Fully-Bayesian Bayesian Optimisation

Supplementary Material

GEORGE DE ATH, University of Exeter, United Kingdom
RICHARD M. EVERSON, University of Exeter, United Kingdom
JONATHAN E. FIELDSEND, University of Exeter, United Kingdom

ACM Reference Format:
George De Ath, Richard M. Everson, and Jonathan E. Fieldsend. 2021. Fully-Bayesian Bayesian Optimisation: Supplementary Material. In *2021 Genetic and Evolutionary Computation Conference Companion (GECCO '21 Companion), July 10–14, 2021, Lille, France.* ACM, New York, NY, USA, 27 pages. https://doi.org/10.1145/3449726.3463164

A INTRODUCTION

In this supplement we include extra results that could not be fit into the main paper due to space constraints. In the following sections, when comparing methods, the best method(s) are determined by whether a method either has the lowest median regret or is statistically indistinguishable from the method with the lowest median regret according to a one-sided, paired Wilcoxon signed-rank test [2] with Holm-bonferroni [1] correction ($p \geq 0.05$).

B INFERENCE SUMMARIES: MAP VS. MCMC

Here we show the inference summary plots with budgets $T \in \{50, 100, 150, 200\}$ function evaluations. Figures 1 and 2 summarise the performance of MAP vs MCMC for each combination of acquisition function and kernel (columns), and test problem (rows).

GECCO '21 Companion, July 10–14, 2021, Lille, France

ACM ISBN 978-1-4503-8351-6/21/07…$15.00
https://doi.org/10.1145/3449726.3463164

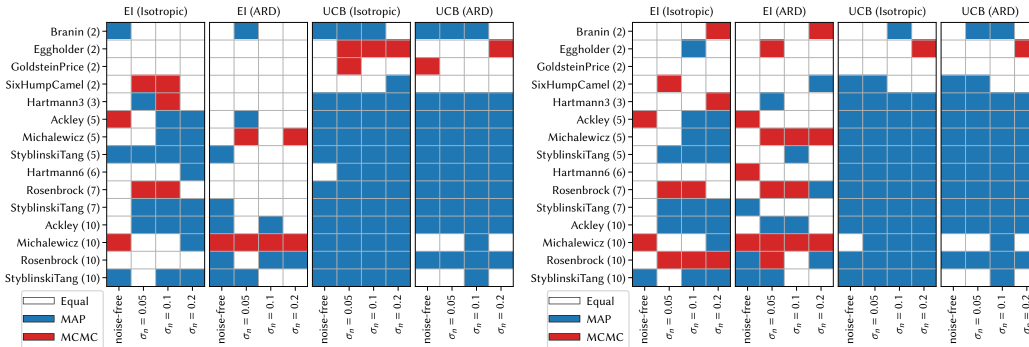

Fig. 1. MAP vs. MCMC inference summary after $T = 50$ (left) and $T = 100$ (right) function evaluations. The colour of each cell corresponds to whether both inference methods were statistically indistinguishable from one another (white), MAP performed better than MCMC (blue) and MCMC performed better than MAP (red).

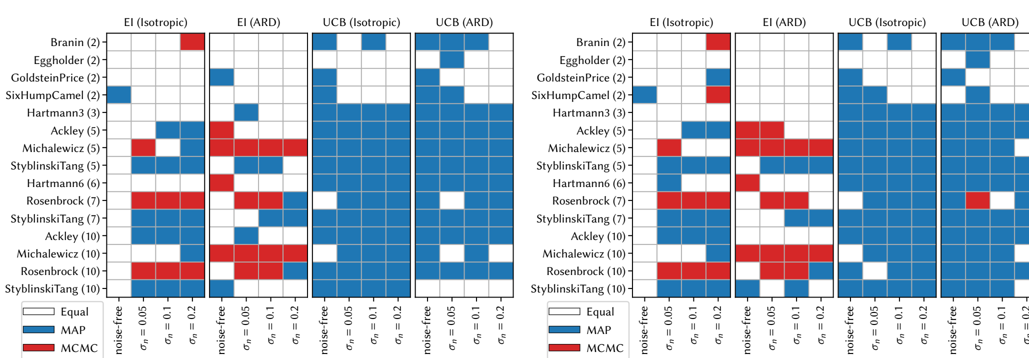

Fig. 2. MAP vs. MCMC inference summary after $T = 150$ (left) and $T = 200$ (right) function evaluations. The colour of each cell corresponds to whether both inference methods were statistically indistinguishable from one another (white), MAP performed better than MCMC (blue) and MCMC performed better than MAP (red).

C OPTIMISATION SUMMARY FOR DIFFERING LEVELS OF NOISE

Figure 3 summarises the performance of each combination of acquisition function, inference method and kernel type for each of the four noise settings. As can be seen from the plots, as the noise level increases, EI becomes less dominant.

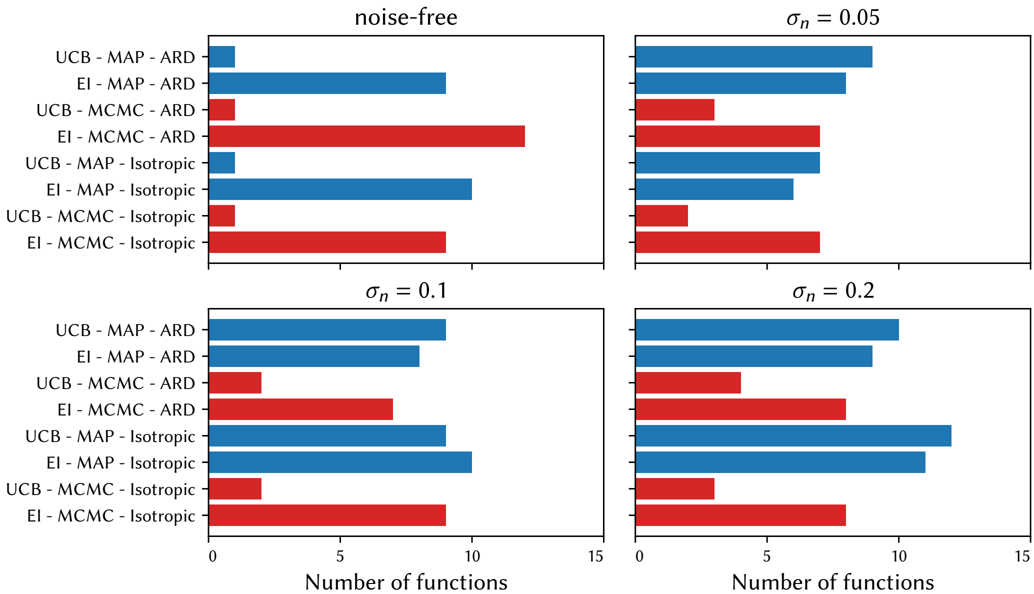

Fig. 3. Optimisation summary for each level of noise. Bar lengths correspond to the number of times each combination of acquisition function, inference method and kernel type was either the best performing or statistically equal to the best performing combination.

D RESULTS TABLES

In this section we show the results tables for each of the experiments. The tables show the median log simple regret as well as the median absolute deviation (MAD) from the median, a robust measure of dispersion. The method with the best (lowest) median regret is shown in dark grey, and those that are statistically indistinguishable from the best method are shown in light grey.

Method	**Branin (2)**		**Eggholder (2)**		**GoldsteinPrice (2)**		**SixHumpCamel (2)**		**Hartmann3 (3)**	
	Median	MAD	Median	MAD	Median	MAD	Median	MAD	Median	MAD
MAP	1.05×10^{-4}	1.22×10^{-4}	6.51×10^{1}	4.83×10^{1}	4.27×10^{-1}	4.07×10^{-1}	3.21×10^{-5}	4.00×10^{-5}	4.91×10^{-5}	4.55×10^{-5}
MCMC	1.68×10^{-4}	2.31×10^{-4}	6.51×10^{1}	1.06×10^{1}	3.42×10^{-1}	4.31×10^{-1}	1.30×10^{-4}	1.50×10^{-4}	8.30×10^{-5}	8.89×10^{-5}
MFVI	6.92×10^{-5}	8.19×10^{-5}	6.58×10^{1}	4.98	3.66×10^{-1}	3.84×10^{-1}	7.41×10^{-5}	8.48×10^{-5}	4.23×10^{-5}	3.82×10^{-5}
FRVI	1.74×10^{-4}	2.20×10^{-4}	6.51×10^{1}	5.57	3.78×10^{-1}	4.51×10^{-1}	6.75×10^{-5}	9.10×10^{-5}	1.00×10^{-4}	1.17×10^{-4}

Method	**Ackley5 (5)**		**Michalewicz5 (5)**		**StyblinskiTang5 (5)**		**Hartmann6 (6)**		**Rosenbrock7 (7)**	
	Median	MAD	Median	MAD	Median	MAD	Median	MAD	Median	MAD
MAP	1.89	1.51	8.21×10^{-1}	6.53×10^{-1}	5.37×10^{-1}	6.21×10^{-1}	4.30×10^{-3}	5.29×10^{-3}	2.44×10^{2}	1.38×10^{2}
MCMC	2.08	9.03×10^{-1}	8.31×10^{-1}	5.64×10^{-1}	6.15×10^{-1}	7.40×10^{-1}	4.07×10^{-3}	5.18×10^{-3}	2.57×10^{2}	1.99×10^{2}
MFVI	2.68	5.92×10^{-1}	1.20	5.52×10^{-1}	3.72	4.16	2.24×10^{-3}	2.38×10^{-3}	5.36×10^{2}	3.41×10^{2}
FRVI	2.71	5.91×10^{-1}	1.11	4.33×10^{-1}	6.48×10^{-1}	7.13×10^{-1}	4.98×10^{-3}	6.67×10^{-3}	2.54×10^{2}	1.74×10^{2}

Method	**StyblinskiTang7 (7)**		**Ackley10 (10)**		**Michalewicz10 (10)**		**Rosenbrock10 (10)**		**StyblinskiTang10 (10)**	
	Median	MAD	Median	MAD	Median	MAD	Median	MAD	Median	MAD
MAP	2.86×10^{1}	1.29×10^{1}	1.24×10^{1}	7.39	5.33	6.25×10^{-1}	8.39×10^{2}	4.22×10^{2}	7.18×10^{1}	2.26×10^{1}
MCMC	2.93×10^{1}	1.79×10^{1}	1.40×10^{1}	5.15	5.26	6.87×10^{-1}	7.52×10^{2}	3.71×10^{2}	6.62×10^{1}	2.91×10^{1}
MFVI	3.17×10^{1}	1.27×10^{1}	3.11	5.06×10^{-1}	4.59	8.53×10^{-1}	2.05×10^{3}	8.47×10^{2}	7.93×10^{1}	2.70×10^{1}
FRVI	2.87×10^{1}	1.76×10^{1}	3.51	1.40	4.54	7.33×10^{-1}	9.97×10^{2}	4.73×10^{2}	7.24×10^{1}	2.32×10^{1}

Table 1. Tabulated results for the EI acquisition function using an isotropic kernel on the noise-free problems. The median log simple regret and the median absolute deviation from the median (MAD) is shown in the left- and right-hand columns respectively. The method with the lowest median performance is shown in dark grey, with those that are statistically equivalent shown in light grey.

Method	**Branin (2)**		**Eggholder (2)**		**GoldsteinPrice (2)**		**SixHumpCamel (2)**		**Hartmann3 (3)**	
	Median	MAD	Median	MAD	Median	MAD	Median	MAD	Median	MAD
MAP	1.13×10^{-1}	1.23×10^{-1}	7.00×10^{1}	7.18	6.13	6.38	2.89×10^{-2}	3.08×10^{-2}	4.99×10^{-3}	3.36×10^{-3}
MCMC	1.25×10^{-1}	8.62×10^{-2}	6.91×10^{1}	5.47	5.85	6.12	2.89×10^{-2}	2.59×10^{-2}	6.23×10^{-3}	5.33×10^{-3}
MFVI	1.60×10^{-1}	1.18×10^{-1}	6.88×10^{1}	5.31	2.76	2.68	4.75×10^{-2}	4.18×10^{-2}	4.56×10^{-3}	4.72×10^{-3}
FRVI	8.35×10^{-2}	8.30×10^{-2}	6.86×10^{1}	4.62	3.99	4.30	3.02×10^{-2}	2.37×10^{-2}	5.28×10^{-3}	3.99×10^{-3}

Method	**Ackley5 (5)**		**Michalewicz5 (5)**		**StyblinskiTang5 (5)**		**Hartmann6 (6)**		**Rosenbrock7 (7)**	
	Median	MAD	Median	MAD	Median	MAD	Median	MAD	Median	MAD
MAP	5.28	2.45	1.23	7.00×10^{-1}	3.25×10^{1}	1.12×10^{1}	7.76×10^{-2}	7.39×10^{-2}	2.41×10^{3}	1.21×10^{3}
MCMC	4.76	1.48	1.06	6.51×10^{-1}	4.70×10^{1}	1.65×10^{1}	9.74×10^{-2}	1.07×10^{-1}	6.47×10^{2}	4.12×10^{2}
MFVI	4.34	1.52	1.25	7.30×10^{-1}	4.32×10^{1}	1.53×10^{1}	8.55×10^{-2}	7.04×10^{-2}	7.13×10^{2}	5.01×10^{2}
FRVI	3.93	1.05	1.48	8.85×10^{-1}	5.18×10^{1}	1.92×10^{1}	7.49×10^{-2}	8.28×10^{-2}	7.26×10^{2}	4.03×10^{2}

Method	**StyblinskiTang7 (7)**		**Ackley10 (10)**		**Michalewicz10 (10)**		**Rosenbrock10 (10)**		**StyblinskiTang10 (10)**	
	Median	MAD	Median	MAD	Median	MAD	Median	MAD	Median	MAD
MAP	6.38×10^{1}	1.89×10^{1}	1.46×10^{1}	2.56	5.32	6.37×10^{-1}	4.70×10^{3}	3.20×10^{3}	1.25×10^{2}	1.87×10^{1}
MCMC	8.14×10^{1}	1.91×10^{1}	1.66×10^{1}	3.11	5.35	6.51×10^{-1}	1.70×10^{3}	8.40×10^{2}	1.35×10^{2}	2.58×10^{1}
MFVI	8.35×10^{1}	2.31×10^{1}	1.66×10^{1}	3.40	5.39	9.63×10^{-1}	1.36×10^{3}	7.91×10^{2}	1.30×10^{2}	2.94×10^{1}
FRVI	8.13×10^{1}	1.76×10^{1}	1.73×10^{1}	2.52	5.27	9.27×10^{-1}	1.54×10^{3}	1.05×10^{3}	1.38×10^{2}	3.00×10^{1}

Table 2. Tabulated results for the EI acquisition function using an isotropic kernel on the $\sigma_n = 0.05$ problems. The median log simple regret and the median absolute deviation from the median (MAD) is shown in the left- and right-hand columns respectively. The method with the lowest median performance is shown in dark grey, with those that are statistically equivalent shown in light grey.

