Affine OneMax

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ABSTRACT

A new class of test functions for black box optimization is introduced. Affine OneMax (AOM) functions are defined as compositions of OneMax and invertible affine maps on bit vectors. The black box complexity of the class is upper bounded by a polynomial of large degree in the dimension. Tunable complexity is achieved by expressing invertible linear maps as finite products of transvections. Finally, experimental results are given to illustrate the performance of search algorithms on AOM functions.

CCS CONCEPTS

• Theory of computation → Theory of randomized search heuristics; **• Mathematics of computing → Combinatorial optimization**; **• Computing methodologies → Discrete space search**.

KEYWORDS

Combinatorial optimization, black box optimization, test functions, tunable complexity, linear group, transvections, black box complexity, discrete Fourier analysis

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1 INTRODUCTION

Theoretical and empirical analyses of search algorithms in the context of black box optimization often require test functions. On the practical side, an algorithm is usually selected by its performance across a collection of diverse test functions such as OneMax, LeadingOnes, MaxSat, etc. In particular, NK landscapes [4] are a class of functions with tunable complexity. In this model, the fitness of an n-dimensional bit vector is the sum of n partial functions, one per variable. Each partial function depends on a variable and its k neighbors. The number k controls the interaction graph hence the complexity of the fitness landscape. NK landscapes have found many applications from theoretical biology to combinatorial optimization. When used as test functions, their flexibility comes at the price of a great number of parameters. Values of partial functions are often sampled from normal or uniform distributions, which generates highly irregular landscapes with unknown maximum, even for small k.

Affine OneMax (AOM) functions are test functions defined as compositions of OneMax and invertible affine maps on bit vectors. They are integer-valued, have a known maximum, and their representations only require a number of bits quadratic in the dimension. The idea of composition of a fitness function and a linear or affine map has already been explored in the context of evolutionary computation [6–8]. The problem addressed in this line of research is to identify a representation of the search space, e.g. an affine map, able to transform a deceptive function into an easy one for genetic algorithms. We propose an algorithm which learns such a representation for AOM functions. Moreover, considering the identity as a linear map, it appears that functions of increasing complexity can be obtained starting with the identity and applying small perturbations to it. In the language of linear algebra, those perturbations are called elementary (or special) transvections. The key parameter of resulting AOM functions is the sequence length which is the analogue of parameter k in NK landscapes.

The paper is organized as follows. Sec. 2 introduces general AOM functions and gives their basic properties. Sec. 3 introduces elementary transvections and their products. In Sec. 4, we report results of experiments involving search algorithms on random instances. Sec. 5 concludes the paper.

2 GENERAL CASE

The OneMax function $\ell : \{0,1\}^n \to \mathbb{R}$ is defined by $\ell(x) = \sum_{i=1}^n x_i$, where the x_i's are seen as real numbers. OneMax takes $n+1$ values $0, 1, \ldots, n$ and, for all $k \in [0..n]$, $|\ell^{-1}(\{k\})| = \binom{n}{k}$. It reaches its maximum n only at $1^n = (1, 1, \ldots, 1)$. From now on, the set $\{0,1\}$ is seen as the finite field $\mathbb{F}_2$ ($1 + 1 \equiv 0 \pmod 2$) and $\{0,1\}^n$ as a linear space over $\mathbb{F}_2$. An affine map $\{0,1\}^n \to \{0,1\}^n$ is defined by $x \mapsto Mx + b$, where M is a $n \times n$ bit matrix and b a $n \times 1$ bit vector. Sums and products in $Mx + b$ are computed in $\mathbb{F}_2$ not in $\mathbb{R}$.

An affine OneMax function f is the composition of OneMax and an invertible affine map. More precisely, it is defined by $f(x) = \ell(Mx + b)$, where M is an invertible matrix and $b \in \{0,1\}^n$. The invertibility of M is important because it ensures that f shares the properties of OneMax outlined above. Firstly, it takes $n + 1$ values $0, 1, \ldots, n$ and, for all $k \in [0..n]$, $|f^{-1}(\{k\})| = \binom{n}{k}$. Secondly, it reaches its maximum n only once at $x^* = M^{-1}(1^n + b)$. The set of all AOM functions will be denoted by $\mathcal{F}$. By construction, $\mathcal{F}$ is closed under invertible affine maps but

Theorem 2.1. *$\mathcal{F}$ is not closed under permutations.*

As a consequence, by a NFL theorem [9], the average performance of an algorithm over all AOM functions depends on the algorithm.

Interestingly, given an AOM function, it is possible to learn its parameters M and b, hence maximize it, using a polynomial number of queries. The idea is to exploit the sparsity of the Fourier spectrum of AOM functions. To do so, we propose to apply a modified version of the Kushilevitz-Mansour algorithm [5] whose original goal is to approximate Boolean functions.

