Runtime Analysis of Population-based Evolutionary Algorithms[1]

Advanced Tutorial at GECCO 2021

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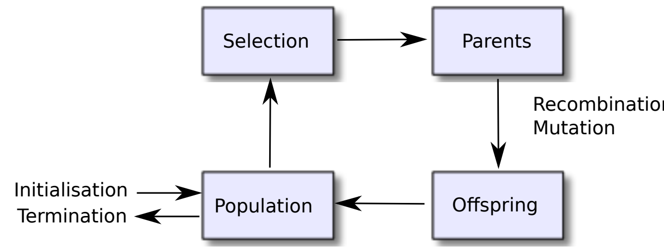
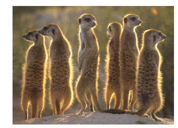
[1] For the latest version of these slides, please go to `http://www.cs.bham.ac.uk/~lehrepk/gecco2021`

Evolutionary Algorithms





Aims and Goals of this Tutorial

- The scope of this tutorial is restricted to population-based
  - generational, non-elitist EAs (first part), and
  - steady-state, elitist EAs (second part).
- This tutorial will provide an overview of
  - the goals of runtime analysis of EAs
  - selected, generally applicable techniques
- You should attend if you wish to
  - theoretically understand the behaviour and performance of the EAs you design
  - familiarise yourself with some of the techniques used
  - pursue research in the area
- enable you or enhance your ability to
  1. understand theoretically population-dynamics of EAs on different problems
  2. perform time complexity analysis of population-based EAs on common toy problems
  3. have the basic skills to start independent research in the area

Outline

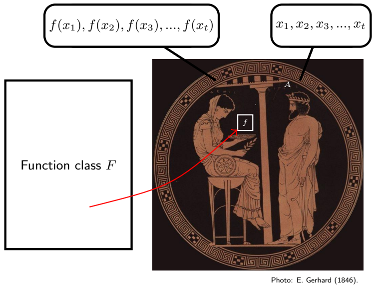
Introduction
Runtime Analysis
Drift Analysis

Upper bounds for Non-elitist, Generational EAs
The Level Based Theorem
(μ,λ) GA on OneMax
Noisy and Uncertain Fitness

Lower Bounds for Non-elitist, Generational EAs
Negative Drift Theorem for Populations
Mutation-Selection Balance
Self-adaptation
Negative Drift with Crossover

Upper Bounds for Elitist, Steady-state EAs
(1+1) EA and Artificial Fitness Levels
(μ+1) Genetic Algorithm and OneMax
(μ+1) Genetic Algorithm and Jump

Black Box Optimisation Algorithms and Runtime



- An unknown optimisation problem f is chosen, possiby adversarily, from a problem class F known to the algorithm.
- For every $t \in \mathbb{N}$, using the obtained information $(x_1, f(x_1)), \ldots, (x_t, f(x_t))$, the algorithm queries a new search point x_{t+1} to obtain $f(x_{t+1})$ from the oracle.

Definition

The **runtime** of algorithm A on fitness function[2] $f : \{0,1\}^n \to \mathbb{R}$ is

$$T_{A,f} := \min_{t \in \mathbb{N}} \{t \mid \forall y \in \{0,1\}^n,\ f(x_t) \geq f(y)\}.$$

The **worst case** expected runtime of algorithm A on problem class F is

$$T_{A,F} := \max_{f \in F} \mathbf{E}\left[T_{A,f}\right]$$

Droste, Jansen, and Wegener [2006]

[2] This definition assumes that the objective is to maximise the fitness function.

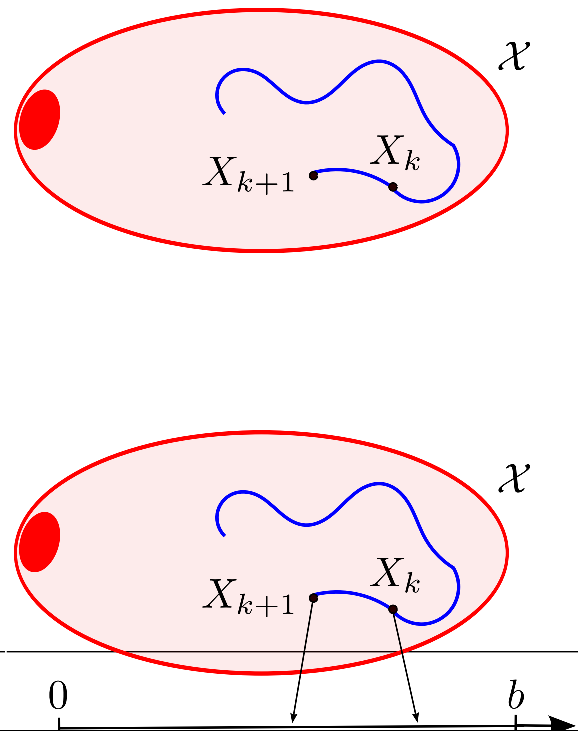
Runtime Analysis of Evolutionary Algorithms

Analysis of Evolutionary Algorithms

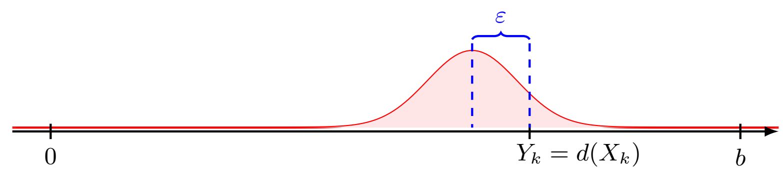
- Infinite population size
- Markov chain analysis He and Yao [2003]
- No parent population, or monomorphic populations
  - (1+1) EA
  - (1+λ) EA Jansen, Jong, and Wegener [2005]
  - (1,λ) EA Rowe and Sudholt [2012]
- Fitness-level techniques
  - (μ+1) EA Witt [2006]
  - (N+N) EAs Chen, He, Sun, Chen, and Yao [2009]
  - non-elitist EAs with unary variation operators Lehre [2011b], Dang and Lehre [2014]
  - (μ+1) GA Corus and Oliveto [2017]
- Stochastic dominance Doerr [2018]
- Drift analysis and martingale theory
  - Fitness proportionate selection Neumann, Oliveto, and Witt [2009a], Oliveto and Witt [2014, 2015]
- Family trees
  - (μ+1) EA Witt [2006]
  - (μ+1) IA Zarges [2009]
- Multi-type branching processes Lehre and Yao [2012]
  - Negative drift theorem for populations Lehre [2011a]
- Level-based analysis Corus, Dang, Eremeev, and Lehre [2014]

Drift Analysis

What is Drift[3] Analysis?



Additive Drift Theorem



(C1+) $\forall k \quad \mathbf{E}[Y_{k+1} - Y_k \mid Y_k > 0] \leq -\varepsilon$

(C1−) $\forall k \quad \mathbf{E}[Y_{k+1} - Y_k \mid Y_k > 0] \geq -\varepsilon$

Theorem (He and Yao [2001], Jägersküpper [2007], Jägersküpper [2008])

Given a stochastic process $Y_1, Y_2, \ldots$ over an interval $[0, b] \subset \mathbb{R}$.
Define $T := \min\{k \geq 0 \mid Y_k = 0\}$, and assume $\mathbf{E}[T] < \infty$.

- *If (C1+) holds for an $\varepsilon > 0$, then $\mathbf{E}[T \mid Y_0] \leq b/\varepsilon$.*
- *If (C1−) holds for an $\varepsilon > 0$, then $\mathbf{E}[T \mid Y_0] \geq Y_0/\varepsilon$.*

LeadingOnes

$$\textsc{LeadingOnes}(x) := \sum_{i=1}^{n} \prod_{j=1}^{i} x_j$$

$$x = \overbrace{1111111111111111}^{\text{Leading 1-bits}}\overbrace{0{*}{*}{*}{*}{*}{*}{*}{*}{*}{*}{*}{*}{*}{*}{*}{*}}^{\text{Remaining bits}}.$$

Left-most 0-bit

Runtime Analysis of (1,λ) EA on LeadingOnes

Theorem

The (1,λ) EA with $\lambda = n$ optimises $\textsc{LeadingOnes}$ *in $O(n^2)$ expected time.*

Proof

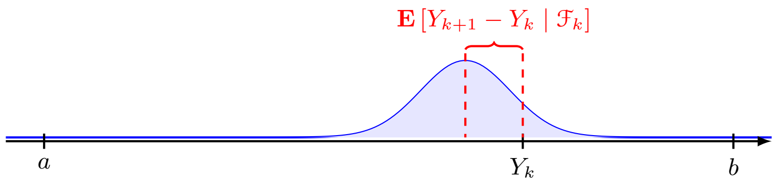
- Distance: let $d(x) = n - i$ where i is the number of leading ones;
- Drift:

$$\begin{aligned}
&\mathbf{E}[d(X_t) - d(X_{t+1}) | d(X_t) = n - i] \\
&\geq 1 \cdot \left(1 - \left(1 - \frac{1}{n}\left(1 - \frac{1}{n}\right)^{n-1}\right)^{\lambda}\right) - n \cdot \left(1 - \left(1 - \frac{1}{n}\right)^{n}\right)^{\lambda} \\
&= c_1 - n \cdot c_2^n = \Omega(1)
\end{aligned}$$

Hence,

$$\mathbf{E}[T] \leq \lambda \cdot \frac{\text{max distance}}{\text{drift}} = \lambda \cdot \frac{n}{\Omega(1)} = O(n^2)$$

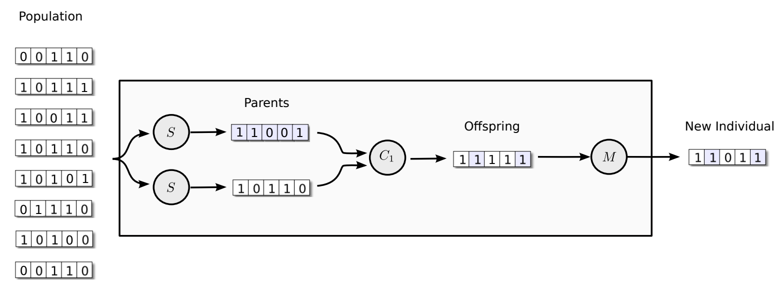
Some Drift Theorems



Drift Condition[4]	Statement	Note
$\mathbf{E}\left[Y_{k+1} \mid \mathcal{F}_k\right] \leq Y_k - \varepsilon_0$	$\mathbf{E}[\tau_a] \leq$	Additive drift [He and Yao, 2001, Jägersküpper, 2008]
	$\Pr(\tau_a > B) \leq$	[Hajek, 1982]
	$\Pr(\tau_b < B) \leq$	Negative drift [Hajek, 1982, Oliveto and Witt, 2010]
$\mathbf{E}\left[Y_{k+1} \mid \mathcal{F}_k\right] \geq Y_k - \varepsilon_0$	$\leq \mathbf{E}[\tau_a]$	Additive drift (lower b.) [He and Yao, 2001, Jägersküpper, 2007]
$\mathbf{E}\left[Y_{k+1} \mid \mathcal{F}_k\right] \leq Y_k$	$\mathbf{E}[\tau_a] \leq$	Supermartingale [Neumann, Sudholt, and Witt, 2009b]
$\mathbf{E}\left[Y_{k+1} \mid \mathcal{F}_k\right] \leq \delta Y_k$	$\mathbf{E}[\tau_a] \leq$	Multiplicative drift [Doerr, Johannsen, and Winzen, 2010, Fischer, Olbrich, and Vöcking, 2008]
	$\Pr(\tau_a > B) \leq$	[Doerr and Goldberg, 2010]
$\mathbf{E}\left[Y_{k+1} \mid \mathcal{F}_k\right] \geq \delta Y_k$	$\leq \mathbf{E}[\tau_a]$	Multiplicative drift (lower b.) [Lehre and Witt, 2012]
$\mathbf{E}\left[Y_{k+1} \mid \mathcal{F}_k\right] \leq h(Y_k)$	$\mathbf{E}[\tau_a] \leq$	Variable drift [Johannsen, 2010]
$\mathbf{E}\left[Y_{k+1} \mid \mathcal{F}_k\right] \leq h(Y_k)$	$\Pr(\tau_a > B) \leq$	[Lehre and Witt, 2014]
	$\Pr(\tau_b < B) \leq$	[Lehre and Witt, 2014]
$\mathbf{E}\left[e^{\lambda Y_{k+1}} \mid \mathcal{F}_k\right] \leq \frac{e^{\lambda Y_k}}{\alpha_0}$	$\Pr(\tau_b < B) \leq$	Population drift [Lehre, 2011a]

[4] Some drift theorems need additional conditions.