Method	Branin (2)		Eggholder (2)		GoldsteinPrice (2)		SixHumpCamel (2)		Hartmann3 (3)	
	Median	MAD	Median	MAD	Median	MAD	Median	MAD	Median	MAD
MAP	6.57×10^{-2}	8.76×10^{-2}	6.84×10^{1}	2.37×10^{1}	8.07	6.88	5.55×10^{-2}	5.32×10^{-2}	1.06×10^{-2}	8.24×10^{-3}
MCMC	1.34×10^{-1}	9.90×10^{-2}	7.11×10^{1}	7.75	4.64	5.81	5.66×10^{-2}	4.23×10^{-2}	9.60×10^{-3}	7.81×10^{-3}
MFVI	9.78×10^{-2}	9.41×10^{-2}	7.13×10^{1}	8.22	5.61	5.62	4.22×10^{-2}	4.07×10^{-2}	1.12×10^{-2}	9.39×10^{-3}
FRVI	1.19×10^{-1}	9.89×10^{-2}	6.86×10^{1}	1.31×10^{1}	6.46	7.15	5.12×10^{-2}	5.42×10^{-2}	1.07×10^{-2}	6.75×10^{-3}

Method	Ackley5 (5)		Michalewicz5 (5)		StyblinskiTang5 (5)		Hartmann6 (6)		Rosenbrock7 (7)	
	Median	MAD	Median	MAD	Median	MAD	Median	MAD	Median	MAD
MAP	8.38	2.74	1.32	6.33×10^{-1}	4.49×10^{1}	1.29×10^{1}	2.11×10^{-1}	1.22×10^{-1}	3.10×10^{3}	2.32×10^{3}
MCMC	9.67	6.68	1.54	6.38×10^{-1}	5.35×10^{1}	1.08×10^{1}	2.20×10^{-1}	8.51×10^{-2}	9.78×10^{2}	9.02×10^{2}
MFVI	9.12	3.75	1.95	7.21×10^{-1}	5.94×10^{1}	1.73×10^{1}	2.13×10^{-1}	1.22×10^{-1}	1.29×10^{3}	1.01×10^{3}
FRVI	9.46	6.85	1.79	5.52×10^{-1}	5.62×10^{1}	1.63×10^{1}	2.01×10^{-1}	8.47×10^{-2}	1.13×10^{3}	9.01×10^{2}

Method	StyblinskiTang7 (7)		Ackley10 (10)		Michalewicz10 (10)		Rosenbrock10 (10)		StyblinskiTang10 (10)	
	Median	MAD	Median	MAD	Median	MAD	Median	MAD	Median	MAD
MAP	6.96×10^{1}	1.89×10^{1}	1.69×10^{1}	1.56	5.78	2.93×10^{-1}	3.98×10^{3}	2.53×10^{3}	1.31×10^{2}	1.89×10^{1}
MCMC	9.68×10^{1}	2.36×10^{1}	1.89×10^{1}	1.25	5.78	5.25×10^{-1}	2.47×10^{3}	1.57×10^{3}	1.50×10^{2}	3.83×10^{1}
MFVI	9.75×10^{1}	2.45×10^{1}	1.96×10^{1}	7.46×10^{-1}	5.96	4.30×10^{-1}	2.37×10^{3}	1.81×10^{3}	1.45×10^{2}	2.80×10^{1}
FRVI	9.52×10^{1}	1.95×10^{1}	1.94×10^{1}	1.04	5.80	5.92×10^{-1}	1.78×10^{3}	1.39×10^{3}	1.46×10^{2}	2.82×10^{1}

Table 3. Tabulated results for the EI acquisition function using an isotropic kernel on the $\sigma_n = 0.1$ problems. The median log simple regret and the median absolute deviation from the median (MAD) is shown in the left- and right-hand columns respectively. The method with the lowest median performance is shown in dark grey, with those that are statistically equivalent shown in light grey.

Method	Branin (2)		Eggholder (2)		GoldsteinPrice (2)		SixHumpCamel (2)		Hartmann3 (3)	
	Median	MAD	Median	MAD	Median	MAD	Median	MAD	Median	MAD
MAP	2.88×10^{-1}	3.05×10^{-1}	7.78×10^{1}	3.45×10^{1}	3.66	4.43	6.64×10^{-2}	5.79×10^{-2}	1.87×10^{-2}	1.34×10^{-2}
MCMC	1.72×10^{-1}	1.56×10^{-1}	7.32×10^{1}	3.85×10^{1}	8.02	7.53	3.36×10^{-2}	3.62×10^{-2}	1.36×10^{-2}	1.31×10^{-2}
MFVI	2.05×10^{-1}	1.78×10^{-1}	8.05×10^{1}	3.18×10^{1}	9.94	8.95	6.17×10^{-2}	5.99×10^{-2}	1.95×10^{-2}	1.59×10^{-2}
FRVI	1.62×10^{-1}	1.76×10^{-1}	7.32×10^{1}	4.33×10^{1}	5.94	5.95	5.47×10^{-2}	5.31×10^{-2}	2.28×10^{-2}	1.88×10^{-2}

Method	Ackley5 (5)		Michalewicz5 (5)		StyblinskiTang5 (5)		Hartmann6 (6)		Rosenbrock7 (7)	
	Median	MAD	Median	MAD	Median	MAD	Median	MAD	Median	MAD
MAP	1.31×10^{1}	5.57	1.88	5.17×10^{-1}	4.27×10^{1}	1.62×10^{1}	3.43×10^{-1}	1.74×10^{-1}	7.68×10^{3}	5.75×10^{3}
MCMC	1.86×10^{1}	2.03	1.90	5.53×10^{-1}	6.88×10^{1}	1.75×10^{1}	2.84×10^{-1}	1.38×10^{-1}	5.85×10^{3}	4.67×10^{3}
MFVI	1.89×10^{1}	1.55	2.20	4.81×10^{-1}	6.62×10^{1}	1.28×10^{1}	2.70×10^{-1}	1.45×10^{-1}	5.60×10^{3}	4.73×10^{3}
FRVI	1.87×10^{1}	1.89	2.12	5.03×10^{-1}	6.62×10^{1}	1.35×10^{1}	3.14×10^{-1}	1.51×10^{-1}	4.98×10^{3}	4.24×10^{3}

Method	StyblinskiTang7 (7)		Ackley10 (10)		Michalewicz10 (10)		Rosenbrock10 (10)		StyblinskiTang10 (10)	
	Median	MAD	Median	MAD	Median	MAD	Median	MAD	Median	MAD
MAP	7.42×10^{1}	2.07×10^{1}	1.80×10^{1}	1.52	6.01	4.62×10^{-1}	2.09×10^{4}	1.27×10^{4}	1.33×10^{2}	1.82×10^{1}
MCMC	1.00×10^{2}	2.39×10^{1}	1.91×10^{1}	8.98×10^{-1}	6.15	3.95×10^{-1}	9.54×10^{3}	7.34×10^{3}	1.67×10^{2}	2.16×10^{1}
MFVI	1.01×10^{2}	2.11×10^{1}	1.95×10^{1}	9.06×10^{-1}	6.10	4.92×10^{-1}	6.07×10^{3}	5.52×10^{3}	1.65×10^{2}	1.86×10^{1}
FRVI	9.24×10^{1}	2.42×10^{1}	1.94×10^{1}	9.59×10^{-1}	5.86	4.64×10^{-1}	8.90×10^{3}	8.74×10^{3}	1.60×10^{2}	3.02×10^{1}

Table 4. Tabulated results for the EI acquisition function using an isotropic kernel on the $\sigma_n = 0.2$ problems. The median log simple regret and the median absolute deviation from the median (MAD) is shown in the left- and right-hand columns respectively. The method with the lowest median performance is shown in dark grey, with those that are statistically equivalent shown in light grey.

Method	Branin (2)		Eggholder (2)		GoldsteinPrice (2)		SixHumpCamel (2)		Hartmann3 (3)	
	Median	MAD	Median	MAD	Median	MAD	Median	MAD	Median	MAD
MAP	1.93×10^{-4}	2.24×10^{-4}	6.51×10^{1}	5.22×10^{1}	2.17×10^{-1}	2.29×10^{-1}	1.26×10^{-4}	1.65×10^{-4}	9.52×10^{-6}	9.64×10^{-6}
MCMC	1.47×10^{-4}	1.68×10^{-4}	3.19×10^{1}	4.73×10^{1}	3.26×10^{-1}	4.32×10^{-1}	1.64×10^{-4}	1.91×10^{-4}	9.37×10^{-6}	1.06×10^{-5}
MFVI	1.04×10^{-4}	1.14×10^{-4}	6.57×10^{1}	1.09	2.96×10^{-1}	3.46×10^{-1}	1.00×10^{-4}	1.05×10^{-4}	1.41×10^{-5}	1.67×10^{-5}
FRVI	8.98×10^{-5}	1.18×10^{-4}	6.51×10^{1}	1.37×10^{1}	3.16×10^{-1}	4.15×10^{-1}	1.85×10^{-4}	2.20×10^{-4}	1.19×10^{-5}	1.35×10^{-5}

Method	Ackley5 (5)		Michalewicz5 (5)		StyblinskiTang5 (5)		Hartmann6 (6)		Rosenbrock7 (7)	
	Median	MAD	Median	MAD	Median	MAD	Median	MAD	Median	MAD
MAP	2.33	7.10×10^{-1}	3.62×10^{-1}	4.04×10^{-1}	6.40	9.40	2.82×10^{-3}	3.20×10^{-3}	2.86×10^{2}	1.41×10^{2}
MCMC	1.96	6.40×10^{-1}	2.41×10^{-1}	1.74×10^{-1}	1.02	1.35	1.31×10^{-3}	1.57×10^{-3}	2.76×10^{2}	1.50×10^{2}
MFVI	2.53	3.84×10^{-1}	5.38×10^{-1}	4.20×10^{-1}	9.44×10^{-1}	1.24	7.90×10^{-4}	9.55×10^{-4}	3.75×10^{2}	2.21×10^{2}
FRVI	2.50	4.51×10^{-1}	3.90×10^{-1}	4.42×10^{-1}	5.69×10^{-1}	5.21×10^{-1}	2.12×10^{-3}	2.70×10^{-3}	2.07×10^{2}	1.22×10^{2}

Method	StyblinskiTang7 (7)		Ackley10 (10)		Michalewicz10 (10)		Rosenbrock10 (10)		StyblinskiTang10 (10)	
	Median	MAD	Median	MAD	Median	MAD	Median	MAD	Median	MAD
MAP	3.16×10^{1}	1.71×10^{1}	3.59	1.11	3.57	8.15×10^{-1}	1.31×10^{3}	5.56×10^{2}	7.57×10^{1}	2.81×10^{1}
MCMC	2.88×10^{1}	1.91×10^{1}	3.52	1.04	2.76	7.34×10^{-1}	1.35×10^{3}	9.11×10^{2}	8.01×10^{1}	2.71×10^{1}
MFVI	3.34×10^{1}	8.67	3.13	3.96×10^{-1}	3.65	7.43×10^{-1}	1.53×10^{3}	5.76×10^{2}	7.49×10^{1}	2.30×10^{1}
FRVI	2.40×10^{1}	1.32×10^{1}	3.19	7.66×10^{-1}	2.79	9.08×10^{-1}	1.14×10^{3}	4.88×10^{2}	6.78×10^{1}	1.73×10^{1}

Table 5. Tabulated results for the EI acquisition function using an ARD kernel on the noise-free problems. The median log simple regret and the median absolute deviation from the median (MAD) is shown in the left- and right-hand columns respectively. The method with the lowest median performance is shown in dark grey, with those that are statistically equivalent shown in light grey.

