1, A_2, \ldots, A_r$
- EAF of $\mathbf{z}$ is the frequency of attaining $\mathbf{z}$ by $A_1, A_2, \ldots, A_r$
- Summary (or $k\%$-) attainment surfaces

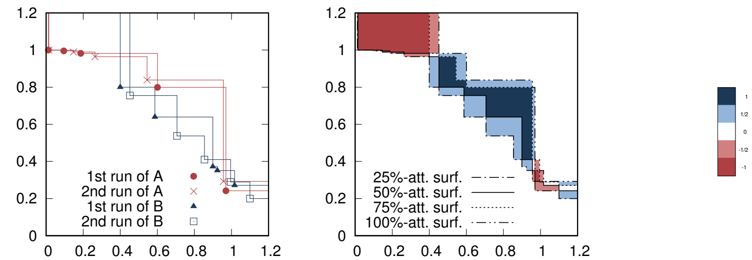


- Visualization with line plots and heat maps

Empirical attainment function

Differences in EAF values [32]

- Algorithm $\mathcal{A}$, approximation sets $A_1, A_2, \ldots, A_r$
- Algorithm $\mathcal{B}$, approximation sets $B_1, B_2, \ldots, B_r$
- Visualize differences between EAF values



- Visualization with heat maps

Visualization of 3-D EAF

Need to compute and visualize a large number (over 10 000) of cuboids

Exact case

- EAF values: Slicing [3], Visualization of facets [12, 21]
- EAF differences: Slicing, Maximum intensity projection [48, 3]

Approximated case

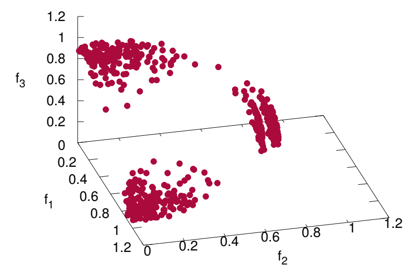
- EAF values: Grid-based sampling [27], Slicing, Direct volume rendering [13, 3]
- EAF differences: Slicing, Maximum intensity projection, Direct volume rendering

Benchmark approximation sets

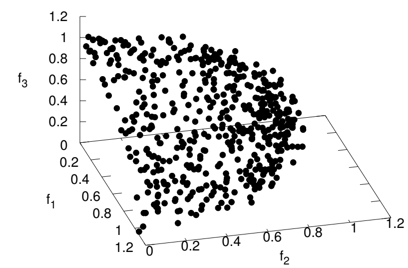
Two groups of spherical approximation sets

- 5 spherical approximation sets with a clustered distribution of solutions (different radii, 100 solutions in each)
- 5 spherical approximation sets with a uniform distribution of solutions (different radii, 100 solutions in each)

Clustered spherical



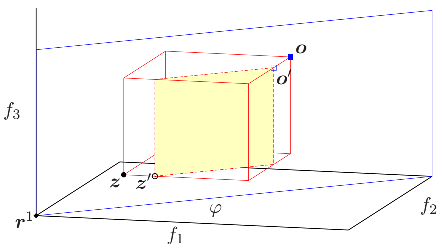
Uniform spherical



Exact 3-D EAF values and differences

Slicing

- Visualize cuboids intersecting the slicing plane
- Need to choose coordinate and angle



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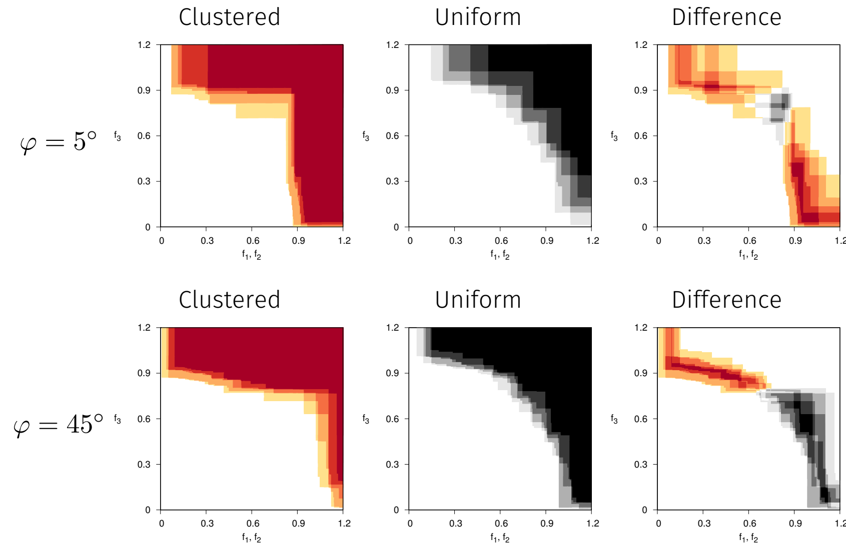
Exact 3-D EAF values and differences