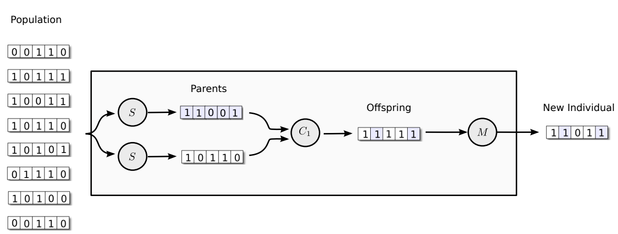
Outline of a Non-elitist, Generational Evolutionary Algorithm[5]



1: **initialise** a population P_0 of λ individuals uniformly at random.
2: **for** $t = 0, 1, 2, \ldots$ until termination condition **do**
3: **evaluate** the individuals in population P_t.
4: **for** $i = 1$ to λ **do**
5: **select** two parents from population P_t.
6: **recombine** the two parents.
7: **mutate** the offspring and add it to population P_{t+1}.

[5] Pseudo-code adapted from **?**.

A Model of Non-elitist, Generational EAs

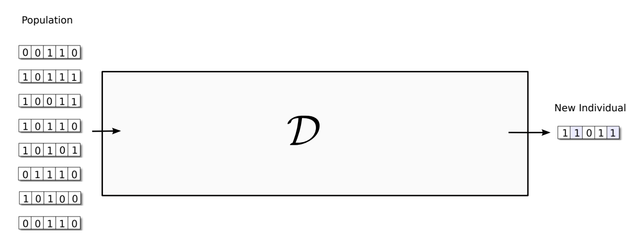


Wide range of evolutionary algorithms...

- selection mechanisms (ranking selection, (μ, λ)-selection, tournament selection, ...)
- fitness models (deterministic, stochastic, dynamic, partial, ...)
- variation operators
- search spaces (e.g. bitstrings, permutations, ...)

We will describe many of these with a general mathematical model.

A Model of Non-elitist, Generational EAs



Require: Search space $\mathcal{X}$ and random operator $\mathcal{D} : \mathcal{X}^\lambda \to \mathcal{X}$
1: $P_0 \sim \mathrm{Unif}(\mathcal{X}^\lambda)$
2: **for** $t = 0, 1, 2, \ldots$ until termination condition **do**
3: **for** $i = 1$ to λ **do**
4: $P_{t+1}(i) \sim \mathcal{D}(P_t)$

Why consider non-elitism?

- Biological plausibility
- Mathematical tractability (disputed)
- Beneficial in uncertain environments
  - Noisy optimisation [Dang and Lehre, 2016a]
  - Dynamical optimisation [Dang, Jansen, and Lehre, 2017b]
- Help populations overcome local optima
  - Exponential speedup with self-adaptation on a a peaked problem [Dang and Lehre, 2016d]
  - Non-elitist EAs excel in fitness landscapes with sparse deceptive regions and dense valleys (GECCO'2021 Theory Track)
- Appropriate algorithm configuration critical with non-elitism.
  - Both (μ,λ) EA and $(\mu+\lambda)$ EA optimises JUMPin[6] $\Theta(n^k)$ [Doerr, 2020]

[6] Under certain conditions on μ, λ, and k.

Level-based Theorem[7]

[7] Corus, Dang, Eremeev, and Lehre [2018]

Level-based Analysis

Problem

- Given any target set $\boldsymbol{B} \subset \boldsymbol{\mathcal{X}}$ (e.g. global optima), let

$$T_B := \min\{\lambda t \mid P_t \cap B \neq \emptyset\}$$

  where $P_0, P_1, \ldots$ are the populations generated by the algorithm.
- How does $\mathbf{E}[T_B]$ depend on $\mathcal{D}$ and λ? Informally, how much time does the algorithm require to discover the target set.

Level-based Theorem (LBT)

- If $\mathcal{D}$ and λ satisfy certain conditions,
  then LBT provides an upper bound for $\mathbf{E}[T_B]$.

Level Partitioning of Search Space $\mathcal{X}$

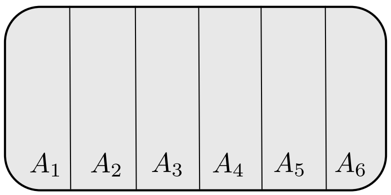
Definition

$(A_1, \ldots, A_m)$ is a level-partitioning of search space $\boldsymbol{\mathcal{X}}$ if

- $\bigcup_{j=1}^{m} A_j = \boldsymbol{\mathcal{X}}$ (together, the levels cover the search space)
- $A_i \cap A_j = \emptyset$ whenever $i \neq j$ (the levels are nonoverlapping)
- the last level A_m covers the optima for the problem

We write $A_{\geq j}$ to denote everything in level j and higher, i.e.,

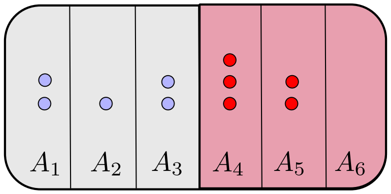
$$A_{\geq j} := \bigcup_{i=j}^{m} A_i.$$



Notation

- For any population $P = (y_1, \ldots, y_\lambda) \in \mathcal{X}^\lambda$ and $j \in [m]$, let
$$|P \cap A_{\geq j}| := |\{i \mid y_i \in A_{\geq j}\}|,$$
i.e, the number of individuals in P that is in subset $A_{\geq j}$.

Example

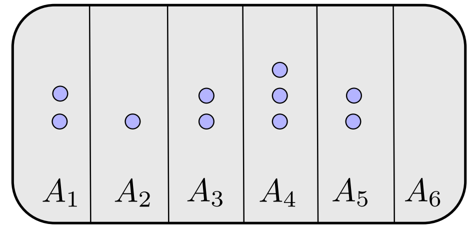


$|P \cap A_{\geq 4}| = 5$ where $A_{\geq 4}$ corresponds to the red region.

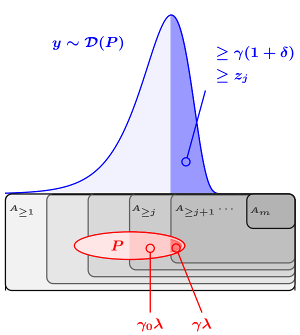
Current level of a population P wrt $\gamma_0 \in (0, 1)$

Definition
The unique integer $j \in [m-1]$ such that
$$|P \cap A_{\geq j}| \geq \gamma_0 \lambda > |P \cap A_{\geq j+1}|$$

Example
Current level wrt $\gamma_0 = \frac{1}{2}$ is ....4.



Level-based Theorem [Corus, Dang, Eremeev, and Lehre, 2018]



If for all populations $P \in \mathcal{X}^\lambda$, an individual $y \sim \mathcal{D}(P)$ has
$$\Pr(y \in A_{\geq j+1}) \geq z_j, \quad \text{(G1)}$$
$$\Pr(y \in A_{\geq j+1}) \geq \gamma(1+\delta), \quad \text{(G2)}$$
where $j \in [m-1]$ is the current level of population P, i.e.,
$$|P \cap A_{\geq j}| \geq \gamma_0 \lambda > |P \cap A_{\geq j+1}| = \gamma\lambda,$$
and the population size λ is bounded from below by
$$\lambda \geq \left(\frac{4}{\gamma_0 \delta^2}\right) \ln\left(\frac{128m}{z_{\min}\delta^2}\right), \quad \text{(G3)}$$
then the algorithm reaches the last level A_m in expected time
$$\mathbf{E}\left[T_{A_m}\right] \leq \left(\frac{8}{\delta^2}\right) \sum_{j=1}^{m-1} \left(\lambda \ln\left(\frac{6\delta\lambda}{4 + z_j\delta\lambda}\right) + \frac{1}{z_j}\right).$$

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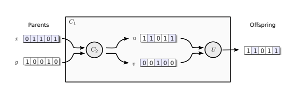
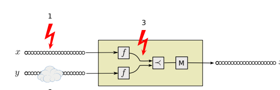
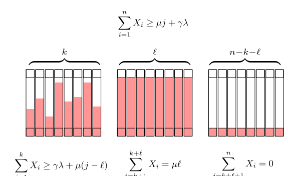
Applications of Level-based Theorem







- **EAs with crossover operator**
  - (Corus, Dang, Eremeev, Lehre, PPSN'14)
- **Theory-led design of EAs**
  - Shortest paths with genetic algorithms (Corus & Lehre, MIC'15)
- **Optimisation under uncertainty**
  - Stochastic fitness functions (Dang & Lehre, FOGA'15)
  - Partially observable fitness functions (Dang & Lehre, GECCO'14)
  - Dynamic fitness functions (Dang, Jansen, Lehre, GECCO'15)
- **Estimation of Distribution Algorithms (EDAs)**
  - UMDA (Dang & Lehre, GECCO'15) (Nguyen & Lehre, GECCO'17)
  - PBIL (Lehre & Nguyen, PPSN'18)
  - GOMEA (Hoog, 2016)
- **Self-adaptive EAs**
  - Self-adaptation of mutation rates (Dang & Lehre, PPSN'16), (Case& Lehre, TEVC2020)

Level-based Theorem via Multiplicative Up-drift[8]

Theorem ([Doerr and Kötzing, 2020] via multiplicative up-drift)
If (G1), (G2), and (G3b): $\lambda \geq \frac{256}{\gamma_0 \delta} \ln(8t_0)$ *where*

$$t_0 := \frac{7000}{\delta}\left(m + \frac{1}{1-\gamma_0}\sum_{j=1}^{m-1}\log_2\left(\frac{2\gamma_0\lambda}{1+\frac{z_j\lambda}{D_0}}\right) + \frac{1}{\lambda}\sum_{j=1}^{m-1}\frac{1}{z_j}\right)$$

the $\mathbf{E}[T] = 8\lambda t_0$.