Method	Branin (2)		Eggholder (2)		GoldsteinPrice (2)		SixHumpCamel (2)		Hartmann3 (3)	
	Median	MAD	Median	MAD	Median	MAD	Median	MAD	Median	MAD
MAP	4.20×10^{-2}	4.24×10^{-2}	6.74×10^{1}	6.89	4.24	4.59	2.09×10^{-2}	2.67×10^{-2}	5.99×10^{-3}	2.74×10^{-3}
MCMC	4.67×10^{-2}	4.87×10^{-2}	6.64×10^{1}	9.27	5.89	5.42	2.09×10^{-2}	2.27×10^{-2}	7.89×10^{-3}	3.81×10^{-3}
MFVI	5.43×10^{-2}	6.16×10^{-2}	6.67×10^{1}	7.33	6.69	6.16	2.82×10^{-2}	3.06×10^{-2}	8.37×10^{-3}	2.14×10^{-3}
FRVI	6.47×10^{-2}	6.07×10^{-2}	6.67×10^{1}	6.67	5.49	5.47	2.76×10^{-2}	2.86×10^{-2}	8.25×10^{-3}	4.69×10^{-3}

Method	Ackley5 (5)		Michalewicz5 (5)		StyblinskiTang5 (5)		Hartmann6 (6)		Rosenbrock7 (7)	
	Median	MAD	Median	MAD	Median	MAD	Median	MAD	Median	MAD
MAP	6.40	1.96	1.77	4.55×10^{-1}	3.38×10^{1}	1.43×10^{1}	8.41×10^{-2}	7.47×10^{-2}	4.74×10^{3}	2.78×10^{3}
MCMC	5.97	2.00	1.29	3.19×10^{-1}	4.32×10^{1}	1.51×10^{1}	8.07×10^{-2}	6.57×10^{-2}	2.00×10^{3}	1.38×10^{3}
MFVI	7.57	1.96	1.71	3.62×10^{-1}	4.66×10^{1}	1.64×10^{1}	7.83×10^{-2}	6.71×10^{-2}	2.19×10^{3}	1.14×10^{3}
FRVI	6.14	1.93	1.71	3.97×10^{-1}	4.36×10^{1}	1.30×10^{1}	7.93×10^{-2}	6.31×10^{-2}	2.32×10^{3}	1.38×10^{3}

Method	StyblinskiTang7 (7)		Ackley10 (10)		Michalewicz10 (10)		Rosenbrock10 (10)		StyblinskiTang10 (10)	
	Median	MAD	Median	MAD	Median	MAD	Median	MAD	Median	MAD
MAP	6.44×10^{1}	1.85×10^{1}	1.51×10^{1}	3.53	5.23	5.46×10^{-1}	9.05×10^{3}	4.82×10^{3}	1.18×10^{2}	2.15×10^{1}
MCMC	6.99×10^{1}	2.55×10^{1}	1.71×10^{1}	2.70	5.09	8.41×10^{-1}	5.65×10^{3}	3.55×10^{3}	1.26×10^{2}	2.16×10^{1}
MFVI	7.86×10^{1}	1.61×10^{1}	1.72×10^{1}	3.16	5.21	7.65×10^{-1}	4.77×10^{3}	2.95×10^{3}	1.21×10^{2}	2.69×10^{1}
FRVI	7.78×10^{1}	1.82×10^{1}	1.53×10^{1}	5.31	5.23	4.12×10^{-1}	5.30×10^{3}	2.56×10^{3}	1.24×10^{2}	3.18×10^{1}

Table 6. Tabulated results for the EI acquisition function using an ARD kernel on the $\sigma_n = 0.05$ problems. The median log simple regret and the median absolute deviation from the median (MAD) is shown in the left- and right-hand columns respectively. The method with the lowest median performance is shown in dark grey, with those that are statistically equivalent shown in light grey.

Method	Branin (2)		Eggholder (2)		GoldsteinPrice (2)		SixHumpCamel (2)		Hartmann3 (3)	
	Median	MAD	Median	MAD	Median	MAD	Median	MAD	Median	MAD
MAP	7.02×10^{-2}	7.25×10^{-2}	5.80×10^{1}	3.41×10^{1}	7.92	7.06	5.01×10^{-2}	4.60×10^{-2}	1.08×10^{-2}	5.00×10^{-3}
MCMC	9.71×10^{-2}	1.13×10^{-1}	6.65×10^{1}	1.94×10^{1}	6.08	5.83	3.63×10^{-2}	3.66×10^{-2}	1.55×10^{-2}	9.23×10^{-3}
MFVI	7.99×10^{-2}	8.86×10^{-2}	6.85×10^{1}	5.04×10^{1}	7.19	7.36	4.68×10^{-2}	4.72×10^{-2}	1.06×10^{-2}	5.45×10^{-3}
FRVI	1.11×10^{-1}	1.12×10^{-1}	6.65×10^{1}	1.94×10^{1}	7.12	8.27	4.57×10^{-2}	5.04×10^{-2}	1.23×10^{-2}	5.31×10^{-3}

Method	Ackley5 (5)		Michalewicz5 (5)		StyblinskiTang5 (5)		Hartmann6 (6)		Rosenbrock7 (7)	
	Median	MAD	Median	MAD	Median	MAD	Median	MAD	Median	MAD
MAP	1.07×10^{1}	4.29	1.95	4.00×10^{-1}	4.07×10^{1}	9.25	1.96×10^{-1}	7.30×10^{-2}	5.35×10^{3}	3.92×10^{3}
MCMC	1.00×10^{1}	3.74	1.45	3.41×10^{-1}	5.20×10^{1}	1.37×10^{1}	1.96×10^{-1}	6.28×10^{-2}	3.06×10^{3}	2.22×10^{3}
MFVI	1.13×10^{1}	6.27	1.75	3.71×10^{-1}	5.50×10^{1}	1.16×10^{1}	2.09×10^{-1}	8.75×10^{-2}	3.36×10^{3}	1.98×10^{3}
FRVI	1.04×10^{1}	6.23	1.80	3.78×10^{-1}	5.44×10^{1}	1.64×10^{1}	2.09×10^{-1}	4.45×10^{-2}	2.47×10^{3}	2.07×10^{3}

Method	StyblinskiTang7 (7)		Ackley10 (10)		Michalewicz10 (10)		Rosenbrock10 (10)		StyblinskiTang10 (10)	
	Median	MAD	Median	MAD	Median	MAD	Median	MAD	Median	MAD
MAP	7.31×10^{1}	1.09×10^{1}	1.93×10^{1}	8.98×10^{-1}	5.71	5.37×10^{-1}	1.36×10^{4}	8.34×10^{3}	1.18×10^{2}	1.83×10^{1}
MCMC	8.07×10^{1}	2.34×10^{1}	1.90×10^{1}	1.05	5.20	3.79×10^{-1}	6.53×10^{3}	3.46×10^{3}	1.32×10^{2}	2.76×10^{1}
MFVI	8.75×10^{1}	1.66×10^{1}	1.98×10^{1}	5.16×10^{-1}	5.72	4.82×10^{-1}	7.76×10^{3}	3.76×10^{3}	1.25×10^{2}	2.27×10^{1}
FRVI	8.62×10^{1}	1.44×10^{1}	1.95×10^{1}	8.24×10^{-1}	5.68	4.39×10^{-1}	7.04×10^{3}	4.72×10^{3}	1.36×10^{2}	2.28×10^{1}

Table 7. Tabulated results for the EI acquisition function using an ARD kernel on the $\sigma_n = 0.1$ problems. The median log simple regret and the median absolute deviation from the median (MAD) is shown in the left- and right-hand columns respectively. The method with the lowest median performance is shown in dark grey, with those that are statistically equivalent shown in light grey.

Method	Branin (2)		Eggholder (2)		GoldsteinPrice (2)		SixHumpCamel (2)		Hartmann3 (3)	
	Median	MAD	Median	MAD	Median	MAD	Median	MAD	Median	MAD
MAP	2.71×10^{-1}	2.46×10^{-1}	8.47×10^{1}	2.48×10^{1}	8.11	7.32	4.90×10^{-2}	4.46×10^{-2}	2.00×10^{-2}	1.66×10^{-2}
MCMC	1.46×10^{-1}	1.72×10^{-1}	7.44×10^{1}	3.92×10^{1}	7.33	6.41	7.22×10^{-2}	5.93×10^{-2}	1.82×10^{-2}	1.36×10^{-2}
MFVI	1.50×10^{-1}	1.60×10^{-1}	7.62×10^{1}	3.94×10^{1}	9.91	1.02×10^{1}	5.26×10^{-2}	5.71×10^{-2}	2.14×10^{-2}	1.59×10^{-2}
FRVI	1.65×10^{-1}	1.56×10^{-1}	7.15×10^{1}	4.63×10^{1}	6.98	8.20	4.75×10^{-2}	4.08×10^{-2}	1.48×10^{-2}	6.87×10^{-3}

Method	Ackley5 (5)		Michalewicz5 (5)		StyblinskiTang5 (5)		Hartmann6 (6)		Rosenbrock7 (7)	
	Median	MAD	Median	MAD	Median	MAD	Median	MAD	Median	MAD
MAP	1.80×10^{1}	2.56	2.31	4.71×10^{-1}	4.32×10^{1}	1.81×10^{1}	2.85×10^{-1}	1.33×10^{-1}	1.12×10^{4}	7.99×10^{3}
MCMC	1.61×10^{1}	2.91	1.93	4.56×10^{-1}	5.57×10^{1}	1.15×10^{1}	2.73×10^{-1}	8.93×10^{-2}	7.69×10^{3}	6.05×10^{3}
MFVI	1.71×10^{1}	3.26	2.08	3.77×10^{-1}	5.50×10^{1}	1.66×10^{1}	3.07×10^{-1}	1.08×10^{-1}	1.20×10^{4}	1.10×10^{4}
FRVI	1.81×10^{1}	2.06	2.11	3.18×10^{-1}	5.55×10^{1}	1.58×10^{1}	2.62×10^{-1}	9.05×10^{-2}	1.19×10^{4}	1.14×10^{4}

Method	StyblinskiTang7 (7)		Ackley10 (10)		Michalewicz10 (10)		Rosenbrock10 (10)		StyblinskiTang10 (10)	
	Median	MAD	Median	MAD	Median	MAD	Median	MAD	Median	MAD
MAP	7.40×10^{1}	1.85×10^{1}	1.95×10^{1}	8.59×10^{-1}	6.11	3.79×10^{-1}	2.59×10^{4}	1.16×10^{4}	1.27×10^{2}	2.12×10^{1}
MCMC	9.05×10^{1}	2.22×10^{1}	1.92×10^{1}	6.95×10^{-1}	5.97	5.23×10^{-1}	3.36×10^{4}	3.07×10^{4}	1.33×10^{2}	3.37×10^{1}
MFVI	8.12×10^{1}	1.55×10^{1}	1.97×10^{1}	7.00×10^{-1}	6.06	4.34×10^{-1}	3.18×10^{4}	2.82×10^{4}	1.40×10^{2}	2.36×10^{1}
FRVI	9.23×10^{1}	1.44×10^{1}	1.96×10^{1}	8.20×10^{-1}	6.04	5.06×10^{-1}	3.22×10^{4}	3.38×10^{4}	1.38×10^{2}	2.76×10^{1}

Table 8. Tabulated results for the EI acquisition function using an ARD kernel on the $\sigma_n = 0.2$ problems. The median log simple regret and the median absolute deviation from the median (MAD) is shown in the left- and right-hand columns respectively. The method with the lowest median performance is shown in dark grey, with those that are statistically equivalent shown in light grey.

Method	Branin (2)		Eggholder (2)		GoldsteinPrice (2)		SixHumpCamel (2)		Hartmann3 (3)	
	Median	MAD	Median	MAD	Median	MAD	Median	MAD	Median	MAD
MAP	1.69×10^{-3}	1.72×10^{-3}	9.50×10^{1}	2.96×10^{1}	1.56	1.42	1.02×10^{-3}	1.23×10^{-3}	6.09×10^{-4}	4.03×10^{-4}
MCMC	3.90×10^{-3}	4.75×10^{-3}	1.06×10^{2}	3.48×10^{1}	2.60	1.88	3.93×10^{-3}	3.84×10^{-3}	6.81×10^{-4}	5.78×10^{-4}
MFVI	2.68×10^{-3}	2.71×10^{-3}	1.04×10^{2}	5.15×10^{1}	2.76	2.06	5.36×10^{-3}	4.83×10^{-3}	1.08×10^{-3}	7.36×10^{-4}
FRVI	4.86×10^{-3}	5.21×10^{-3}	8.55×10^{1}	6.97×10^{1}	1.85	1.96	4.86×10^{-3}	3.35×10^{-3}	7.76×10^{-4}	6.42×10^{-4}

Method	Ackley5 (5)		Michalewicz5 (5)		StyblinskiTang5 (5)		Hartmann6 (6)		Rosenbrock7 (7)	
	Median	MAD	Median	MAD	Median	MAD	Median	MAD	Median	MAD
MAP	3.74	5.42×10^{-1}	1.15	2.59×10^{-1}	2.79×10^{1}	8.73	8.94×10^{-2}	1.21×10^{-1}	6.61×10^{3}	4.81×10^{3}
MCMC	4.67	1.02	1.40	2.67×10^{-1}	4.36×10^{1}	6.22	2.74×10^{-1}	1.16×10^{-1}	4.30×10^{3}	4.71×10^{3}
MFVI	5.08	7.63×10^{-1}	1.42	2.67×10^{-1}	4.23×10^{1}	6.90	3.05×10^{-1}	1.19×10^{-1}	5.82×10^{3}	5.49×10^{3}
FRVI	4.86	9.71×10^{-1}	1.40	4.06×10^{-1}	4.28×10^{1}	8.43	2.95×10^{-1}	1.51×10^{-1}	4.34×10^{3}	3.62×10^{3}

Method	StyblinskiTang7 (7)		Ackley10 (10)		Michalewicz10 (10)		Rosenbrock10 (10)		StyblinskiTang10 (10)	
	Median	MAD	Median	MAD	Median	MAD	Median	MAD	Median	MAD
MAP	5.93×10^{1}	1.01×10^{1}	5.94	6.35×10^{-1}	4.68	4.53×10^{-1}	2.21×10^{4}	1.77×10^{4}	1.13×10^{2}	2.33×10^{1}
MCMC	1.20×10^{2}	2.24×10^{1}	7.72	1.07	4.87	4.29×10^{-1}	6.30×10^{4}	3.00×10^{4}	1.44×10^{2}	3.91×10^{1}
MFVI	1.24×10^{2}	1.68×10^{1}	7.25	9.96×10^{-1}	4.67	7.75×10^{-1}	6.71×10^{4}	1.58×10^{4}	1.69×10^{2}	4.60×10^{1}
FRVI	1.27×10^{2}	1.78×10^{1}	7.89	1.01	4.82	6.42×10^{-1}	6.01×10^{4}	1.93×10^{4}	1.96×10^{2}	2.00×10^{1}

Table 9. Tabulated results for the UCB acquisition function using an isotropic kernel on the noise-free problems. The median log simple regret and the median absolute deviation from the median (MAD) is shown in the left- and right-hand columns respectively. The method with the lowest median performance is shown in dark grey, with those that are statistically equivalent shown in light grey.