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Theorem 2.2. *Let $f \in \mathcal{F}$ and $\delta \in (0, 1)$. There is a randomized algorithm which exactly learns and maximizes f with at most $m_1 m_2 n^2 + m_3 n$ evaluations and probability at least $1 - \delta$, where $m_1 = \Theta\left(n^4 \log(n^2/\delta)\right)$, $m_2 = \Theta\left(n^4 \log(n^2 m_1/\delta)\right)$, $m_3 = \Theta\left(n^2 \log(n/\delta)\right)$.*

Using this algorithm, we can provide an upper bound of the (unrestricted) black box complexity [1, 2] of AOM functions. The black box complexity $B_{\mathcal{F}}$ of AOM functions is defined as $\inf_{A \in \mathcal{A}} \sup_{f \in \mathcal{F}} \mathbb{E}(T(A, f))$, where $\mathcal{A}$ is the set of randomized search algorithms and $\mathbb{E}(T(A, f))$ the expected runtime of algorithm A on f. We have

Corollary 2.3. $B_{\mathcal{F}} = O(n^{10} \log^2 n)$.

3 SEQUENCES OF TRANSVECTIONS

We would like to sort AOM functions from the easiest to the hardest ones to maximize or at least provide them with some structure. We will achieve this goal using elementary transvections. For all $i, j \in [1..n]$, with $i \neq j$, let τ_{ij} denote an elementary transvection. For all $x \in \{0, 1\}^n$, $\tau_{ij} x$ is obtained from x by adding x_j to x_i or $x_i \leftarrow x_i + x_j$, leaving other bits unchanged. At most one bit is changed. It is clear that τ_{ij} is a linear map. The key property is that elementary transvections are generators of the group of invertible matrices.

Having defined transvections, we consider finite products of transvections and the corresponding classes of AOM functions. Let t be a positive integer and G_t be the set of all products of t transvections. Let $\mathcal{F}_t$ be the set of functions $x \mapsto \ell(Mx + b)$, where $M \in G_t$ and $b \in \{0, 1\}^n$. Functions in classes $\mathcal{F}_t$ will be refered to as transvection sequence AOM (TS-AOM) functions.

4 EXPERIMENTS

The first experiment[1] is a fixed-budget experiment in which search algorithms have been applied to random instances of AOM functions ($n = 100$, $3 \cdot 10^5$ evaluations, 20 runs). In particular, for each run, a new invertible 100×100 matrix has been sampled. We have considered the following search algorithms: random search, random local search, hill climbing, simulated annealing, GA, $(1+1)$ EA, $(10+1)$ EA, PBIL, MIMIC, UMDA, HBOA, LTGA, and P3. For all of them, the median function value is not greater than 73, whereas the function maximum is 100. It should be noted that random search performs as well as other algorithms.

On the contrary to AOM functions, TS-AOM functions with small sequence length make good test functions. They reveal how search algorithms resist an increasing number t of perturbations (elementary transvections). Fig. 1 shows the results for $n = 100$. MIMIC is the first algorithm to fail to maximize an instance (at $t = 10$) whereas P3 is the last one (at $t = 130$).

5 CONCLUSION

We have introduced affine OneMax functions which are test functions for search algorithms. They are defined as compositions of OneMax and invertible affine maps on bit vectors. They have a simple representation and a known maximum. Tunable complexity is achieved by expressing invertible linear maps as finite products of transvections. The complexity is controlled by the length

[1] All experiments have been produced with the HNCO framework [3].

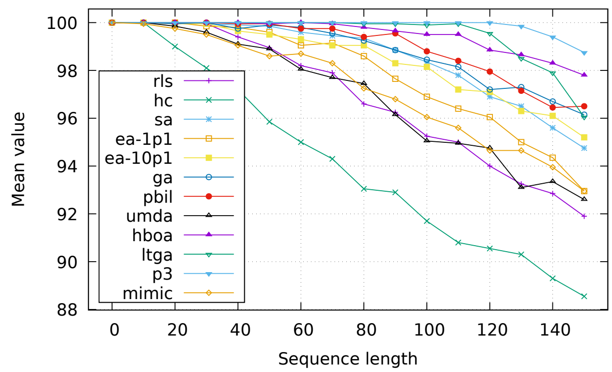

Figure 1: Mean fixed-budget performance of search algorithms on TS-AOM functions as a function of sequence length $t \in [0..150]$ ($n = 100$, $3 \cdot 10^5$ evaluations, 20 runs).

of transvection sequences and their properties. Transvection sequence AOM functions with small sequence length are of practical interest in the benchmarking of search algorithms. We have shown by means of Fourier analysis that the black box complexity of AOM functions is upper bounded by a high degree polynomial. However, it can be as low as logarithmic for the simplest AOM functions.

Many open questions remain. The gap between the lower and upper bounds of the black box complexity in the general case is significant and should be reduced. A rigorous analysis of the runtime of $(1 + 1)$ EA on AOM functions would be of great interest. OneMax is one of the simplest nontrivial functions used in black box optimization. It seems legitimate to consider alternative functions to define new classes in the same way as for AOM functions. Candidate functions include pseudo-linear functions or LeadingOnes. However, since LeadingOnes is not sparse, the method presented in this paper does not apply.

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