Theorem ([Corus et al., 2018])
If (G1), (G2), and (G3): $\lambda \geq \left(\frac{4}{\gamma_0 \delta^2}\right)\ln\left(\frac{128m}{z_{min}\delta^2}\right)$ *then*

$$\mathbf{E}[T] \leq \left(\frac{8}{\delta^2}\right)\sum_{j=1}^{m-1}\left(\lambda \ln\left(\frac{6\delta\lambda}{4+z_j\delta\lambda}\right) + \frac{1}{z_j}\right).$$

[8] Note that "multiplicative up-drift" was already used in Corus et al. [2018] (Corollary 22).

Suggested recipe for application of level-based theorem

1. Find a partition $(A_1, \ldots, A_m)$ of $\mathcal{X}$ that reflects the state of the algorithm, and where A_m consists of all goal states.
2. Find parameters γ_0 and δ and a configuration of the algorithm (e.g., mutation rate, selective pressure) such that whenever $|P \cap A_{\geq j+1}| = \gamma\lambda > 0$, condition (G2) holds, i.e.,
$$\mathbf{Pr}\left(y \in A_{\geq j+1}\right) \geq \gamma(1+\delta)$$
3. For each level $j \in [m-1]$, estimate a lower bound $z_j \in (0,1)$ such that whenever $|P \cap A_{\geq j+1}| = 0$, condition (G1) holds, i.e.,
$$\mathbf{Pr}\left(y \in A_{\geq j+1}\right) \geq z_j$$
4. Calculate the sufficient population size λ from condition (G3).
5. Read off the bound on expected runtime.

The Level-Based Theorem (LBT) is "tight"[9]

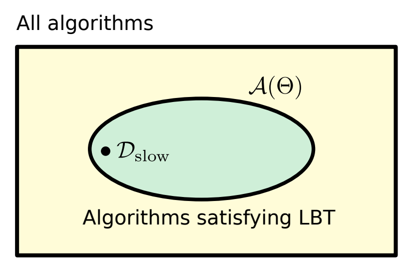
Theorem
For any valid set of parameters $\Theta = (A, \gamma_0, \delta, z)$ *for LBT, there exists a mapping* $\mathcal{D}_{slow}$ *satisfying (G1) and (G2) of LBT st.*

$$\mathbf{E}[T_{A_m}] \geq \left(\frac{2}{3\delta}\right)\sum_{j=1}^{m-2}\left(\lambda \ln\left(\frac{\gamma_0\delta\lambda}{1/\delta + 2z_j\delta\lambda}\right) + \frac{1}{z_j}\right)$$

$$\mathbf{E}[T_{A_m}] \leq \left(\frac{8}{\delta^2}\right)\sum_{j=1}^{m-1}\left(\lambda \ln\left(\frac{6\delta\lambda}{4+z_j\delta\lambda}\right) + \frac{1}{z_j}\right).$$

[9] Corus, Dang, Eremeev and Lehre (IEEE TEVC 2018) `https://arxiv.org/abs/1407.7663`

More info about $\mathcal{D}$ required for more precise bounds



Logical structure of LBT
For all parameter settings Θ, and all mappings $\mathcal{D} \in \mathcal{A}(\Theta)$

$$\mathbf{E}[T_{\mathcal{D}}] \leq f(\Theta) + \varepsilon.$$

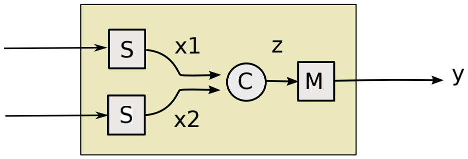
Also, there exists $\mathcal{D} \in \mathcal{A}(\Theta)$ st

$$\mathbf{E}[T_{\mathcal{D}}] \geq f(\Theta) - \varepsilon$$

Assume you have applied the LBT to your algorithm, how precise is the bound?

- The only LBT knows about your algorithm $\mathcal{D}$ is that it satisfies the conditions for the parameters Θ. (Many other processes satisfy the conditions for the same Θ.)
- The lower bound implies that the LBT gives the best possible ($\pm\varepsilon$) runtime bound for your algorithm given the information that is available.
- Some algorithms in $\mathcal{A}(\Theta)$, including yours, could be faster than $f(\Theta)$. However, more information about the algorithm required to prove so, i.e.,
  - a more precise set of parameters Θ', or
  - a different way of characterising algorithms than $\mathcal{A}(\Theta)$

Example application – (μ,λ) GA on Onemax



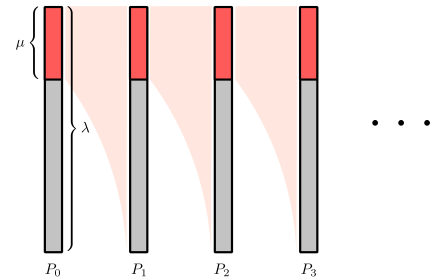
$$\textsc{OneMax}(x) := \sum_{i=1}^{n} x_i.$$

(μ,λ) Genetic Algorithm (GA)

for $t = 0, 1, 2, \ldots$ until termination condition **do**
  for $i = 1$ to λ **do**
    Select a parent x from population P_t acc. to (μ, λ)-selection
    Select a parent y from population P_t acc. to (μ, λ)-selection
    Apply uniform crossover to x and y, i.e. $z :=$ crossover(x, y)
    Create $P_{t+1}(i)$ by flipping each bit in z with probability χ/n.

Theorem
If $\lambda > c\ln(n)$ for a sufficiently large constant $c > 0$, and $\frac{\lambda}{\mu} > 2e^{\chi}(1+\delta)$ for any constant $\delta > 0$, then the expected runtime of (μ,λ) GA on $\textsc{OneMax}$ *is $O(n\lambda)$.*

(μ, λ)-selection mechanism



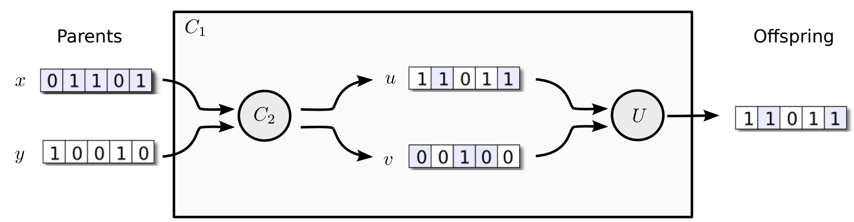
1. Sort the current population $P = (x_1, \ldots, x_\lambda)$ such that
$$f(x_1) \geq f(x_2) \geq \ldots \geq f(x_\lambda)$$
2. **return** $\mathrm{Unif}(x_1, \ldots, x_\mu)$

Partition of Search Space into Levels

Partition into $m := n + 1$ levels $A_0, \ldots, A_n$

$$A_j := \{x \in \{0,1\}^n \mid \textsc{OneMax}(x) = j\}$$

Analysis of Crossover Operator



Proof.
Assume that $x \in A_{\geq j+1}$ and $y \in A_{\geq j}$, and w.l.o.g. that $|u| \geq |v|$

$$\begin{aligned} 2j + 1 &\leq |x| + |y| \\ &= |u| + |v| \\ &\leq 2|u|. \end{aligned}$$

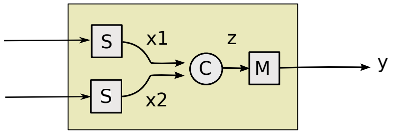
Therefore $\mathbf{Pr}\left(u \in A_{\geq j+1}\right) = 1$ and

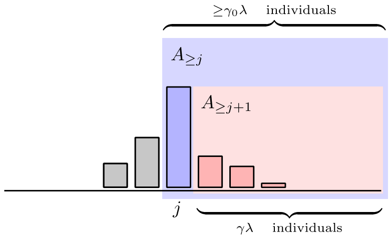
$$\mathbf{Pr}\left(\text{crossover}(x, y) \in A_{\geq j+1} \mid x \in A_{\geq j+1} \text{ and } y \in A_{\geq j}\right) \geq \frac{1}{2} =: \varepsilon.$$

$\square$

Verification of Condition (G2)

Assume that $|P \cap A_{\geq j}| \geq \gamma_0 \lambda > |P \cap A_{\geq j+1}|$ where $0 < \gamma < \gamma_0 := \mu/\lambda$.





If $\lambda \geq 2\mu\left(1 - \frac{\chi}{n}\right)^{-n}(1+\delta) \approx 2\mu e^{\chi}$, we have

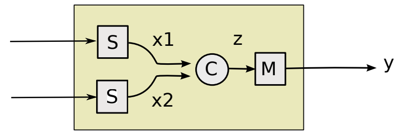
$$\begin{aligned}\Pr\left(z \in A_{\geq j+1}\right) &= \Pr\left(x_1 \in A_{\geq j} \wedge x_2 \in A_{\geq j+1}\right)\Pr\left(z \in A_{\geq j+1} \mid x_1 \in A_{\geq j} \wedge x_2 \in A_{\geq j+1}\right)\\ &\geq 1 \cdot \frac{\gamma\lambda}{\mu} \cdot (1/2) \geq \gamma(1+\delta)\left(1 - \frac{\chi}{n}\right)^{-n}\end{aligned}$$

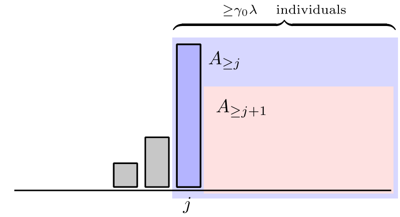
hence, we get

$$\begin{aligned}\Pr\left(y \in A_{\geq j+1}\right) &= \Pr\left(z \in A_{\geq j+1}\right)\Pr\left(y \in A_{\geq j+1} \mid z \in A_{\geq j+1}\right)\\ &\geq \Pr\left(z \in A_{\geq j+1}\right)\left(1 - \frac{\chi}{n}\right)^{n}\\ &\geq \gamma(1+\delta).\end{aligned}$$

Verification of Condition (G1)

Assume that $|P \cap A_{\geq j}| \geq \gamma_0 \lambda > |P \cap A_{\geq j+1}|$ where $0 \leq \gamma < \gamma_0 := \mu/\lambda$.