Method	Branin (2)		Eggholder (2)		GoldsteinPrice (2)		SixHumpCamel (2)		Hartmann3 (3)	
	Median	MAD	Median	MAD	Median	MAD	Median	MAD	Median	MAD
MAP	8.33×10^{-2}	9.12×10^{-2}	8.36×10^{1}	3.62×10^{1}	8.09	8.18	2.95×10^{-2}	2.66×10^{-2}	5.48×10^{-3}	3.29×10^{-3}
MCMC	1.23×10^{-1}	1.02×10^{-1}	7.93×10^{1}	6.09×10^{1}	6.20	7.42	4.40×10^{-2}	3.46×10^{-2}	1.30×10^{-2}	8.03×10^{-3}
MFVI	1.09×10^{-1}	8.55×10^{-2}	8.45×10^{1}	3.46×10^{1}	6.67	6.47	4.29×10^{-2}	5.20×10^{-2}	1.38×10^{-2}	1.15×10^{-2}
FRVI	1.03×10^{-1}	1.12×10^{-1}	1.01×10^{2}	2.94×10^{1}	6.70	6.44	5.21×10^{-2}	4.26×10^{-2}	1.45×10^{-2}	9.79×10^{-3}

Method	Ackley5 (5)		Michalewicz5 (5)		StyblinskiTang5 (5)		Hartmann6 (6)		Rosenbrock7 (7)	
	Median	MAD	Median	MAD	Median	MAD	Median	MAD	Median	MAD
MAP	4.23	8.57×10^{-1}	1.52	4.67×10^{-1}	3.56×10^{1}	1.28×10^{1}	1.03×10^{-1}	1.01×10^{-1}	2.62×10^{3}	1.58×10^{3}
MCMC	6.64	1.52	2.35	3.48×10^{-1}	8.14×10^{1}	1.46×10^{1}	3.51×10^{-1}	1.55×10^{-1}	3.97×10^{3}	1.10×10^{3}
MFVI	6.73	1.53	2.41	3.28×10^{-1}	8.14×10^{1}	1.46×10^{1}	3.53×10^{-1}	2.08×10^{-1}	3.55×10^{3}	1.11×10^{3}
FRVI	6.89	1.31	2.23	3.56×10^{-1}	8.14×10^{1}	1.46×10^{1}	3.26×10^{-1}	2.02×10^{-1}	4.30×10^{3}	1.30×10^{3}

Method	StyblinskiTang7 (7)		Ackley10 (10)		Michalewicz10 (10)		Rosenbrock10 (10)		StyblinskiTang10 (10)	
	Median	MAD	Median	MAD	Median	MAD	Median	MAD	Median	MAD
MAP	6.99×10^{1}	2.02×10^{1}	1.26×10^{1}	5.90	5.32	6.60×10^{-1}	1.03×10^{4}	7.02×10^{3}	1.27×10^{2}	2.11×10^{1}
MCMC	1.22×10^{2}	1.92×10^{1}	1.93×10^{1}	1.68	6.26	5.45×10^{-1}	8.93×10^{3}	8.71×10^{3}	1.85×10^{2}	2.28×10^{1}
MFVI	1.21×10^{2}	2.00×10^{1}	1.86×10^{1}	2.71	6.09	5.46×10^{-1}	5.13×10^{3}	3.50×10^{3}	1.87×10^{2}	2.49×10^{1}
FRVI	1.21×10^{2}	2.02×10^{1}	1.51×10^{1}	4.98	5.99	3.77×10^{-1}	8.37×10^{3}	7.42×10^{3}	1.87×10^{2}	2.57×10^{1}

Table 10. Tabulated results for the UCB acquisition function using an isotropic kernel on the $\sigma_n = 0.05$ problems. The median log simple regret and the median absolute deviation from the median (MAD) is shown in the left- and right-hand columns respectively. The method with the lowest median performance is shown in dark grey, with those that are statistically equivalent shown in light grey.

Method	Branin (2)		Eggholder (2)		GoldsteinPrice (2)		SixHumpCamel (2)		Hartmann3 (3)	
	Median	MAD	Median	MAD	Median	MAD	Median	MAD	Median	MAD
MAP	6.99×10^{-2}	7.82×10^{-2}	9.60×10^{1}	2.10×10^{1}	6.49	7.33	3.39×10^{-2}	2.84×10^{-2}	7.29×10^{-3}	5.50×10^{-3}
MCMC	1.03×10^{-1}	1.01×10^{-1}	1.02×10^{2}	2.74×10^{1}	8.04	8.45	4.27×10^{-2}	2.60×10^{-2}	1.93×10^{-2}	1.50×10^{-2}
MFVI	1.24×10^{-1}	1.30×10^{-1}	8.84×10^{1}	3.39×10^{1}	8.84	8.22	5.39×10^{-2}	4.41×10^{-2}	1.28×10^{-2}	7.39×10^{-3}
FRVI	1.18×10^{-1}	1.01×10^{-1}	1.01×10^{2}	3.15×10^{1}	4.91	5.99	3.81×10^{-2}	4.66×10^{-2}	1.43×10^{-2}	7.29×10^{-3}

Method	Ackley5 (5)		Michalewicz5 (5)		StyblinskiTang5 (5)		Hartmann6 (6)		Rosenbrock7 (7)	
	Median	MAD	Median	MAD	Median	MAD	Median	MAD	Median	MAD
MAP	7.63	2.57	1.61	5.03×10^{-1}	4.51×10^{1}	1.69×10^{1}	1.96×10^{-1}	1.31×10^{-1}	2.34×10^{3}	2.11×10^{3}
MCMC	1.85×10^{1}	2.51	2.32	4.83×10^{-1}	8.01×10^{1}	2.09×10^{1}	4.93×10^{-1}	1.53×10^{-1}	4.74×10^{3}	2.77×10^{3}
MFVI	1.93×10^{1}	1.51	2.27	3.39×10^{-1}	7.80×10^{1}	2.25×10^{1}	4.35×10^{-1}	2.06×10^{-1}	4.60×10^{3}	2.30×10^{3}
FRVI	1.85×10^{1}	2.46	2.42	4.38×10^{-1}	8.01×10^{1}	2.17×10^{1}	3.55×10^{-1}	1.99×10^{-1}	4.52×10^{3}	1.90×10^{3}

Method	StyblinskiTang7 (7)		Ackley10 (10)		Michalewicz10 (10)		Rosenbrock10 (10)		StyblinskiTang10 (10)	
	Median	MAD	Median	MAD	Median	MAD	Median	MAD	Median	MAD
MAP	7.94×10^{1}	1.51×10^{1}	1.81×10^{1}	2.33	5.39	4.79×10^{-1}	1.01×10^{4}	5.67×10^{3}	1.31×10^{2}	2.64×10^{1}
MCMC	1.17×10^{2}	1.94×10^{1}	2.01×10^{1}	4.37×10^{-1}	6.22	4.73×10^{-1}	6.08×10^{4}	1.12×10^{4}	1.90×10^{2}	3.31×10^{1}
MFVI	1.17×10^{2}	2.13×10^{1}	2.01×10^{1}	4.36×10^{-1}	6.23	5.84×10^{-1}	6.10×10^{4}	1.03×10^{4}	1.84×10^{2}	3.17×10^{1}
FRVI	1.17×10^{2}	2.12×10^{1}	2.02×10^{1}	3.75×10^{-1}	6.18	6.00×10^{-1}	6.19×10^{4}	1.39×10^{4}	1.90×10^{2}	3.87×10^{1}

Table 11. Tabulated results for the UCB acquisition function using an isotropic kernel on the $\sigma_n = 0.1$ problems. The median log simple regret and the median absolute deviation from the median (MAD) is shown in the left- and right-hand columns respectively. The method with the lowest median performance is shown in dark grey, with those that are statistically equivalent shown in light grey.

Method	Branin (2)		Eggholder (2)		GoldsteinPrice (2)		SixHumpCamel (2)		Hartmann3 (3)	
	Median	MAD	Median	MAD	Median	MAD	Median	MAD	Median	MAD
MAP	2.48×10^{-1}	3.04×10^{-1}	9.33×10^{1}	3.61×10^{1}	5.75	7.43	3.75×10^{-2}	3.72×10^{-2}	1.41×10^{-2}	1.17×10^{-2}
MCMC	3.07×10^{-1}	2.65×10^{-1}	7.92×10^{1}	5.58×10^{1}	3.84	4.42	6.21×10^{-2}	5.73×10^{-2}	3.16×10^{-2}	2.02×10^{-2}
MFVI	3.77×10^{-1}	3.43×10^{-1}	7.28×10^{1}	5.10×10^{1}	5.73	5.09	4.21×10^{-2}	4.39×10^{-2}	2.19×10^{-2}	1.80×10^{-2}
FRVI	4.42×10^{-1}	4.44×10^{-1}	8.51×10^{1}	4.13×10^{1}	2.75	2.98	7.65×10^{-2}	7.36×10^{-2}	2.25×10^{-2}	1.74×10^{-2}

Method	Ackley5 (5)		Michalewicz5 (5)		StyblinskiTang5 (5)		Hartmann6 (6)		Rosenbrock7 (7)	
	Median	MAD	Median	MAD	Median	MAD	Median	MAD	Median	MAD
MAP	1.08×10^{1}	6.89	1.89	4.77×10^{-1}	4.54×10^{1}	1.55×10^{1}	4.04×10^{-1}	1.67×10^{-1}	6.51×10^{3}	6.57×10^{3}
MCMC	1.93×10^{1}	1.02	2.60	4.37×10^{-1}	8.71×10^{1}	1.96×10^{1}	5.76×10^{-1}	2.64×10^{-1}	2.66×10^{4}	3.09×10^{4}
MFVI	1.93×10^{1}	1.22	2.65	2.64×10^{-1}	8.67×10^{1}	1.99×10^{1}	5.30×10^{-1}	1.67×10^{-1}	1.49×10^{4}	1.62×10^{4}
FRVI	1.93×10^{1}	1.15	2.74	3.22×10^{-1}	8.67×10^{1}	1.99×10^{1}	5.22×10^{-1}	1.43×10^{-1}	3.00×10^{4}	3.37×10^{4}

Method	StyblinskiTang7 (7)		Ackley10 (10)		Michalewicz10 (10)		Rosenbrock10 (10)		StyblinskiTang10 (10)	
	Median	MAD	Median	MAD	Median	MAD	Median	MAD	Median	MAD
MAP	7.41×10^{1}	2.02×10^{1}	1.90×10^{1}	1.15	5.62	3.93×10^{-1}	4.68×10^{4}	3.88×10^{4}	1.25×10^{2}	3.15×10^{1}
MCMC	1.28×10^{2}	2.19×10^{1}	2.01×10^{1}	5.79×10^{-1}	6.30	5.92×10^{-1}	8.89×10^{4}	5.54×10^{4}	1.80×10^{2}	2.87×10^{1}
MFVI	1.26×10^{2}	1.99×10^{1}	2.01×10^{1}	5.77×10^{-1}	6.39	5.01×10^{-1}	9.25×10^{4}	3.86×10^{4}	1.79×10^{2}	2.77×10^{1}
FRVI	1.26×10^{2}	2.08×10^{1}	1.99×10^{1}	6.47×10^{-1}	6.33	6.01×10^{-1}	9.25×10^{4}	3.73×10^{4}	1.81×10^{2}	2.29×10^{1}

Table 12. Tabulated results for the UCB acquisition function using an isotropic kernel on the $\sigma_n = 0.2$ problems. The median log simple regret and the median absolute deviation from the median (MAD) is shown in the left- and right-hand columns respectively. The method with the lowest median performance is shown in dark grey, with those that are statistically equivalent shown in light grey.