Slicing



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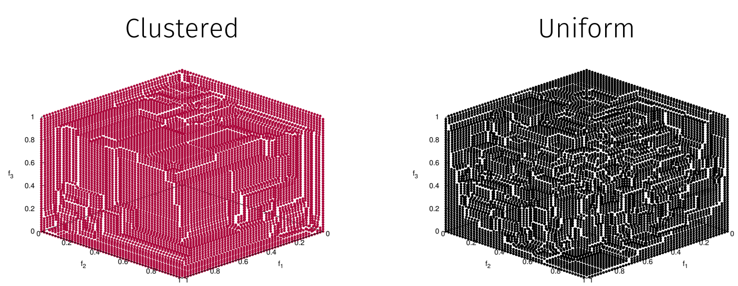
Approximated attainment surfaces

Grid-based sampling

Repeat for all $f_i f_j$, $i < j$ (i.e. $f_1 f_2$, $f_1 f_3$ and $f_2 f_3$):

- Construct a $k \times k$ grid on the plane $f_i f_j$
- Compute intersections between the attainment surface and the axis-aligned lines on the grid



Median attainment surfaces

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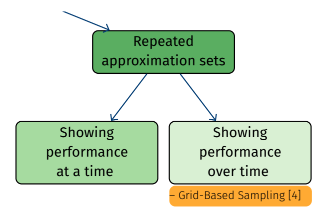
Visualizing repeated approximation sets



- Showing performance at a time with the Empirical Attainment Function (EAF) [20]
- Showing performance over time with the Average Runtime Attainment Function (aRTA) [4]

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Average Runtime Attainment Function

aRTA value

- Algorithm $\mathcal{A}$ run r times
- All solutions that are nondominated at creation are recorded
- aRTA($\mathbf{z}$) is the average number of evaluations needed to attain $\mathbf{z}$

aRTA ratio

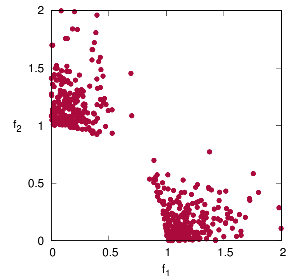
- Algorithms $\mathcal{A}$ and $\mathcal{B}$
- Visualize ratio between aRTA($\mathbf{z}$) values for $\mathcal{A}$ and $\mathcal{B}$

Benchmark approximation sets

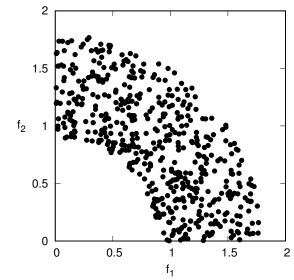
Two groups of sets mimicking convergence to a spherical front

- 5 sets mimicking logarithmic convergence to a spherical front with a clustered distribution (100 solutions each)
- 5 sets mimicking linear convergence to a spherical front with a linear distribution (100 solutions each)

Clustered spherical with logarithmic convergence



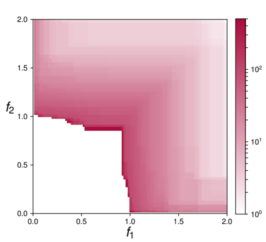
Uniform spherical with linear convergence



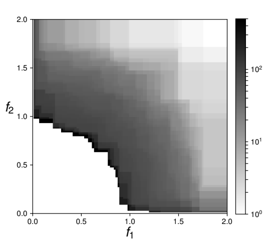
Average Runtime Attainment Function

Grid-based sampling

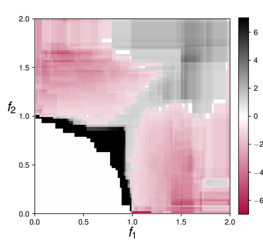
Clustered



Uniform



Ratio



Visualizing problem landscapes

Visualizing problem landscapes

General idea

- 2-D decision space (projection) approximated with a $k \times k$ grid
- Color (or the third dimension) used to show a value

Visualizing ranks

- Multiobjective cost landscapes [16]

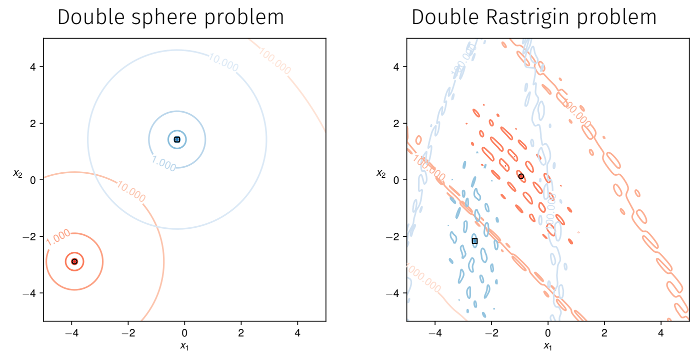
Visualizing cumulative gradients

- Cumulated gradient field landscapes [26]