We have

$$\begin{aligned}\Pr\left(z \in A_{\geq j}\right) &= \Pr\left(x_1 \in A_{\geq j} \wedge x_2 \in A_{\geq j}\right)\Pr\left(z \in A_{\geq j} \mid x_1 \in A_{\geq j} \wedge x_2 \in A_{\geq j}\right)\\ &\geq 1 \cdot 1/2\end{aligned}$$

hence, we get

$$\begin{aligned}\Pr\left(y \in A_{\geq j+1}\right) &= \Pr\left(z \in A_{\geq j}\right)\Pr\left(y \in A_{\geq j+1} \mid z \in A_{\geq j}\right)\\ &\geq \Pr\left(z \in A_{\geq j}\right) \cdot (n-j) \cdot \left(\frac{\chi}{n}\right)\left(1 - \frac{\chi}{n}\right)^{n-1}\\ &\geq (\chi/2) \cdot (1 - j/n)\left(1 - \frac{\chi}{n}\right)^{n-1}\\ &\geq \frac{(1-\delta)\chi}{2e^{\chi}} \cdot (1 - j/n) =: z_j\end{aligned}$$

Application of the Level-based Theorem

If for all populations $P \in \mathcal{X}^{\lambda}$, an individual $y \sim \mathcal{D}(P)$ has

$$\Pr\left(y \in A_{\geq j+1}\right) \geq z_j, \qquad \text{(G1)}$$
$$\Pr\left(y \in A_{\geq j+1}\right) \geq \gamma(1+\delta), \qquad \text{(G2)}$$

where $j \in [m-1]$ is the current level of population P, i.e.,

$$|P \cap A_{\geq j}| \geq \gamma_0 \lambda > |P \cap A_{\geq j+1}| = \gamma\lambda,$$

and the population size λ is bounded from below by

$$\lambda \geq \left(\frac{4}{\gamma_0 \delta^2}\right)\ln\left(\frac{128m}{z_{\min}\delta^2}\right), \qquad \text{(G3)}$$

then the algorithm reaches the last level A_m in expected time

$$\mathbf{E}\left[T_{A_m}\right] \leq \left(\frac{8}{\delta^2}\right)\sum_{j=1}^{m-1}\left(\lambda \ln\left(\frac{6\delta\lambda}{4 + z_j\delta\lambda}\right) + \frac{1}{z_j}\right).$$

Bounding the first term (first attempt, imprecise)

$$\sum_{j=0}^{n-1}\ln\left(\frac{6\delta\lambda}{4 + z_j\delta\lambda}\right) < \sum_{j=0}^{n-1}\ln\left(\frac{6\delta\lambda}{4}\right) = \mathcal{O}(n\ln(\lambda)).$$

- This upper bound is imprecise because it does not exploit that the upgrade probabilities z_j are large for small j.

Bounding the first term (second attempt, more precise)

$$\sum_{j=0}^{n-1} \ln\left(\frac{6\delta\lambda}{4+z_j\delta\lambda}\right) < \sum_{j=0}^{n-1} \ln\left(\frac{6}{z_j}\right)$$

using $\ln(a)+\ln(b)=\ln(ab)$ and defining $c := \frac{12e^{\chi}}{(1-\delta)\chi}$

$$= \ln\left(\prod_{j=0}^{n-1} \frac{cn}{n-j}\right) = \ln\left(\frac{(cn)^n}{n!}\right)$$

and using the lower bound $n! > (n/e)^n$

$$< \ln\left(\frac{(cn)^n e^n}{n^n}\right) = n\ln(ec) = \mathcal{O}(n).$$

Bounding the second term

Recall the definition of the n-th Harmonic number

$$H_n := \sum_{i=1}^{n} \frac{1}{i} = \mathcal{O}(\ln(n)).$$

The second term can therefore be bounded as

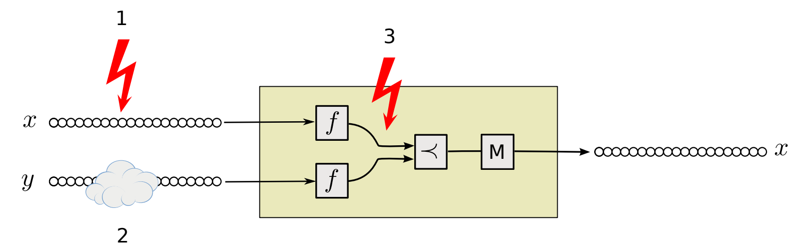
$$\sum_{j=0}^{n-1} \frac{1}{z_j} = \mathcal{O}\left(\sum_{j=0}^{n-1} \frac{n}{n-j}\right) = \mathcal{O}(n\ln(n))$$

Completing the proof

Theorem
If $\lambda > c\ln(n)$ for a sufficiently large constant $c>0$, and $\frac{\lambda}{\mu} > 2e^{\chi}(1+\delta)$ for any constant $\delta > 0$, then the expected runtime of (μ,λ) GA on OneMax *is*

$$\left(\frac{8}{\delta^2}\right)\left(\lambda\sum_{j=0}^{n-1}\ln\left(\frac{6\delta\lambda}{4+z_j\delta\lambda}\right) + \sum_{j=0}^{n-1}\frac{1}{z_j}\right)$$
$$= \mathcal{O}(n\lambda) + \mathcal{O}(n\ln n) = \mathcal{O}(n\lambda).$$

Uncertainty in Comparison-based PSVAs



Sources of uncertainty

1. Droste noise model (Droste, 2004)
2. Partial evaluation
3. Noisy fitness (Prügel-Bennet, Rowe, Shapiro, 2015)

Sufficient with mutation rate $\delta/(3n)$ and

$$\Pr\left(x \text{ choosen} \mid f(x) > f(y)\right) \geq \frac{1}{2} + \delta \quad \text{with } 1/\delta \in \mathrm{poly}(n)$$

Dang and Lehre [2015] and Dang and Lehre [2016b]

Lower Bounds

Lower Bounds

Problem
Consider a sequence of populations $P_1, \ldots$ over a search space $\mathcal{X}$, and a target region $A \subset \mathcal{X}$ (e.g., the set of optimal solutions), let

$$T_A := \min\{\ \lambda t \ \mid\ P_t \cap A \neq \emptyset\ \}$$

We would like to prove statements on the form

$$\Pr\left(T_A \leq t(n)\right) \leq e^{-\Omega(n)}. \qquad (1)$$

- i.e., with overwhelmingly high probability, the target region A has not been found in $t(n)$ evaluations
- lower bounds often harder to prove than upper bounds
- will present an easy to use method that is applicable in many situations

Algorithms considered for lower bounds

Definition (Non-elitist EA with selection and mutation)

for $t = 0, 1, 2, \ldots$ until termination condition **do**
  for $i = 1$ to λ **do**
    Select parent x from population P_t according to p_{sel}
    Flip each position in x independently with probability χ/n.
    Let the i-th offspring be $P_{t+1}(i) := x$.
    (i.e., create offspring by mutating the parent)

Assumptions
- population size $\lambda \in \text{poly}(n)$, i.e. not exponentially large
- bitwise mutation with probability χ/n, but no crossover.
- results hold for any non-elitist selection scheme p_{sel} that satisfy some mild conditions to be described later.

Reproductive rate[10]

Definition
For any population $P = (x_1, \ldots, x_\lambda)$ let $p_{\text{sel}}(x_i)$ be the probability that individual x_i is selected from the population P

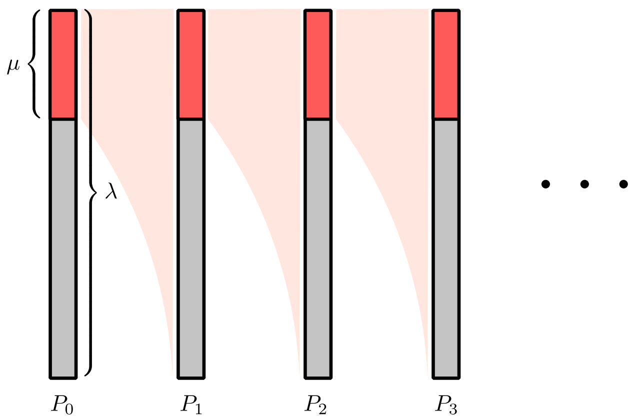
- The reproductive rate of individual x_i is $\lambda \cdot p_{\text{sel}}(x_i)$.
- The reproductive rate of a selection mechanism is bounded from above by α_0 if

$$\forall P \in \mathcal{X}^\lambda,\ \ \forall x \in P \quad \lambda \cdot p_{\text{sel}}(x)\ \leq\ \alpha_0$$

(i.e., no individual gets more than α_0 offspring in expectation)

[10] The reproductive rate of an individual as defined here corresponds to the notion of "fitness" as used in the field of population genetics, i.e., the expected number of offspring.

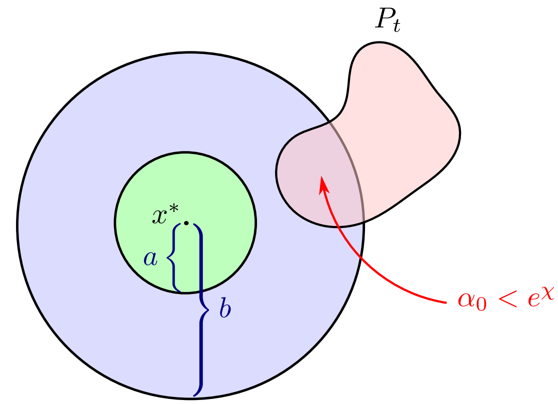
(μ, λ)-selection mechanism



Probability of selecting i-th individual is $p_i \in \{0, \frac{1}{\mu}\}$.

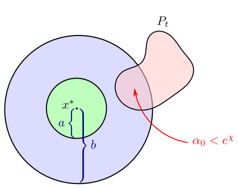
- reproductive rate bounded by $\alpha_0 = \lambda/\mu$

Negative Drift Theorem for Populations (informal)



If individuals closer than b of target has reproductive rate $\alpha_0 < e^{\chi}$, then it takes exponential time $e^{c(b-a)}$ to reach within a of target.

Negative Drift Thm. for Populations [Lehre, 2011a]



Consider the non-elitist EA with

- population size $\lambda = \text{poly}(n)$
- bitwise mutation rate χ/n for $0 < \chi < n$

let $T := \min\{t \mid H(P_t, x^*) \leq a\}$ for any $x^* \in \{0,1\}^n$.

If there are constants $\alpha_0 \geq 1$, $\delta > 0$ and integers $a(n)$ and $b(n) < \frac{n}{\chi}$ where $b(n) - a(n) = \omega(\ln n)$, st.

(C1) If $a(n) < H(x, x^*) < b(n)$ then $\lambda \cdot p_{\text{sel}}(x) \leq \alpha_0$.

(C2) $\psi := \ln(\alpha_0)/\chi + \delta < 1$

(C3) $b(n) < \min\left\{\frac{n}{5}, \frac{n}{2}\left(1 - \sqrt{\psi(2-\psi)}\right)\right\}$

then there exist constants $c, c' > 0$ such that

$$\Pr\left(T \leq e^{c(b(n)-a(n))}\right) \leq e^{-c'(b(n)-a(n))}.$$

The worst individuals have low reproductive rate

Lemma

Consider any selection mechanism which for $x, y \in P$ satisfies

(a) *If $f(x) > f(y)$, then $p_{sel}(x) \geq p_{sel}(y)$.*
(selection probabilities are monotone wrt fitness)

(b) *If $f(x) = f(y)$, then $p_{sel}(x) = p_{sel}(y)$.*
(ties are drawn randomly)

If $f(x) = \min_{y \in P} f(y)$, then $p_{sel}(x) \leq 1/\lambda$.
(individuals with lowest fitness have reproductive rate ≤ 1)

Proof.