Method	Branin (2)		Eggholder (2)		GoldsteinPrice (2)		SixHumpCamel (2)		Hartmann3 (3)	
	Median	MAD	Median	MAD	Median	MAD	Median	MAD	Median	MAD
MAP	2.98×10^{-3}	3.12×10^{-3}	8.44×10^{1}	2.73×10^{1}	1.88	1.26	1.43×10^{-3}	1.29×10^{-3}	9.75×10^{-4}	5.34×10^{-4}
MCMC	6.10×10^{-3}	7.76×10^{-3}	9.44×10^{1}	4.78×10^{1}	2.66	1.45	2.17×10^{-3}	2.41×10^{-3}	1.75×10^{-3}	4.76×10^{-4}
MFVI	7.48×10^{-3}	9.78×10^{-3}	9.26×10^{1}	5.58×10^{1}	1.86	1.88	1.33×10^{-3}	1.32×10^{-3}	1.92×10^{-3}	2.83×10^{-4}
FRVI	5.87×10^{-3}	7.57×10^{-3}	1.05×10^{2}	4.96×10^{1}	1.93	1.69	2.12×10^{-3}	1.92×10^{-3}	1.84×10^{-3}	3.21×10^{-4}

Method	Ackley5 (5)		Michalewicz5 (5)		StyblinskiTang5 (5)		Hartmann6 (6)		Rosenbrock7 (7)	
	Median	MAD	Median	MAD	Median	MAD	Median	MAD	Median	MAD
MAP	3.73	4.94×10^{-1}	1.64	4.85×10^{-1}	4.72×10^{1}	1.12×10^{1}	5.76×10^{-3}	5.15×10^{-3}	9.12×10^{3}	5.98×10^{3}
MCMC	4.86	7.40×10^{-1}	2.09	3.95×10^{-1}	5.51×10^{1}	1.13×10^{1}	3.98×10^{-2}	5.49×10^{-2}	1.68×10^{4}	1.29×10^{4}
MFVI	5.02	7.67×10^{-1}	2.12	4.73×10^{-1}	5.42×10^{1}	9.16	1.60×10^{-2}	2.25×10^{-2}	1.79×10^{4}	1.34×10^{4}
FRVI	4.87	7.62×10^{-1}	2.11	3.65×10^{-1}	5.42×10^{1}	1.32×10^{1}	4.03×10^{-2}	5.62×10^{-2}	1.59×10^{4}	1.16×10^{4}

Method	StyblinskiTang7 (7)		Ackley10 (10)		Michalewicz10 (10)		Rosenbrock10 (10)		StyblinskiTang10 (10)	
	Median	MAD	Median	MAD	Median	MAD	Median	MAD	Median	MAD
MAP	7.12×10^{1}	1.34×10^{1}	6.05	7.66×10^{-1}	4.94	3.91×10^{-1}	1.32×10^{4}	1.31×10^{4}	1.10×10^{2}	2.37×10^{1}
MCMC	1.06×10^{2}	2.09×10^{1}	7.99	1.03	5.41	6.20×10^{-1}	7.13×10^{4}	1.98×10^{4}	1.27×10^{2}	2.69×10^{1}
MFVI	1.15×10^{2}	1.62×10^{1}	8.36	1.23	5.48	5.30×10^{-1}	7.28×10^{4}	2.27×10^{4}	1.56×10^{2}	3.00×10^{1}
FRVI	1.19×10^{2}	1.94×10^{1}	8.17	9.96×10^{-1}	5.14	5.31×10^{-1}	7.15×10^{4}	2.22×10^{4}	1.17×10^{2}	1.88×10^{1}

Table 13. Tabulated results for the UCB acquisition function using an ARD kernel on the noise-free problems. The median log simple regret and the median absolute deviation from the median (MAD) is shown in the left- and right-hand columns respectively. The method with the lowest median performance is shown in dark grey, with those that are statistically equivalent shown in light grey.

Method	Branin (2)		Eggholder (2)		GoldsteinPrice (2)		SixHumpCamel (2)		Hartmann3 (3)	
	Median	MAD	Median	MAD	Median	MAD	Median	MAD	Median	MAD
MAP	3.51×10^{-2}	3.14×10^{-2}	8.61×10^{1}	3.75×10^{1}	4.74	6.11	2.58×10^{-2}	2.02×10^{-2}	4.68×10^{-3}	2.85×10^{-3}
MCMC	8.47×10^{-2}	9.31×10^{-2}	1.02×10^{2}	5.17×10^{1}	4.07	5.72	4.84×10^{-2}	4.47×10^{-2}	9.81×10^{-3}	2.12×10^{-3}
MFVI	9.01×10^{-2}	9.74×10^{-2}	9.00×10^{1}	4.54×10^{1}	7.22	6.37	3.64×10^{-2}	2.84×10^{-2}	1.11×10^{-2}	3.70×10^{-3}
FRVI	9.60×10^{-2}	1.08×10^{-1}	1.02×10^{2}	4.61×10^{1}	4.71	4.87	2.84×10^{-2}	2.37×10^{-2}	1.04×10^{-2}	3.00×10^{-3}

Method	Ackley5 (5)		Michalewicz5 (5)		StyblinskiTang5 (5)		Hartmann6 (6)		Rosenbrock7 (7)	
	Median	MAD	Median	MAD	Median	MAD	Median	MAD	Median	MAD
MAP	4.36	6.15×10^{-1}	1.71	4.53×10^{-1}	5.25×10^{1}	1.67×10^{1}	3.40×10^{-2}	2.18×10^{-2}	1.04×10^{4}	7.73×10^{3}
MCMC	7.05	2.02	2.25	3.26×10^{-1}	7.73×10^{1}	1.36×10^{1}	1.97×10^{-1}	6.65×10^{-2}	3.46×10^{3}	3.40×10^{3}
MFVI	7.61	1.33	2.15	2.88×10^{-1}	7.58×10^{1}	1.32×10^{1}	2.15×10^{-1}	4.29×10^{-2}	4.97×10^{3}	5.00×10^{3}
FRVI	6.73	1.64	2.25	2.59×10^{-1}	7.15×10^{1}	1.46×10^{1}	2.17×10^{-1}	4.96×10^{-2}	4.87×10^{3}	5.13×10^{3}

Method	StyblinskiTang7 (7)		Ackley10 (10)		Michalewicz10 (10)		Rosenbrock10 (10)		StyblinskiTang10 (10)	
	Median	MAD	Median	MAD	Median	MAD	Median	MAD	Median	MAD
MAP	7.08×10^{1}	2.18×10^{1}	1.07×10^{1}	3.55	5.14	5.68×10^{-1}	2.16×10^{4}	1.60×10^{4}	1.16×10^{2}	1.60×10^{1}
MCMC	1.21×10^{2}	1.99×10^{1}	1.81×10^{1}	2.87	5.49	4.99×10^{-1}	5.86×10^{4}	2.61×10^{4}	1.35×10^{2}	3.11×10^{1}
MFVI	1.19×10^{2}	1.72×10^{1}	1.94×10^{1}	1.47	5.62	5.55×10^{-1}	6.75×10^{4}	1.40×10^{4}	1.62×10^{2}	3.33×10^{1}
FRVI	1.21×10^{2}	1.96×10^{1}	1.82×10^{1}	3.25	5.61	4.02×10^{-1}	6.75×10^{4}	2.00×10^{4}	1.43×10^{2}	2.09×10^{1}

Table 14. Tabulated results for the UCB acquisition function using an ARD kernel on the $\sigma_n = 0.05$ problems. The median log simple regret and the median absolute deviation from the median (MAD) is shown in the left- and right-hand columns respectively. The method with the lowest median performance is shown in dark grey, with those that are statistically equivalent shown in light grey.

Method	Branin (2)		Eggholder (2)		GoldsteinPrice (2)		SixHumpCamel (2)		Hartmann3 (3)	
	Median	MAD	Median	MAD	Median	MAD	Median	MAD	Median	MAD
MAP	3.28×10^{-2}	3.58×10^{-2}	8.31×10^{1}	3.21×10^{1}	5.56	5.21	3.07×10^{-2}	3.72×10^{-2}	9.47×10^{-3}	3.06×10^{-3}
MCMC	6.85×10^{-2}	7.48×10^{-2}	8.35×10^{1}	4.09×10^{1}	5.60	5.91	4.04×10^{-2}	4.53×10^{-2}	1.72×10^{-2}	7.04×10^{-3}
MFVI	7.32×10^{-2}	8.50×10^{-2}	9.11×10^{1}	5.66×10^{1}	5.77	4.79	4.08×10^{-2}	4.86×10^{-2}	1.45×10^{-2}	5.84×10^{-3}
FRVI	7.25×10^{-2}	7.16×10^{-2}	1.01×10^{2}	4.46×10^{1}	5.22	6.22	5.07×10^{-2}	4.34×10^{-2}	1.44×10^{-2}	7.01×10^{-3}

Method	Ackley5 (5)		Michalewicz5 (5)		StyblinskiTang5 (5)		Hartmann6 (6)		Rosenbrock7 (7)	
	Median	MAD	Median	MAD	Median	MAD	Median	MAD	Median	MAD
MAP	7.26	2.26	1.97	4.00×10^{-1}	5.44×10^{1}	1.31×10^{1}	2.08×10^{-1}	6.69×10^{-2}	7.24×10^{3}	5.46×10^{3}
MCMC	1.70×10^{1}	3.59	2.26	3.44×10^{-1}	6.91×10^{1}	1.99×10^{1}	2.78×10^{-1}	6.03×10^{-2}	9.33×10^{3}	7.78×10^{3}
MFVI	1.66×10^{1}	4.48	2.20	3.13×10^{-1}	7.12×10^{1}	2.01×10^{1}	3.00×10^{-1}	8.37×10^{-2}	7.22×10^{3}	7.60×10^{3}
FRVI	1.71×10^{1}	3.42	2.18	3.27×10^{-1}	7.46×10^{1}	1.98×10^{1}	2.72×10^{-1}	3.84×10^{-2}	7.33×10^{3}	7.13×10^{3}

Method	StyblinskiTang7 (7)		Ackley10 (10)		Michalewicz10 (10)		Rosenbrock10 (10)		StyblinskiTang10 (10)	
	Median	MAD	Median	MAD	Median	MAD	Median	MAD	Median	MAD
MAP	7.14×10^{1}	1.63×10^{1}	1.89×10^{1}	1.87	5.30	5.98×10^{-1}	2.86×10^{4}	2.15×10^{4}	1.14×10^{2}	2.19×10^{1}
MCMC	1.11×10^{2}	1.30×10^{1}	2.01×10^{1}	4.59×10^{-1}	5.66	4.59×10^{-1}	7.07×10^{4}	3.47×10^{4}	1.23×10^{2}	2.68×10^{1}
MFVI	1.11×10^{2}	2.62×10^{1}	2.02×10^{1}	3.67×10^{-1}	6.05	5.43×10^{-1}	6.92×10^{4}	3.08×10^{4}	1.48×10^{2}	2.90×10^{1}
FRVI	1.11×10^{2}	1.69×10^{1}	2.01×10^{1}	4.75×10^{-1}	5.82	6.18×10^{-1}	7.25×10^{4}	2.21×10^{4}	1.36×10^{2}	3.00×10^{1}

Table 15. Tabulated results for the UCB acquisition function using an ARD kernel on the $\sigma_n = 0.1$ problems. The median log simple regret and the median absolute deviation from the median (MAD) is shown in the left- and right-hand columns respectively. The method with the lowest median performance is shown in dark grey, with those that are statistically equivalent shown in light grey.

Method	Branin (2)		Eggholder (2)		GoldsteinPrice (2)		SixHumpCamel (2)		Hartmann3 (3)	
	Median	MAD	Median	MAD	Median	MAD	Median	MAD	Median	MAD
MAP	2.06×10^{-1}	2.63×10^{-1}	1.01×10^{2}	3.75×10^{1}	6.62	7.14	1.86×10^{-2}	2.22×10^{-2}	1.43×10^{-2}	7.74×10^{-3}
MCMC	1.96×10^{-1}	1.88×10^{-1}	8.62×10^{1}	4.28×10^{1}	8.83	7.79	3.54×10^{-2}	3.72×10^{-2}	2.72×10^{-2}	1.70×10^{-2}
MFVI	1.87×10^{-1}	1.94×10^{-1}	1.02×10^{2}	1.77×10^{1}	7.54	6.96	4.77×10^{-2}	4.59×10^{-2}	2.15×10^{-2}	1.46×10^{-2}
FRVI	1.39×10^{-1}	1.67×10^{-1}	1.01×10^{2}	1.91×10^{1}	7.66	7.30	7.25×10^{-2}	7.45×10^{-2}	2.21×10^{-2}	1.65×10^{-2}

Method	Ackley5 (5)		Michalewicz5 (5)		StyblinskiTang5 (5)		Hartmann6 (6)		Rosenbrock7 (7)	
	Median	MAD	Median	MAD	Median	MAD	Median	MAD	Median	MAD
MAP	1.78×10^{1}	2.29	2.05	4.24×10^{-1}	5.26×10^{1}	1.54×10^{1}	3.34×10^{-1}	1.73×10^{-1}	1.32×10^{4}	1.02×10^{4}
MCMC	1.88×10^{1}	1.70	2.23	2.89×10^{-1}	6.87×10^{1}	1.36×10^{1}	5.28×10^{-1}	1.77×10^{-1}	3.06×10^{4}	2.63×10^{4}
MFVI	1.80×10^{1}	1.70	2.35	4.20×10^{-1}	6.86×10^{1}	2.22×10^{1}	5.40×10^{-1}	1.18×10^{-1}	2.66×10^{4}	2.76×10^{4}
FRVI	1.86×10^{1}	1.77	2.38	2.72×10^{-1}	6.88×10^{1}	2.25×10^{1}	4.52×10^{-1}	1.72×10^{-1}	2.71×10^{4}	2.27×10^{4}

Method	StyblinskiTang7 (7)		Ackley10 (10)		Michalewicz10 (10)		Rosenbrock10 (10)		StyblinskiTang10 (10)	
	Median	MAD	Median	MAD	Median	MAD	Median	MAD	Median	MAD
MAP	8.12×10^{1}	1.37×10^{1}	1.92×10^{1}	1.48	5.76	6.10×10^{-1}	3.99×10^{4}	3.54×10^{4}	1.30×10^{2}	1.98×10^{1}
MCMC	1.17×10^{2}	2.93×10^{1}	2.01×10^{1}	5.44×10^{-1}	5.94	6.28×10^{-1}	8.18×10^{4}	4.87×10^{4}	1.34×10^{2}	2.84×10^{1}
MFVI	1.17×10^{2}	2.67×10^{1}	2.02×10^{1}	5.70×10^{-1}	6.32	4.60×10^{-1}	8.53×10^{4}	5.07×10^{4}	1.56×10^{2}	2.55×10^{1}
FRVI	1.17×10^{2}	2.22×10^{1}	2.00×10^{1}	6.05×10^{-1}	6.24	5.56×10^{-1}	9.40×10^{4}	5.61×10^{4}	1.28×10^{2}	2.26×10^{1}

Table 16. Tabulated results for the UCB acquisition function using an ARD kernel on the $\sigma_n = 0.2$ problems. The median log simple regret and the median absolute deviation from the median (MAD) is shown in the left- and right-hand columns respectively. The method with the lowest median performance is shown in dark grey, with those that are statistically equivalent shown in light grey.