Benchmark problems

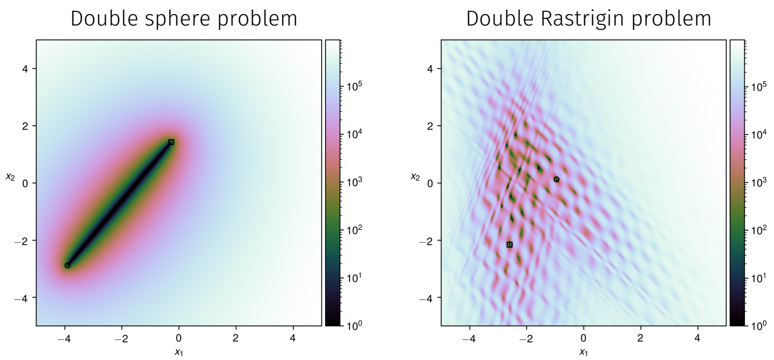
Two problems from the `bbob-biobj` test suite [5]

- Double sphere problem ($F_1 = (f_1, f_1)$ in 2-D, instance 1)
- Double Rastrigin problem ($F_{46} = (f_{15}, f_{15})$ in 2-D, instance 4)



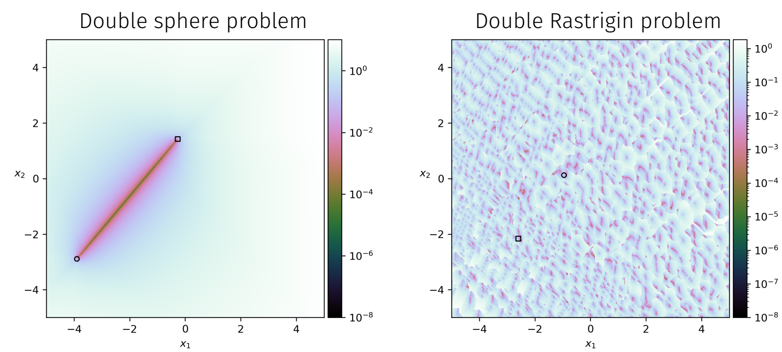
Visualizing ranks

- Rank = number of grid points that dominate the current point
- All nondominted points have rank $= 0$
- 1000×1000 grid
- Visualize rank $+ 1$ in logarithmic scale



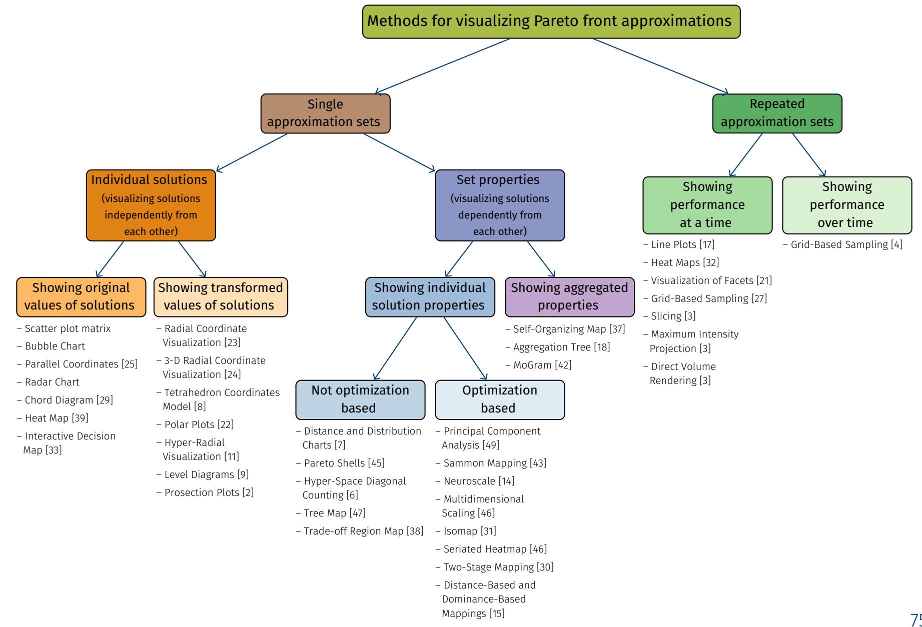
Visualizing cumulative gradients

- From a grid point, follow the path in the direction of the bi-objective gradient
- Sum all bi-objective gradient values along the path
- 1000×1000 grid



Summary

Summary



Summary

- Visualization in multiobjective optimization useful for various purposes
- Customized methods are needed to address the peculiarities of approximation set visualization as well as problem landscape visualization
- New visualization methods should first be analyzed using some approximation sets with known properties

Acknowledgement

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