- By (a) and (b), $p_{\text{sel}}(x) = \min_{y \in P} p_{\text{sel}}(y)$.
- $1 = \sum_{x \in P} p_{\text{sel}}(x) \geq \lambda \cdot \min_{y \in P} p_{\text{sel}}(y) = \lambda \cdot p_{\text{sel}}(x)$.

$\square$

Example 1: Needle in the haystack

Definition

$$\textsc{Needle}(x) = \begin{cases} 1 & \text{if } x = 1^n \\ 0 & \text{otherwise.} \end{cases}$$

Theorem

The optimisation time of the non-elitist EA with any selection mechanism satisfying the properties above[11] on $\textsc{Needle}$ *is at least* e^{cn} *with probability* $1 - e^{-\Omega(n)}$ *for some constant* $c > 0$.

[11] From black-box complexity theory, it is known that $\textsc{Needle}$ is hard for all search heuristics (Droste et al 2006).

Example 1: Needle in the haystack (proof[12])

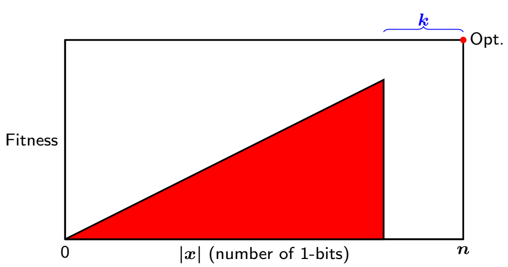
- Apply negative drift theorem with $a(n) := 1$.
- By previous lemma, can choose $\alpha_0 = 1$ for any $b(n)$, hence $\psi = \ln(\alpha)/\chi + \delta = \delta < 1$ for all χ and $\delta < 1$.
- Choosing the parameters $\delta := 1/10$ and $b(n) := n/6$ give

$$\min\left\{\frac{n}{5}, \frac{n}{2}\left(1 - \sqrt{\psi(2-\psi)}\right)\right\} = \frac{n}{5} > b(n).$$

- It follows that $\Pr\left(T \le e^{c(b(n)-a(n))}\right) \le e^{-\Omega(n)}$.

[12] For simplicity, we assume that $\chi \le 6$, thus $b(n) = n/6 \le n/\chi$ holds.

Exercise: Optimisation time on Jump_k



Recipe

- $a(n) = 1$
- $b(n) = k$
- $\alpha_0 = 1$ as before
- small δ

$$\textsc{Jump}_k(x) := \begin{cases} 0 & \text{if } n - k \le |x| < n, \\ |x| & \text{otherwise.} \end{cases}$$

When the best individuals have low reproductive rate

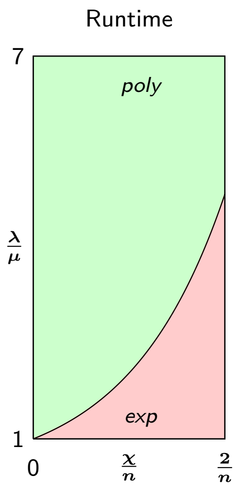
Remark

- The negative drift conditions hold trivially if $\alpha_0 < e^{\chi}$ holds for all individuals.

Example (Insufficient selective pressure)

Selection mechanism	Parameter settings
Linear ranking selection	$\eta < e^{\chi}$
k-tournament selection	$k < e^{\chi}$
(μ,λ)-selection	$\lambda < \mu e^{\chi}$
Any in cellular EAs	$\Delta(G) < e^{\chi}$

Error threshold



Example

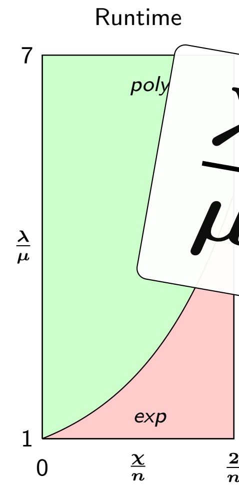
The runtime T of a non-elitist EA with

- (μ, λ)-selection
- bit-wise mutation rate χ/n
- population size $\lambda > c\log(n)$

on LeadingOnes has expected value

$$\mathbf{E}[T] = \begin{cases} e^{\Omega(n)} & \text{if } \lambda < \mu e^{\chi} \\ O(n\lambda\ln(\lambda) + n^2) & \text{if } \lambda > \mu e^{\chi} \end{cases}$$

Error threshold



$$\frac{\lambda}{\mu} > e^{\chi}$$

Example

The runtime T of a non-elitist EA with

- (μ, λ)-selection
- bit-wise mutation rate χ/n
- population size $\lambda > c\log(n)$

on LeadingOnes has expected value

$$\mathbf{E}[T] = \begin{cases} e^{\Omega(n)} & \text{if } \lambda < \mu e^{\chi} \\ O(n\lambda\ln(\lambda) + n^2) & \text{if } \lambda > \mu e^{\chi} \end{cases}$$

Other Example Applications

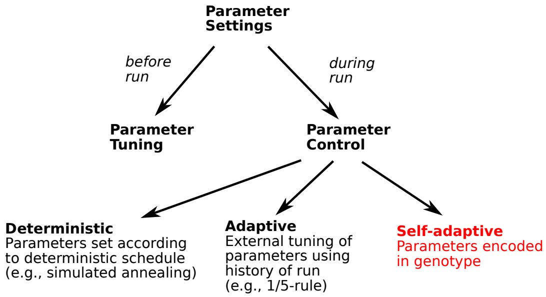
Expected runtime of EA with bit-wise mutation rate χ/n

Selection Mechanism	High Selective Pressure	Low Selective Pressure
Fitness Proportionate	$\nu > f_{\max}\ln(2e^{\chi})$	$\nu < \chi/\ln 2$ and $\lambda \geq n^3$
Linear Ranking	$\eta > e^{\chi}$	$\eta < e^{\chi}$
k-Tournament	$k > e^{\chi}$	$k < e^{\chi}$
(μ, λ)	$\lambda > \mu e^{\chi}$	$\lambda < \mu e^{\chi}$
Cellular EAs		$\Delta(G) < e^{\chi}$
OneMax	$O(n\lambda)$	$e^{\Omega(n)}$
LeadingOnes	$O(n\lambda\ln(\lambda) + n^2)$	$e^{\Omega(n)}$
Linear Functions	$O(n\lambda\ln(\lambda) + n^2)$	$e^{\Omega(n)}$
r-Unimodal	$O(r\lambda\ln(\lambda) + nr)$	$e^{\Omega(n)}$
Jump$_r$	$O(n\lambda + (n/\chi)^r)$	$e^{\Omega(n)}$

Self-adaptive Evolutionary Algorithms[13]

[13] Dang and Lehre [2016c]

Parameter settings in optimisation algorithms

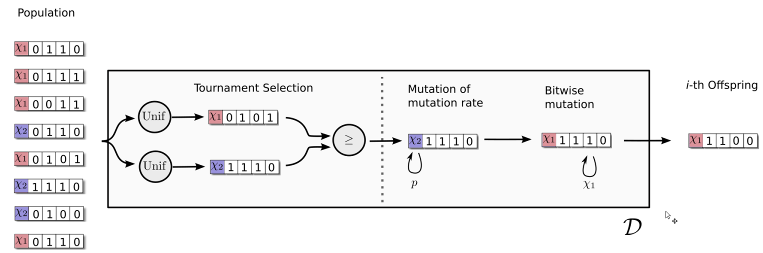


Eiben, Hinterding, and Michalewicz [1999]

Can self-adaptive EAs control parameters effectively, i.e., on a problem

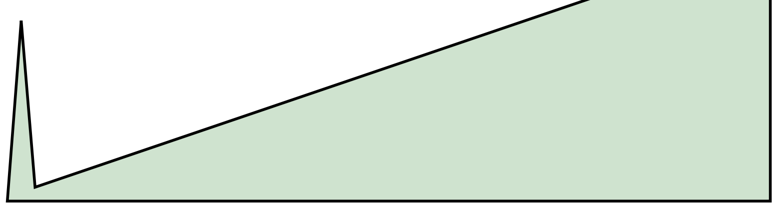
- any fixed parameter setting $\Longrightarrow$ exponential runtime
- uniform mixing of parameters $\Longrightarrow$ exponental runtime
- self-adaptation over parameters $\Longrightarrow$ polynomial runtime

Self-adaptive, non-elitist EA without crossover



- search space is $\{\chi_1, \chi_2\} \times \{0,1\}^n$, where $\chi_1, \chi_2 \in \Theta(n)$ are two mutation rates
- parent selection via binary tournament selection
- mutation rate switched with probability p, and obtain offspring by mutation with new mutation rate

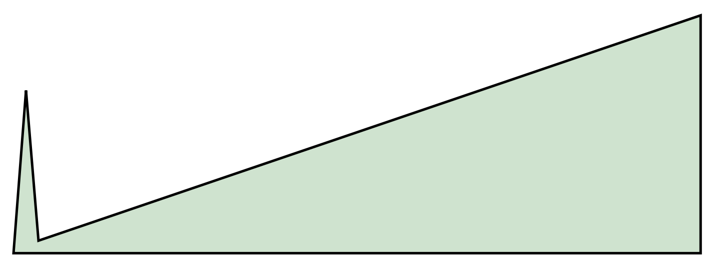
Peaked Fitness Landscape



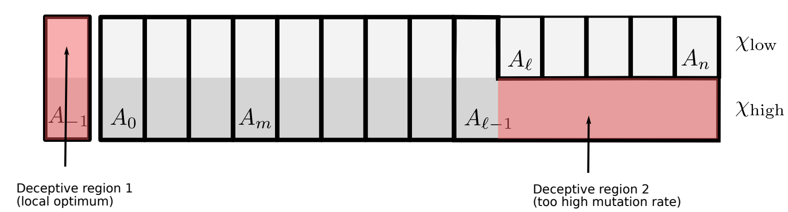
Intuition

- Low mutation rate (or elitist selection mechanism) $\Longrightarrow$ exponential time to escape local optimum
- Mutation rate above error threshold $\Longrightarrow$ exponential runtime via negative population drift (L., PPSN 2010)

Fitness function and level structure



$$\textsc{PeakedLO}(x) := \begin{cases} m & \text{if } x = 0^n \\ \sum_{i=1}^{n} \prod_{j=1}^{i} x_j & \text{otherwise.} \end{cases}$$



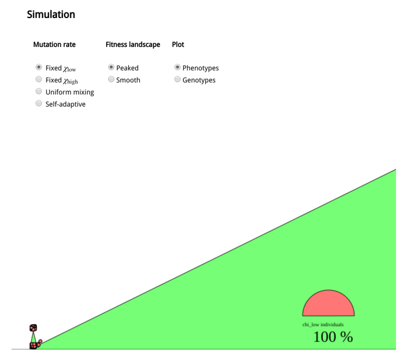
Results imply benefit of non-elitism and self-adaptation[14]

Mutation control	Runtime	Proof idea
Fixed rate χ_{low}	$e^{\Omega(n)}$	Most individuals remain on the peak. Too low selective pressure among sub-optimal individuals. (Negative drift in populations).
Fixed rate χ_{high}	$e^{\Omega(n)}$	Most individuals fall off the peak, but mutation rate is too high wrt selective pressure to reach opt. (Negative drift in populations).
Uniform mixing	$e^{\Omega(n)}$	Most individuals fall off the peak, but the effective mutation rate is too high wrt selective pressure. (Negative drift in populations).
Self-adaptation	$\mathcal{O}(n^2)$	Most individuals fall off the peak. Peak individuals do not dominate. A sub-population surviving off the peak switches to low mutation rate. (Level-based analysis).
(μ+λ) EA	$e^{\Omega(n)}$	Elitism prevents escape from peak.