E CONVERGENCE AND DISTANCE PLOTS

The figures in this section show the convergence and distance plots for each combination of acquisition function, kernel type and level of function noise. The convergence plots show the median log simple regret, with shading representing the interquartile range over 51 runs, and the dashed vertical line indicating the end of the initial LHS phase. The distance plots show the normalised Euclidean distance between consecutively selected locations over the optimisation run. For each d-dimensional problem, distances are normalised by the largest possible distance possible, i.e. $\sqrt{d}$, so that distances reside in $[0, 1]$.

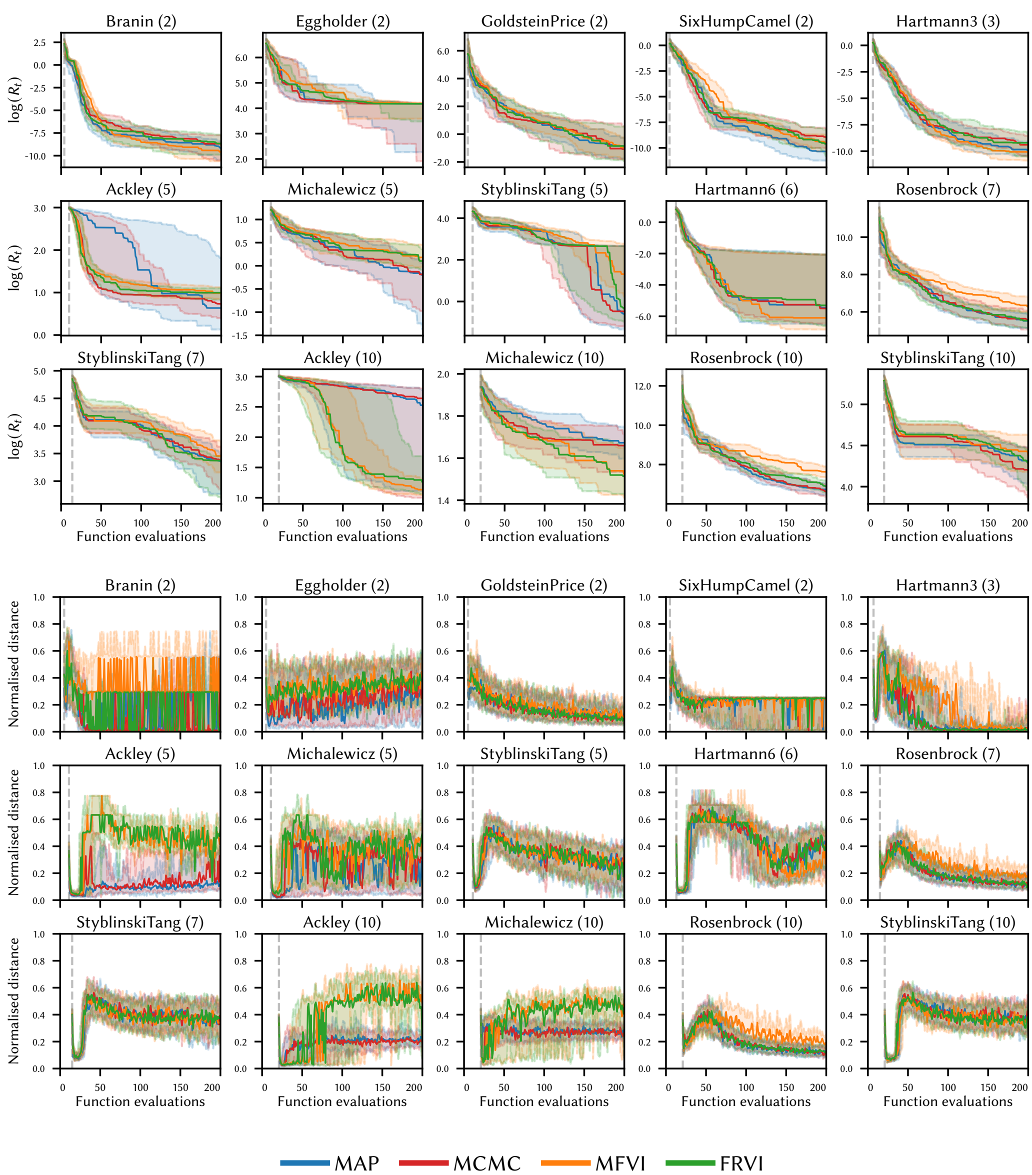

Fig. 4. Convergence (upper) and distance (lower) plots for the EI acquisition function with an isotropic kernel on the noise-free problems.

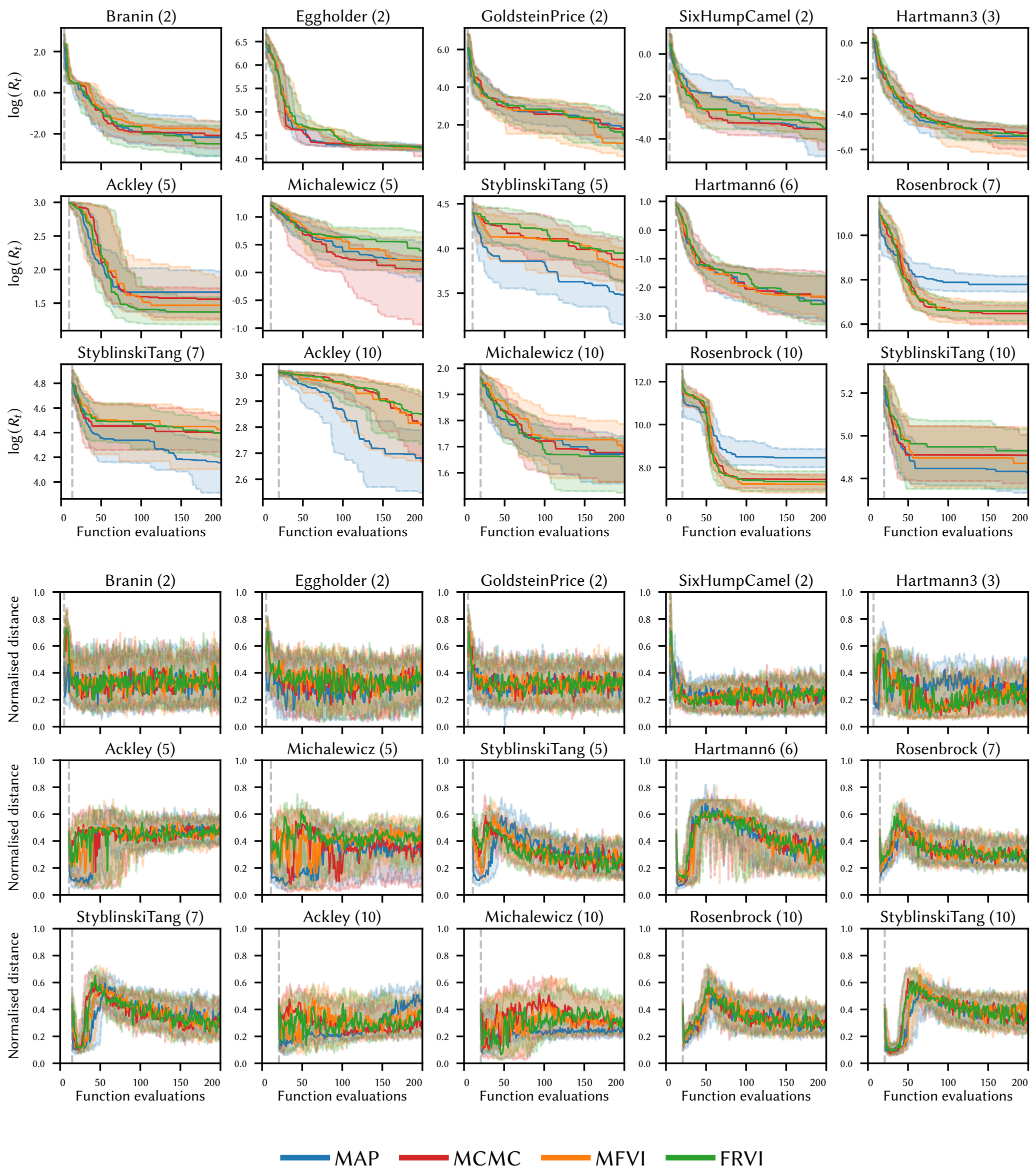

Fig. 5. Convergence (upper) and distance (lower) plots for the EI acquisition function with an isotropic kernel on the $\sigma_n = 0.05$ problems.

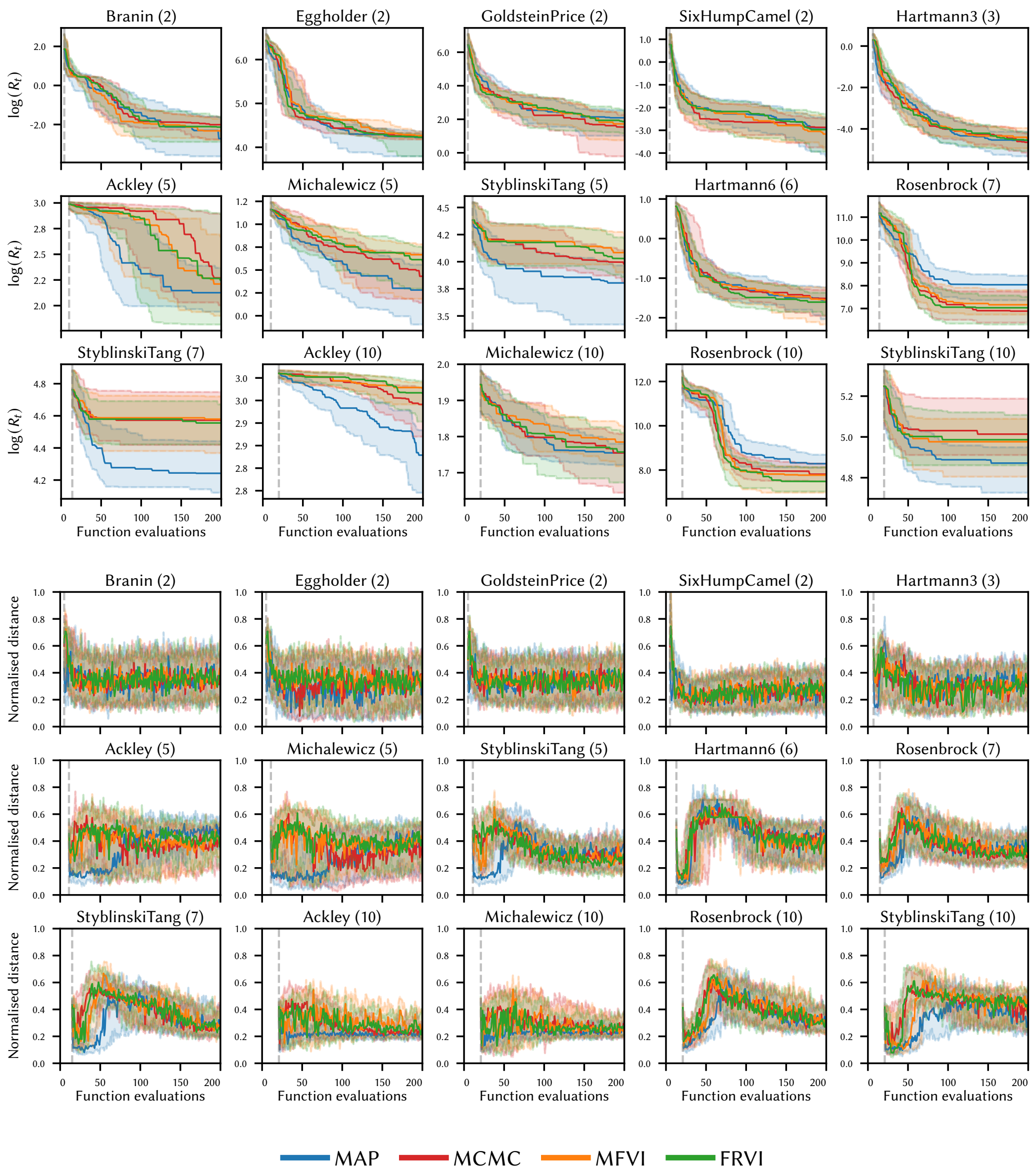

Fig. 6. Convergence (upper) and distance (lower) plots for the EI acquisition function with an isotropic kernel on the $\sigma_n = 0.1$ problems.