Dang and Lehre [2016c]

[14] The results assume appropriate choices of the mutation rates χ_1 and χ_2, the strategy parameter p, and the problem parameter m.

Interactive Simulation of Results



http://www.cs.bham.ac.uk/~lehrepk/selfadapt/

Fitness proportional selection + crossover Oliveto and Witt [2014, 2015]

Definition (Simple Genetic Algorithm (SGA) (Goldberg 1989))

for $t = 0, 1, 2, \ldots$ until termination condition **do**
  for $i = 1$ to λ **do**
    Select two parents x and y from P_t proportionally to fitness
    Obtain z by applying uniform crossover to x and y with $p = 1/2$
    Flip each position in z independently with $p = 1/n$.
    Let the i-th offspring be $P_{t+1}(i) := x$.
    (i.e., create offspring by crossover followed by mutation)

Application to OneMax

Expected Behaviour
- Backward drift due to mutation close to the optimum
- no positive drift due to crossover
- selection too weak to keep positive fluctuations

Difficulties When Introducing Crossover:
- Variance of offspring distribution
- # flipping bits due to mutation Poisson-distributed → variance $O(1)$
- # of one-bits created by crossover binomially distributed according to Hamming distance of parents and $1/2$ → deviation $\Omega(\sqrt{n})$ possible

Negative Drift Theorem With Scaling

Let X_t, $t \geq 0$, random variable describing a stochastic process over finite state space $S \subseteq \mathbb{R}$;

If there $\exists$ interval $[a, b]$ and, possibly depending on $\ell := b - a$, bound $\epsilon(\ell) > 0$ and scaling factor $r(\ell)$ st.

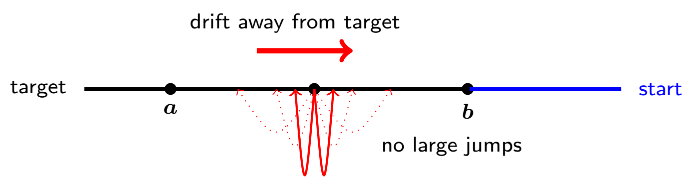
(C1) $E(X_{t+1} - X_t \mid X_0, \dots, X_t \;\wedge\; a < X_t < b) \;\geq\; \epsilon$,

(C2) $\mathrm{Prob}(|X_{t+1} - X_t| \geq jr \mid X_0, \dots, X_t \;\wedge\; a < X_t) \;\leq\; e^{-j}$ for $j \in \mathbb{N}_0$,

(C3) $1 \leq r \leq \min\{\epsilon^2 \ell, \sqrt{\epsilon\ell/(132\log(\epsilon\ell))}\}$.

then

$$\mathrm{Pr}\left(T \leq e^{\epsilon\ell/(132r^2)}\right) = O(e^{-\epsilon\ell/(132r^2)}).$$



Potential Function

For drift theorem, capture whole population in one value: For $X = \{x_1, \dots, x_\mu\}$ let $g(X) := \sum_{i=1}^{\mu} e^{\kappa \textsc{OneMax}(x_i)}$.

Problem: maybe $r(\ell) = \Omega(\sqrt{\ell})$

Solution

Find bits that are "converged" within population, i.e., either ones or zeros only. Crossover is irrelevant for these.

Diversity

X_t: $\#$ individuals with 1 in some fixed position at time t

Assume uniform selection (and no mutation). Then:

- The probability crossover produces an individual with 1 in the fixed position is $(X_t = k)$:
- $\frac{k}{\mu} \cdot \frac{k}{\mu} + 2 \cdot \frac{1}{2} \cdot \frac{k(\mu-k)}{\mu^2} = \frac{k}{\mu}$
- $\{X_t\} \approx B(\mu, k/\mu) \rightsquigarrow E(X_t \mid X_{t-1} = k) = k$ (martingale)
- But random fluctuations $\rightsquigarrow$ absorbing state 0 or μ due to the variance ($E(T_{0 \vee \mu}) = O(\mu \log \mu)$ [drift analysis]).
- Progress by crossover is at most $n^{1/2+\epsilon}$ w.o.p. (Chernoff Bounds when ones are $n/2$).
- If $\mu \leq n^{1/2-\epsilon}$ a bit has converged to 0 before optimum is found w.o.p.

Diversity

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- $\frac{k}{\mu} \cdot \frac{k}{\mu} + 2 \cdot \frac{1}{2} \cdot \frac{k(\mu-k)}{\mu^2} = \frac{k}{\mu}$
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- But random fluctuations $\rightsquigarrow$ absorbing state 0 or μ due to the variance

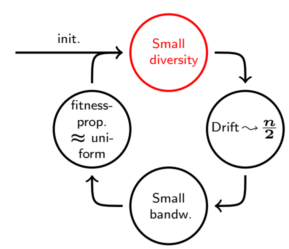
Compare fitness-prop. and uniform selection:

- Basically no difference for small population *bandwidth* (difference of best and worst $\textsc{OneMax}$-value in pop.)
- $E(X_t \mid X_{t-1} = k) = k \pm 1/(7\mu)$

Result

Let $\mu \leq n^{1/8-\epsilon}$ for an arbitrarily small constant $\epsilon > 0$. Then with probability $1 - 2^{-\Omega(n^{\epsilon/9})}$, the SGA on $\textsc{OneMax}$ does not create individuals with more than $(1+c)\frac{n}{2}$ or less than $(1-c)\frac{n}{2}$ one-bits, for arbitrarily small constant $c > 0$, within the first $2^{n^{\epsilon/10}}$ generations. In particular, it does not reach the optimum then.

Overall Proof Structure



Not a loop, but in each step only exponentially small failure prob.

Steady-state GA

Algorithm 1: (μ+1) GA

$P_1 \leftarrow \mu$ individuals, uniformly at random from $\{0,1\}^n$;
$t \leftarrow \mu$;
repeat
    Select $x, y \in P_t$ uniformly at random with replacement ;
    $z \leftarrow$ *uniform crossover*(x, y);
    $z \leftarrow$ *mutate*(z);
    $P_{t+1} \leftarrow P_t \cup \{z\}$;
    Remove the element with lowest fitness from P_{t+1}, breaking ties at random;
    $t \leftarrow t + 1$;
until *optimum is found*;

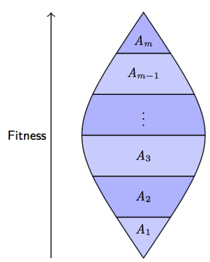
Uniform crossover:

Picks the value for each bit position from one parent or the other uniformly at random (i.e., from each parent with probability 1/2).

Algorithm 3 (1+1) EA

Sample one search points $x \in \{0,1\}^n$ uniformly at random.
for $t \leftarrow 1, 2, \ldots$ **do**
    Create y by flipping each bit of x independently with probability $1/n$.
    If $f(y) \geq f(x)$ then $x := y$.
end for

Artificial Fitness Levels [DJW 2002]



Definition

A tuple $(A_1, A_2, \ldots, A_m)$ is an f**-based partition** of $f : \mathcal{X} \to \mathbb{R}$ if

1. $A_1 \cup A_2 \cup \cdots \cup A_m = \mathcal{X}$
2. $A_i \cap A_j = \emptyset$ for $i \neq j$
3. $f(A_1) < f(A_2) < \cdots < f(A_m)$
4. $f(A_m) = \max_x f(x)$

Example

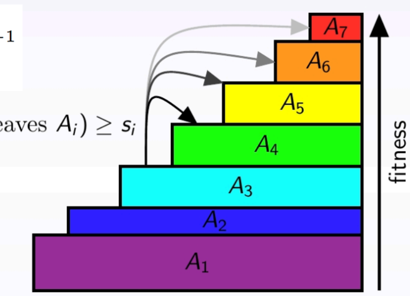
Partition of OneMax into $n+1$ levels

$$A_j := \{x \in \{0,1\}^n \mid \textsc{OneMax}(x) = j\}$$

(1+1)-EA for OneMax via Artificial Fitness Levels

$$E(T) \leq \sum_{i=1}^{m-1} s_i^{-1}$$

$\Pr((1{+}1)\text{ EA leaves } A_i) \geq s_i$



Ai : number of 0-bits;

$$s_i \geq i \cdot \frac{1}{n}\left(1 - \frac{1}{n}\right)^{n-1} \geq \frac{i}{en}$$

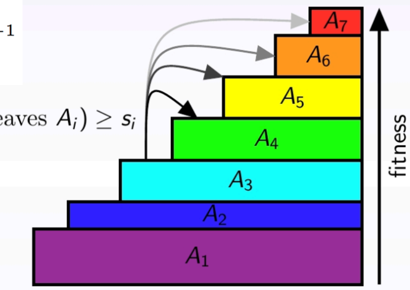
$$E(T) \leq \sum_{i=1}^{m-1} s_i^{-1} \leq \sum_{i=1}^{n} \frac{en}{i} \leq e \cdot n \sum_{i=1}^{m-1} \frac{1}{i} \leq e \cdot n \cdot (\ln n + 1)$$

By showing that with high probability the algorithm does not skip too many levels then it can be proven that E(T) is also at least e *n ln n* + *O*(*n*); [Sudholt, IEEE TEVC 2013]

(μ+1)-EA for OneMax via Artificial Fitness Levels

$$E(T) \le \sum_{i=1}^{m-1} s_i^{-1}$$

$$\Pr((1+1)\text{ EA leaves } A_i) \ge s_i$$



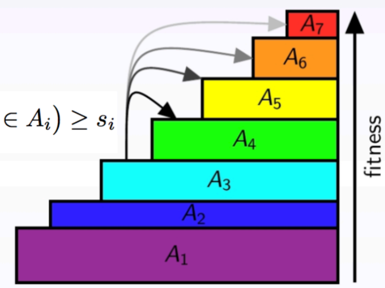
Ai : number of 0-bits;

$$s_i \ge \frac{1}{\mu} \cdot i \cdot \frac{1}{n}\left(1-\frac{1}{n}\right)^{n-1} \ge \frac{i}{e\mu n}$$

$$E(T) \le \sum_{i=1}^{m-1} s_i^{-1} \le \sum_{i=1}^{n} \frac{e\mu n}{i} \le e\mu n \sum_{i=1}^{n} \le e\mu n(\ln + 1) = O(\mu n \ln n)$$

(μ+1)-EA for OneMax via Artificial Fitness Levels (2)

$$Pr((\mu+1)\ EA \text{ leaves } A_i \mid \chi(i) \in A_i) \ge s_i$$



Let:

- T_o be the expected time for a fraction $\chi(i)$ of the population to be in level A_i
- s_i be the probability to leave level A_i for A_j with $j > i$ given $\chi(i)$ in level A_i
- **Then:**

$$E(T) \le \sum_{i=1}^{m-1}\left(\frac{1}{s_i} + T_o\right)$$

(μ+1)-EA for OneMax via Artificial Fitness Levels

Theorem

The expected runtime of the $(\mu+1)$-EA for ONEMAX *is $O(\mu n + n \log n)$.*

Given that there are i copies,
the probability of creating a new copy is:

$$Pr(\text{copy} \mid i \in A_j) \ge \frac{i}{\mu}\left(1-\frac{1}{n}\right)^n \ge \frac{i}{2e\mu}$$

The expected time to create
at most $\min\{n/j, \mu\}$ copies is:

$$T_0 \le \sum_{i=1}^{n/j} \frac{2e\mu}{i} \le 2e\mu(\ln(n/j))$$

And,

$$E\left(\sum_{j=1}^{n} T_0\right) \le \sum_{j=1}^{n-1} 2e\mu \ln\left(\frac{n}{j}\right) \le 2e\mu \sum_{j=1}^{n-1} \ln\left(\frac{n}{j}\right) \le 4e\mu n$$

Overall,

$$E(T) \le \sum_{j=1}^{n}\left(\frac{1}{s_j} + T_0\right) \le \sum_{j=1}^{n}\frac{1}{s_j} + \sum_{j=1}^{n} T_0$$

$$\le \sum_{j=1}^{n} \frac{\mu}{\min\{n/j,\mu\}} \cdot \frac{en}{j} + 4e\mu n \le en\ln n + e\mu n + 4e\mu n$$

$$= O(n \ln n + \mu n)$$

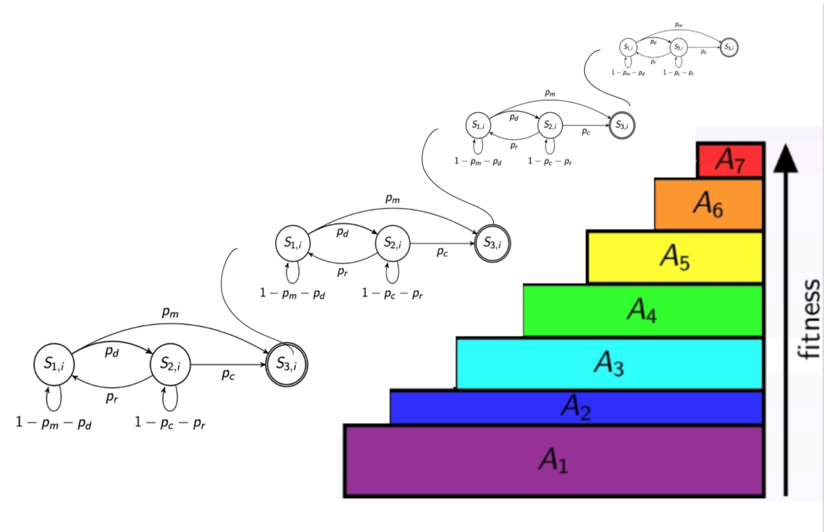
Steady-state GA

Algorithm 1: $(\mu+1)$ GA

```
P_1 ← μ individuals, uniformly at random from {0,1}^n;
t ← μ;
repeat
    Select x, y ∈ P_t uniformly at random with replacement ;
    z ← uniform crossover(x, y);
    z ← mutate(z);
    P_{t+1} ← P_t ∪ {z};
    Remove the element with lowest fitness from P_{t+1}, breaking ties at random;
    t ← t + 1;
until optimum is found;
```

x: 010101 f(x) = 3
y: 011100 f(y) = 3
z: 011101 f(z) = 4 (p=1/4)

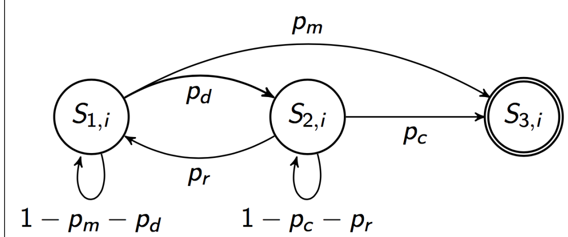
(μ+1)-GA for OneMax via Artificial Fitness Levels



Proof Idea

1) We divide the search space in canonical fitness levels $L_i = \{x \in \{0,1\}^n | \textsc{OneMax}(x) = i\}$;

2) Each level *i* is represented by a Markov Chain (and all individuals are at least in *Li*) ;

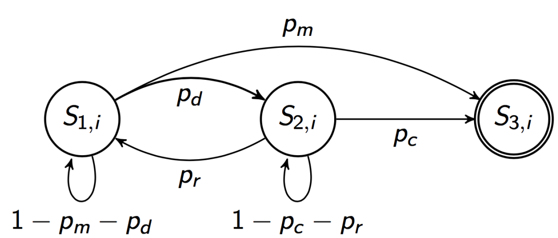


S_1: no diversity;
S_2: at least one diverse individual;
S_3: at least one individual in *Li+1* or higher.

3) The runtime is upper bounded by the time it takes to discover the next level E[*Li*] + the time it takes for the entire population to take over the level ($E[T_{takeover}] = O(\mu \log \mu)$) .

$$E[T] \le \sum_{i=0}^{n-1} \Big(E[L_i] + E[T_{takeover}] \Big)$$

Proof Idea (2)



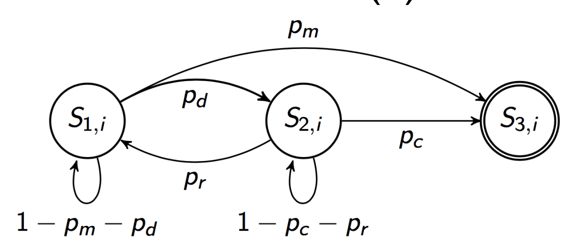
Absorbing time from State S_1:

$$E[T_1] = \frac{p_c + p_r + p_d}{p_c p_d + p_c p_m + p_m p_r} \le \frac{p_c + p_r}{p_c p_d + p_c p_m + p_m p_r} + \frac{1}{p_c}$$

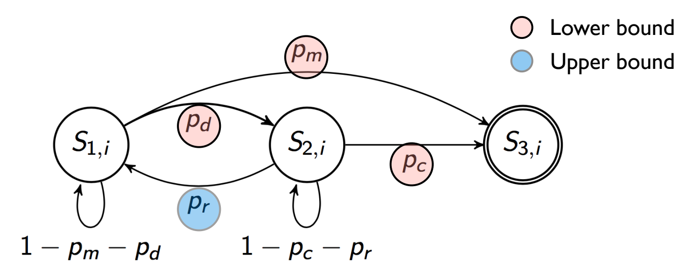
But the exact transition probabilities are tedious to calculate!

Proof Idea (3)

M_1:

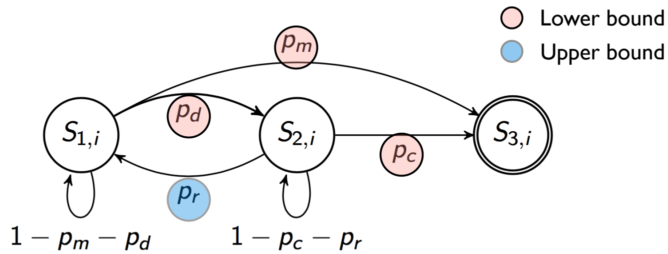


M_2:



As long as $p_m < p_c$ then $E[T_{M_1}] \le E[T_{M_2}]$

Proof Idea (4)



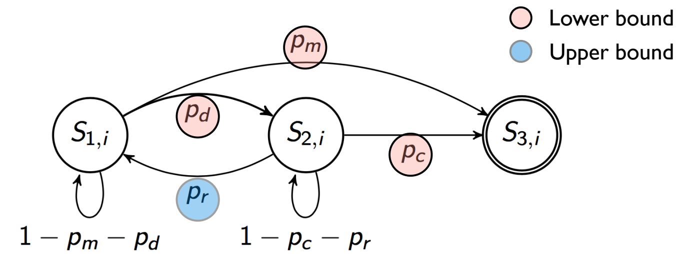
Lemma

The transition probabilities p_d, p_c, p_r and p_m are bounded as follows:

$$p_d \geq \frac{\mu}{(\mu+1)} \frac{i(n-i)c^2}{n^2(e^c + \mathcal{O}(1/n))} \qquad p_c \geq \frac{\mu - 1}{2\mu^2(e^c + \mathcal{O}(1/n))}$$

$$p_r \leq \frac{(\mu-1)(2\mu - 1 + \mathcal{O}(1/n))}{2e^c\mu^2(\mu+1)} \qquad p_m \geq \frac{c(n-i)}{n(e^c + \mathcal{O}(1/n))}.$$

Proof Idea (5)



$$E[T_i] \leq \frac{p_c + p_r}{p_c p_d + p_c p_m + p_m p_r} + \frac{1}{p_c} \leq \frac{e^c n}{c(n-i)} \frac{3}{3+c} + \mathcal{O}(\mu)$$

$$E[T] \leq \sum_{i=n-n/(4c(1+e^c))}^{n} \left(\frac{e^c n}{c(n-i)} \frac{3}{3+c} + \mathcal{O}(\mu) + \mathcal{O}(\mu \log \mu) \right)$$

$$\leq \frac{3e^c n \log n}{c(3+c)} + \mathcal{O}(n\mu \log \mu)$$

Main Result [Corus,Oliveto, IEEE TEVC 2017]

Theorem

The expected runtime of the $(\mu+1)$ GA with $\mu \geq 3$ and mutation rate c/n for any constant c on OneMax *is:*

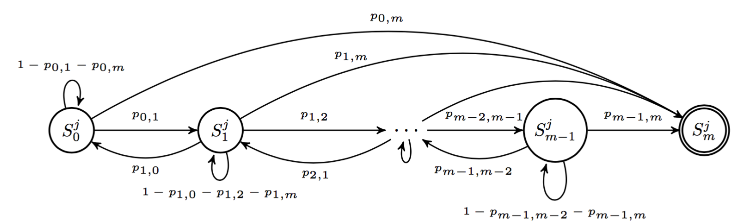
$$E[T] \leq \frac{3e^c n \log n}{c(3+c)} + \mathcal{O}(n\mu \log \mu).$$

For $\mu = o(\log n / \log \log n)$, the bound reduces to:

$$E[T] \leq \frac{3}{c(3+c)} e^c n \log n (1 + o(1)).$$

- Mutation rate 1/n: 3/4 e n log n (1 + o(1)) (GA) versus e n log n (1+ o(1)) (no crossover);
- Higher mutation rates may be beneficial: 0.72 e n log n (1 + o(1)) ~1.3/n (GA);
- Populations may be beneficial: for μ=2 the bound is 9/11 e n log n (1 + o(1)).

More Careful Analysis [Corus,Oliveto - GECCO 2019]



1) Show that the expected absorption time of the Markov chain is larger than that of the actual algorithm; (Drift analysis on a non-negative potential function; drift >1)

2) Bound the absorption time of the Markov chain by inverting its fundamental matrix, the result of which is a *strongly dominant tridiagonal matrix*.