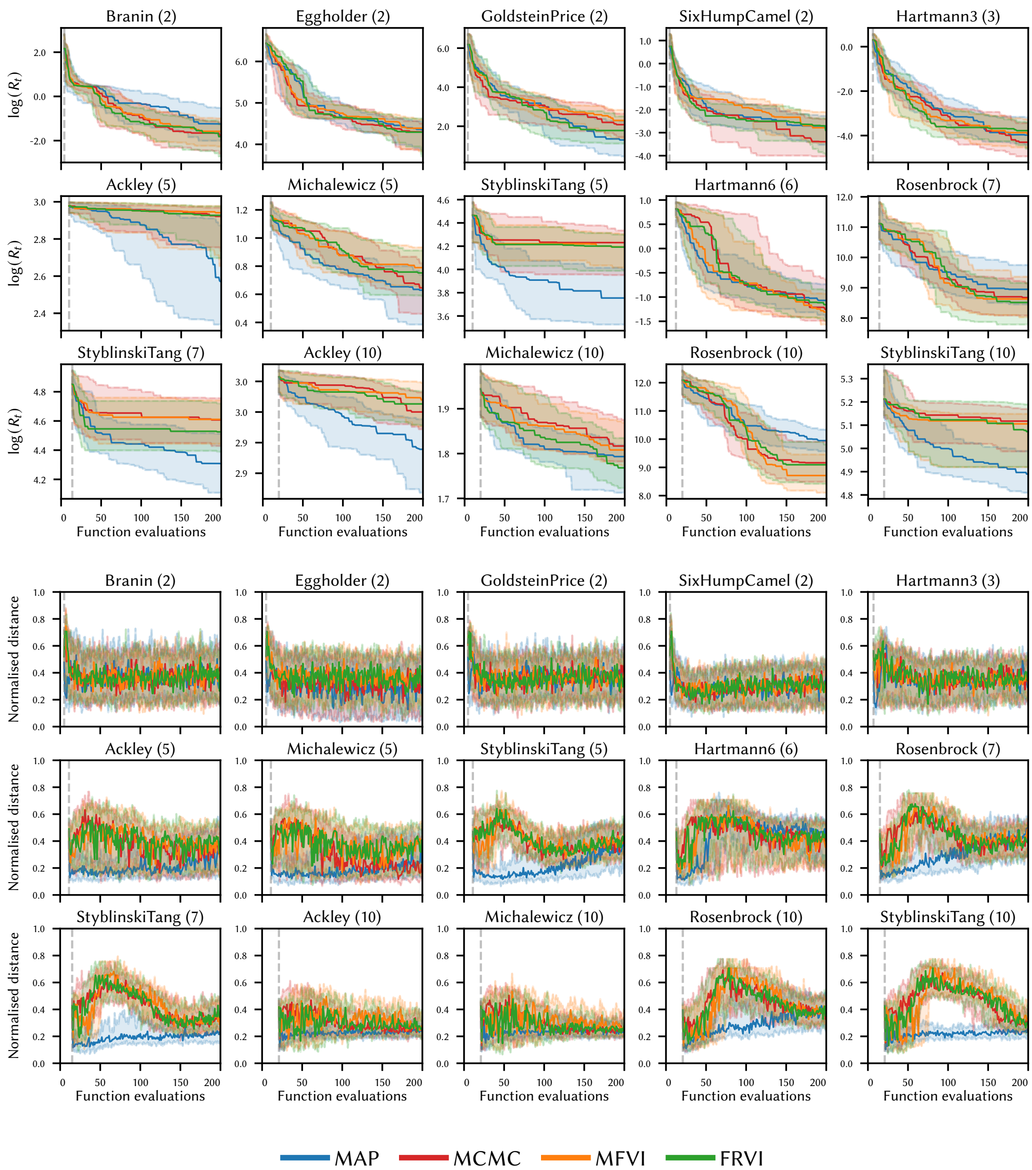

Fig. 7. Convergence (upper) and distance (lower) plots for the EI acquisition function with an isotropic kernel on the $\sigma_n = 0.2$ problems.

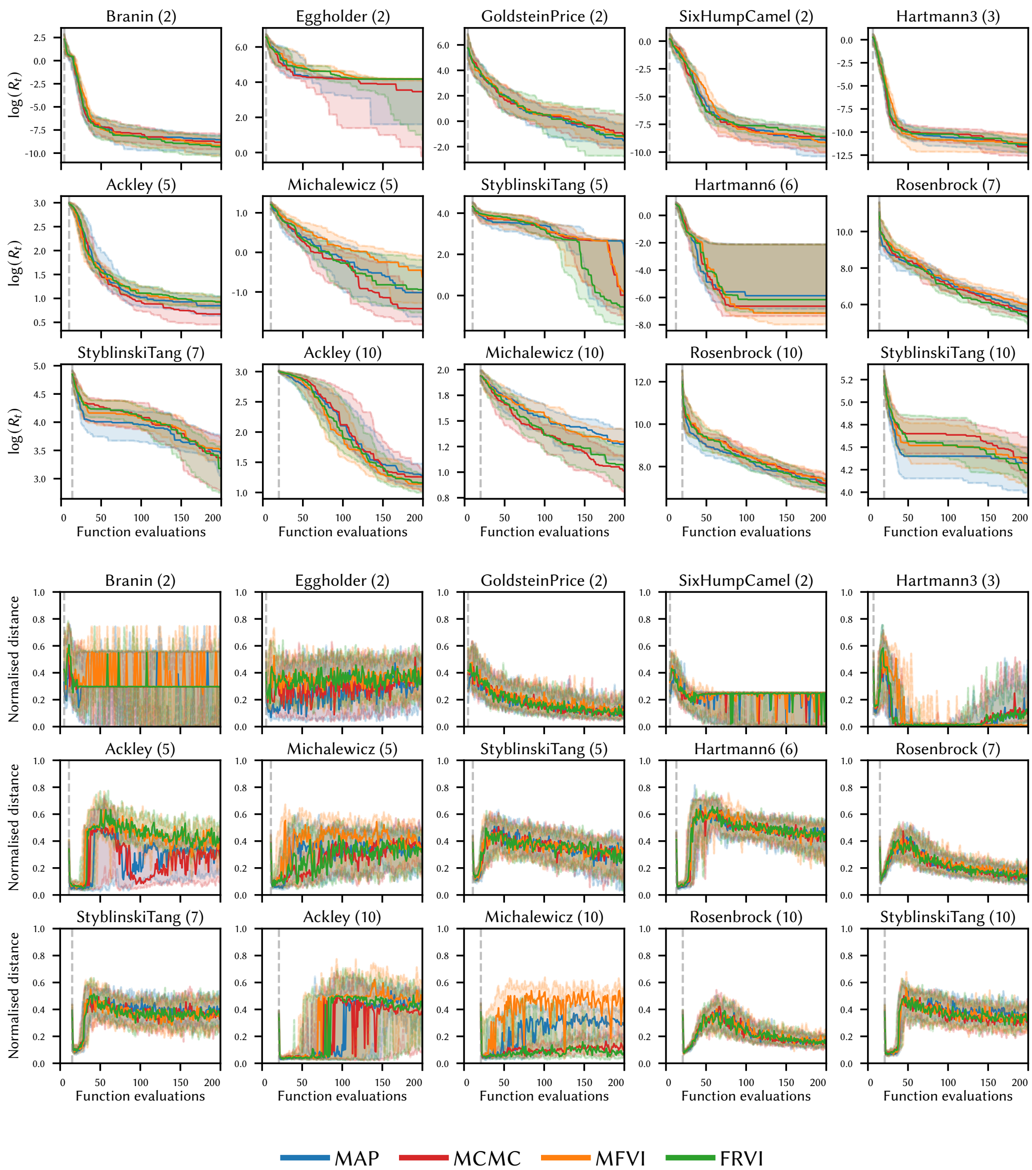

Fig. 8. Convergence (upper) and distance (lower) plots for the EI acquisition function with an ARD kernel on the noise-free problems.

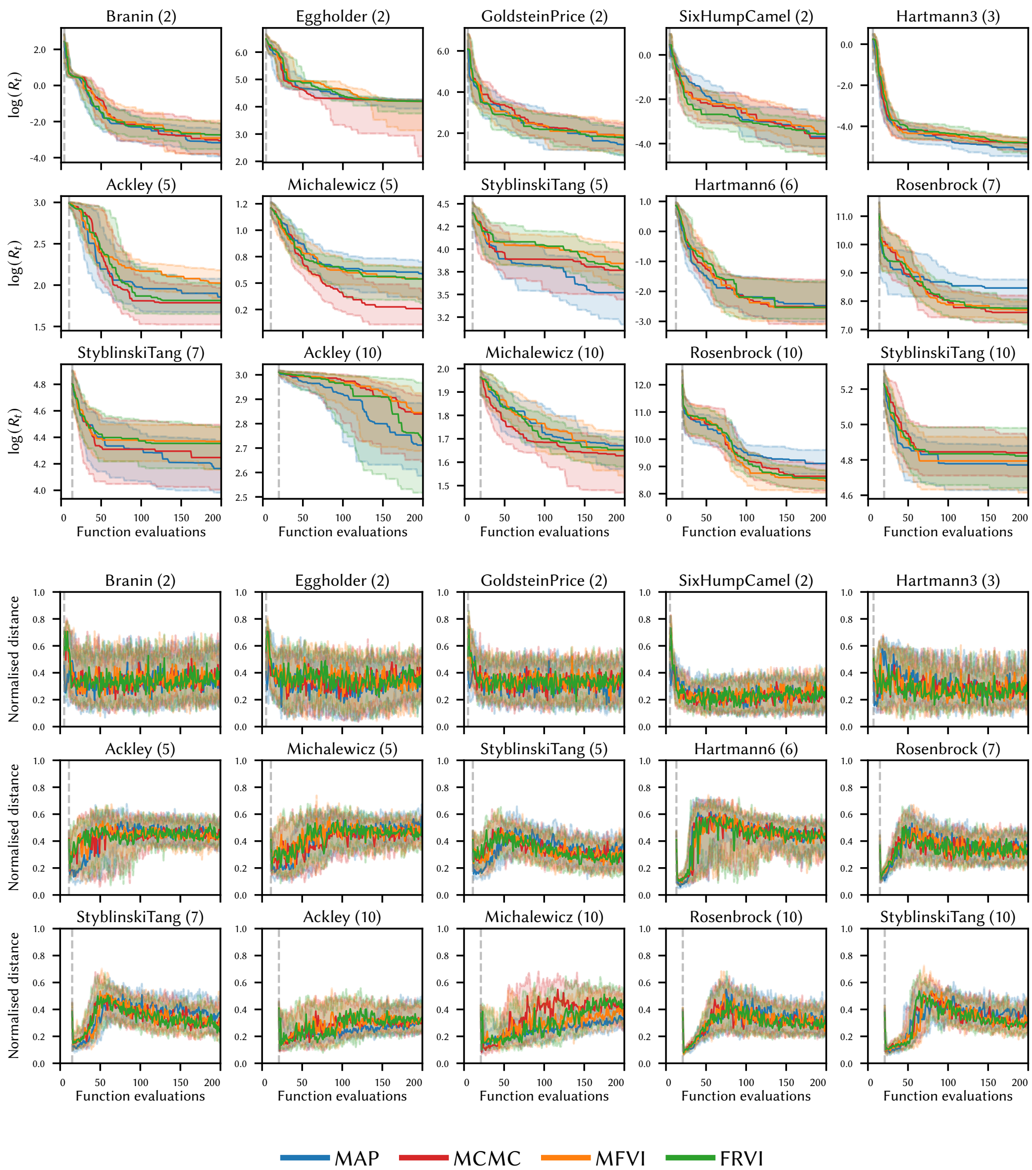

Fig. 9. Convergence (upper) and distance (lower) plots for the EI acquisition function with an ARD kernel on the $\sigma_n = 0.05$ problems.

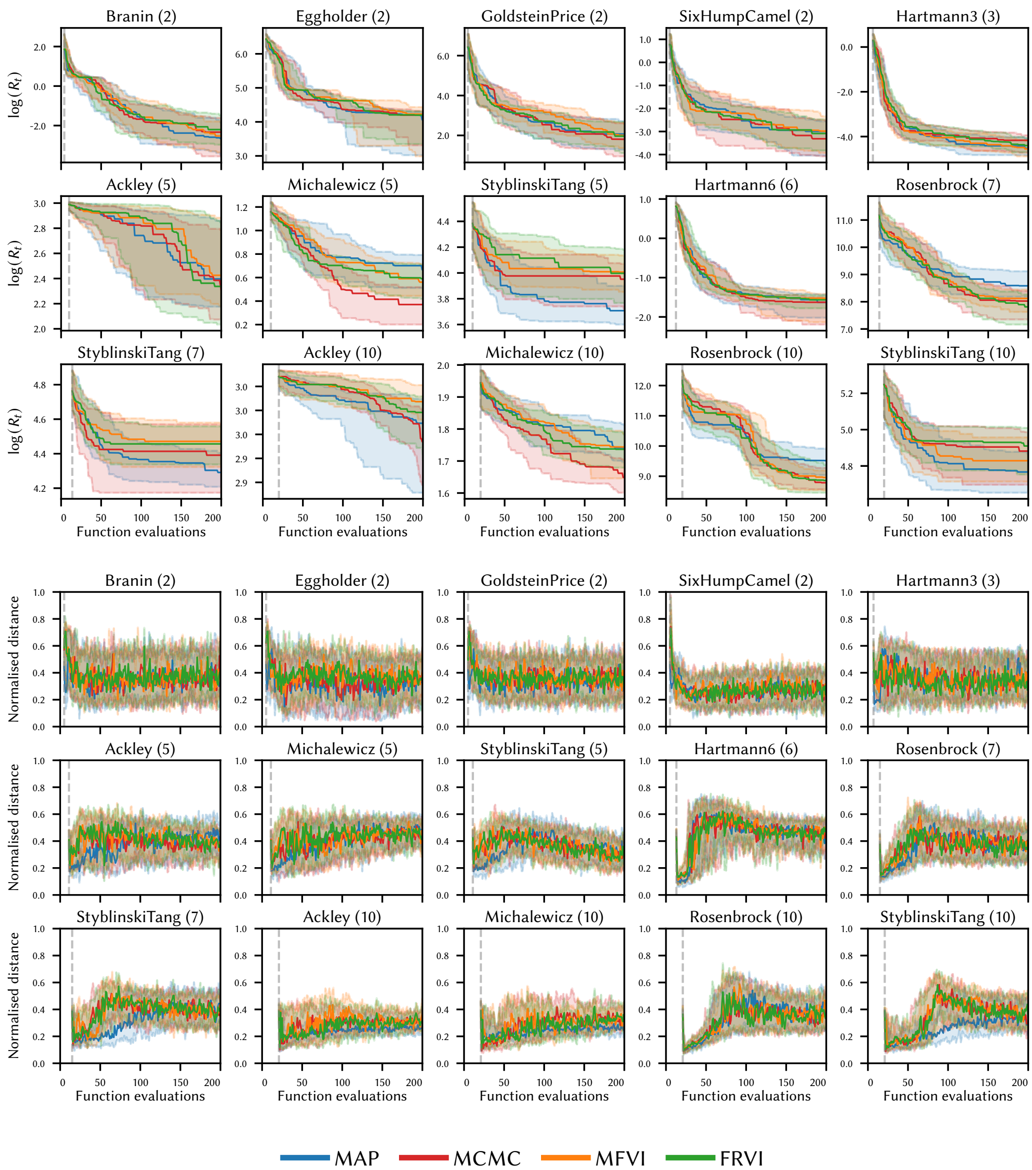

Fig. 10. Convergence (upper) and distance (lower) plots for the EI acquisition function with an ARD kernel on the $\sigma_n = 0.1$ problems.