The Runtime of the Steady-state GA [Corus,Oliveto - GECCO 2019]

▶ **Theorem 1.** *The expected runtime for the $(\mu+1)$ GA with $\mu = o(\sqrt{\log n})$ using an unbiased mutation operator $mutate(x)$ that flips i bits with probability p_i with $p_0 \in \Omega(1)$ and $p_1 \in \Omega(1)$ to optimise the* OneMax *function is:*

1. $E[T] \leq (1+o(1))n \ln n \frac{1}{p_1 + p_2 \frac{2(1-\xi_2)\mu}{(\mu+1)}}$ *if the quality of each offspring is evaluated,*
2. $E[T] \leq (1+o(1))n \ln n \frac{(1-p_0)}{p_1 + p_2 \frac{2(1-\xi_2)\mu}{(\mu+1)}}$ *if the quality of offspring identical to their parents is not evaluated for their quality is known;*

1) The statements are very general as they provide upper bounds on the expected runtime for each value of the population size (up to $\mu = o(\sqrt{\log n})$) and any unbiased mutation operator.

2) The preciseness of the analysis allows to appreciate the importance of the population for optimising unimodal functions (the upper bounds **decrease** with the population size)!

3) For each population size we can **derive the mutation rate** that provides the best upper bound on the expected runtime!

Best Upper Bounds ($p_2 = 1-\varepsilon$, $p_1 = \varepsilon/2$ $p_0 = \varepsilon/2$)

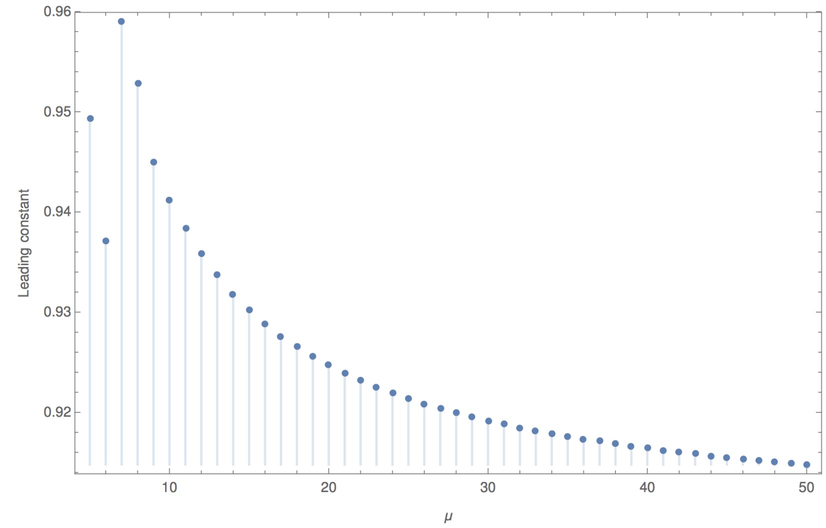


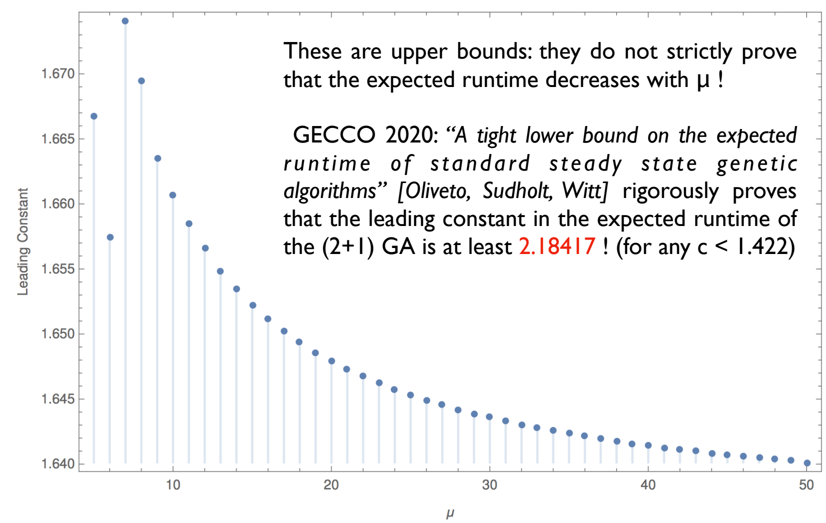
Figure 1 The leading constant from the second statement of Theorem 1 versus the population size. The best leading constant achievable by any unary unbiased algorithm is 1.

Corollary: Standard Bit Mutation

▶ **Corollary 2.** *Let ξ_2 be as defined in Theorem 1. The expected runtime for the $(\mu+1)$ GA with $\mu = o(\sqrt{\log n})$ using standard bit-mutation with mutation rate c/n, $c = \Theta(1)$ to optimise the* OneMax *function is:*

1. $E[T] \leq (1+o(1))n \ln n \frac{e^c}{c + \frac{c^2 \mu}{(\mu+1)}(1-\xi_2)}$ *if the quality of each offspring is evaluated,*
2. $E[T] \leq (1+o(1))n \ln n \frac{(1-e^{-c})e^c}{c + \frac{c^2 \mu}{(\mu+1)}(1-\xi_2)}$ *if the quality of offspring identical to their parents is not evaluated for their quality is known;*

Standard bit mutation: c/n with best c for each μ

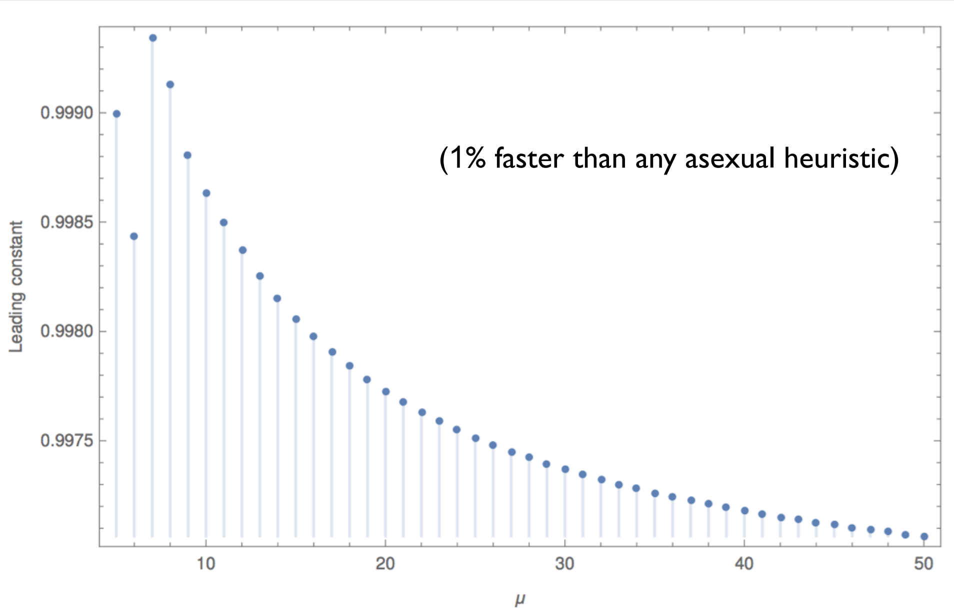


These are upper bounds: they do not strictly prove that the expected runtime decreases with μ !

GECCO 2020: *"A tight lower bound on the expected runtime of standard steady state genetic algorithms" [Oliveto, Sudholt, Witt]* rigorously proves that the leading constant in the expected runtime of the (2+1) GA is at least 2.18417 ! (for any $c < 1.422$)

Figure 2 The leading constant from the first statement of Corollary 2 versus the population size. The best mutation-only variant has a the leading constant of $e \approx 2.71$.

If the offspring has the same genotype as one of its parents then it is not necessary to evaluate it again!



An open problem for a long time (1994)

When Will a Genetic Algorithm Outperform Hill Climbing?

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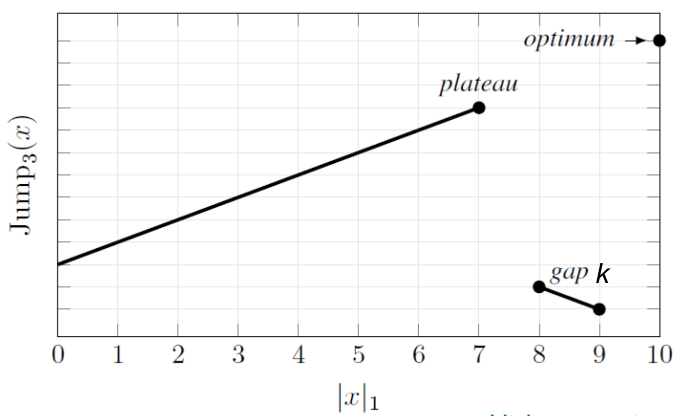
Abstract

We analyze a simple hill-climbing algorithm (RMHC) that was previously shown to outperform a genetic algorithm (GA) on a simple "Royal Road" function. We then analyze an "idealized" genetic algorithm (IGA) that is significantly faster than RMHC and that gives a lower bound for GA speed. We identify the features of the IGA that give rise to this speedup, and discuss how these features can be incorporated into a real GA.

In J. D. Cowan, G. Tesauro, and J. Alspector (editors), *Advances in Neural Information Processing Systems 6.* San Mateo, CA: Morgan Kaufmann, 1994.

Jump Benchmark Function [DFKKLoSS., IEEE TEVC 2017]



Higher mutation rates again!

Expected Runtimes ($k > 2$)

- $(\mu+1)$ EA with $p_m = 1/n$: $\Theta(n^k)$ (i.e., no crossover);
- $(\mu+1)$ GA with $p_m = 1/n$: $O(n^{k-1}\log n)$ [$\mu = \Theta(n)$];
- $(\mu+1)$ GA with $p_m = (1+\delta)/n$ is $O(n^{k-1})$ [$\mu = \Theta(\log n)$].

Reason: The interplay between mutation and crossover can create diversity on top of the plateau; then crossover + mutation can jump more quickly!

Conclusions

Steady state GAs are faster than any standard bit mutation-only evolutionary algorithm;

Evidence that evolving populations via sexual recombination is beneficial;

Evidence that higher mutation rates than standard are beneficial.

Summary

- Runtime analysis of evolutionary algorithms
  - mathematically rigorous statements about EA performance
  - most previous results on simple EAs, such as (1+1) EA
  - special techniques developed for population-based EAs
- Drift Analysis
- Level-based method [Corus et al., 2014]
  - EAs analysed from the perspective of EDAs
  - Upper bounds on expected optimisation time
  - Example applications include crossover, noise, and self-adaptation
- Negative drift theorem [Lehre, 2011a]
  - reproductive rate vs selective pressure
  - exponential lower bounds
  - mutation-selection balance
- Diversity + Bandwidth analysis for fitness proportional selection [Oliveto and Witt, 2014, 2015]
  - analysis of crossover
  - low selection pressure
  - exponential lower bounds
- Speed-up via crossover for steady state GAs to hillclimb OneMax and escape local optima [Dang, Friedrich, Kötzing, Krejca, Lehre, Oliveto, Sudholt, and Sutton, 2017a, Corus and Oliveto, 2017]

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EPSRC Engineering and Physical Sciences Research Council

RIGOROUS RESEARCH

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