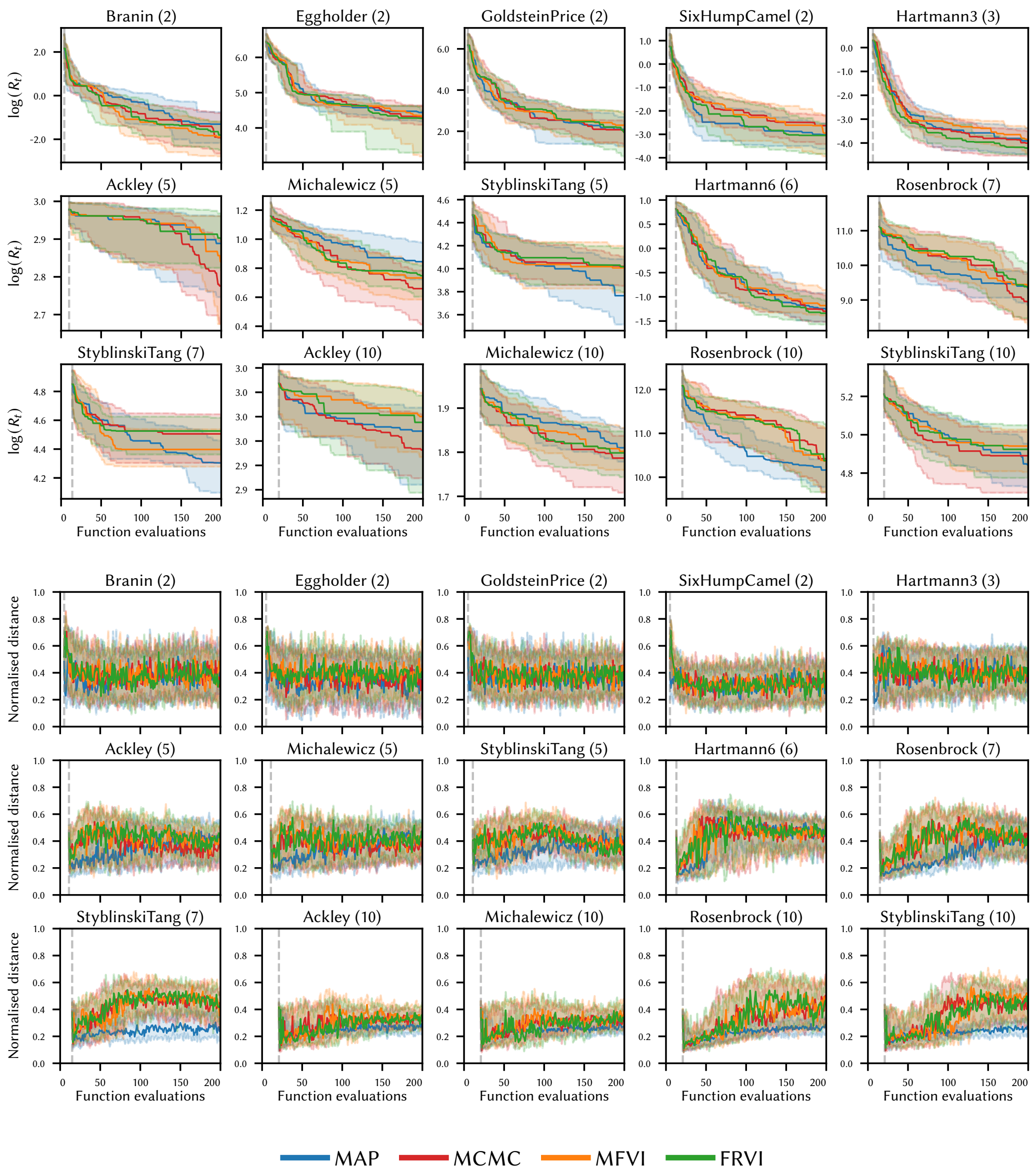

Fig. 11. Convergence (upper) and distance (lower) plots for the EI acquisition function with an ARD kernel on the $\sigma_n = 0.2$ problems.

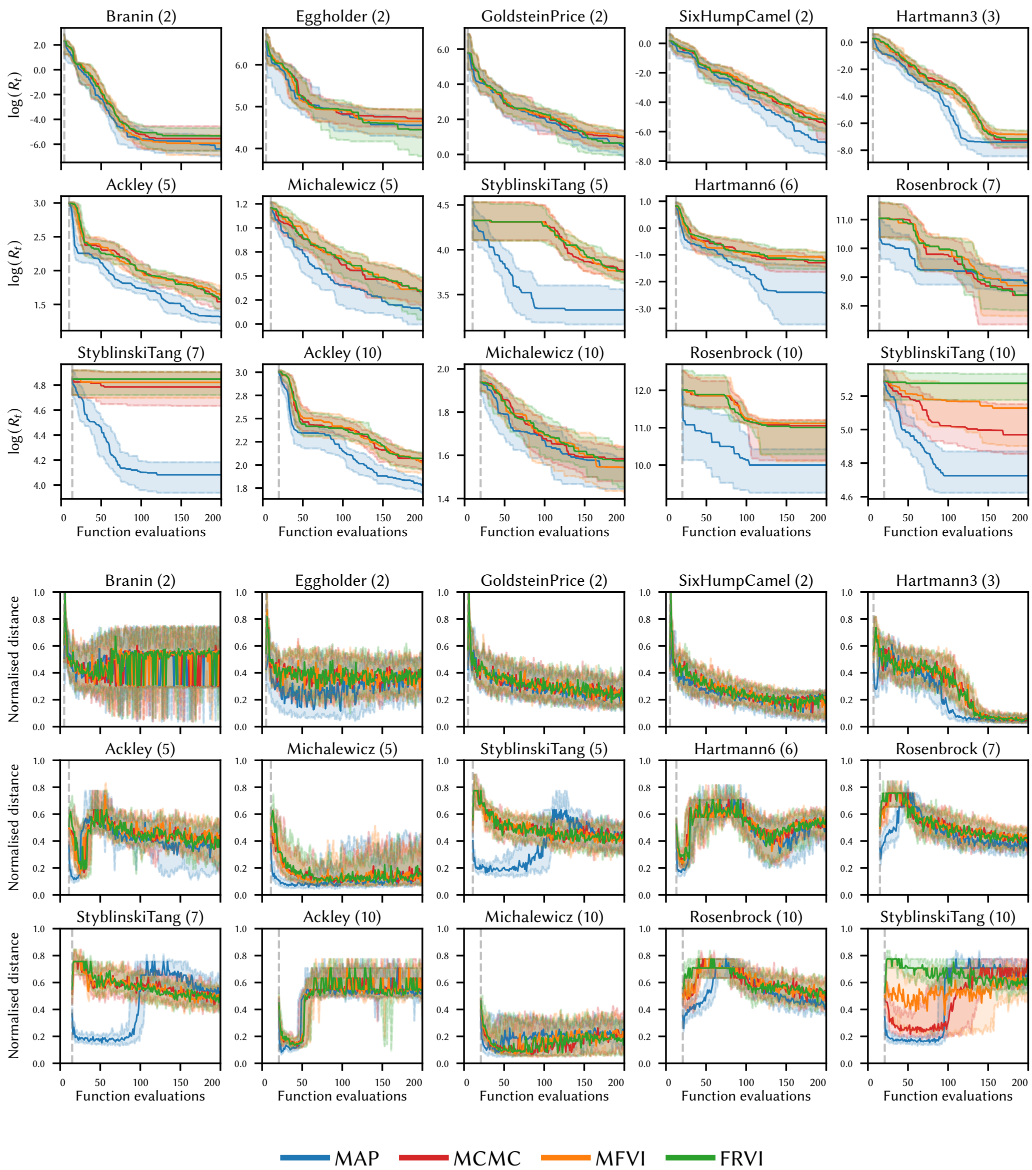

Fig. 12. Convergence (upper) and distance (lower) plots for the UCB acquisition function with an isotropic kernel on the noise-free problems.

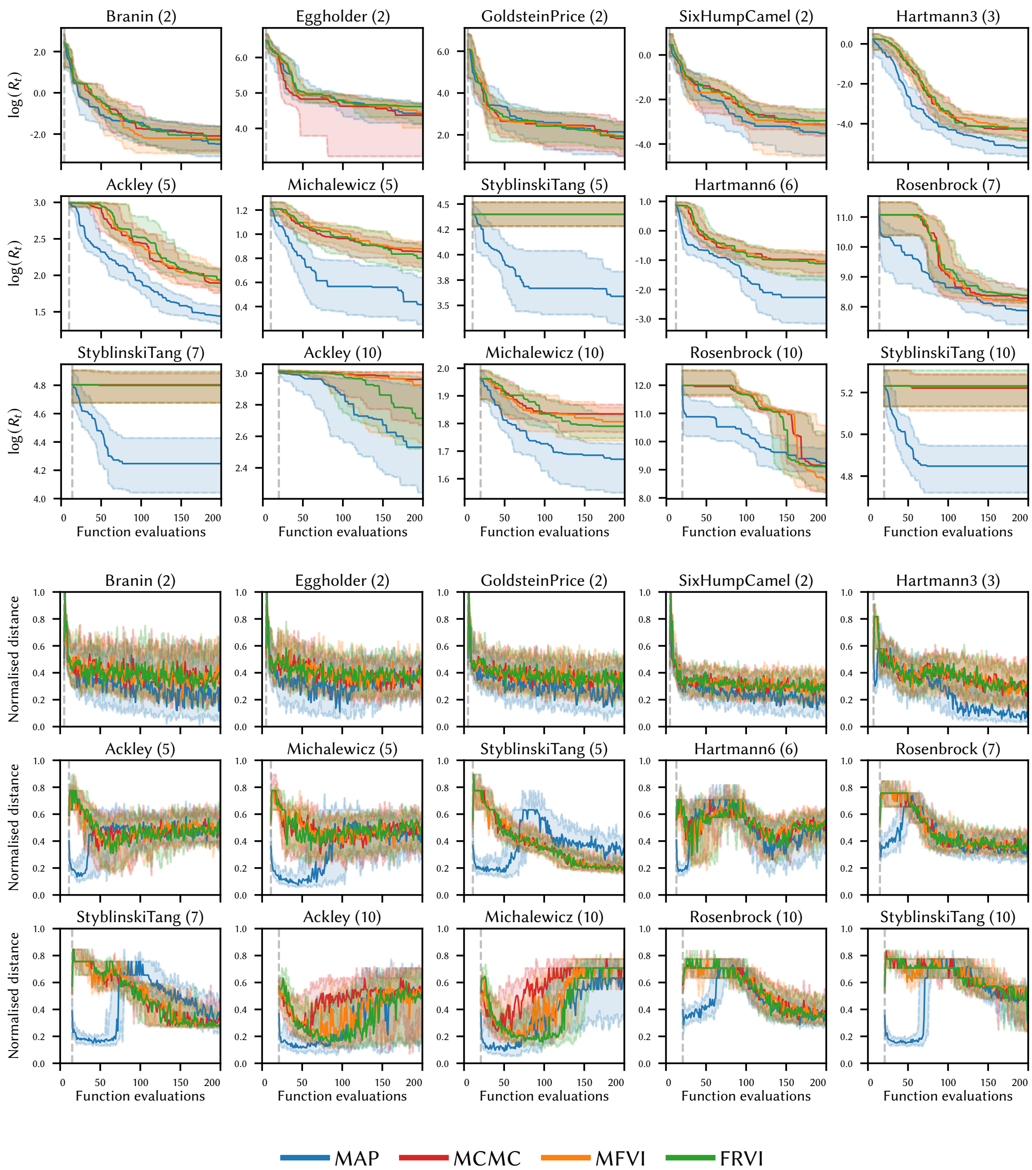

Fig. 13. Convergence (upper) and distance (lower) plots for the UCB acquisition function with an isotropic kernel on the $\sigma_n = 0.05$ problems.

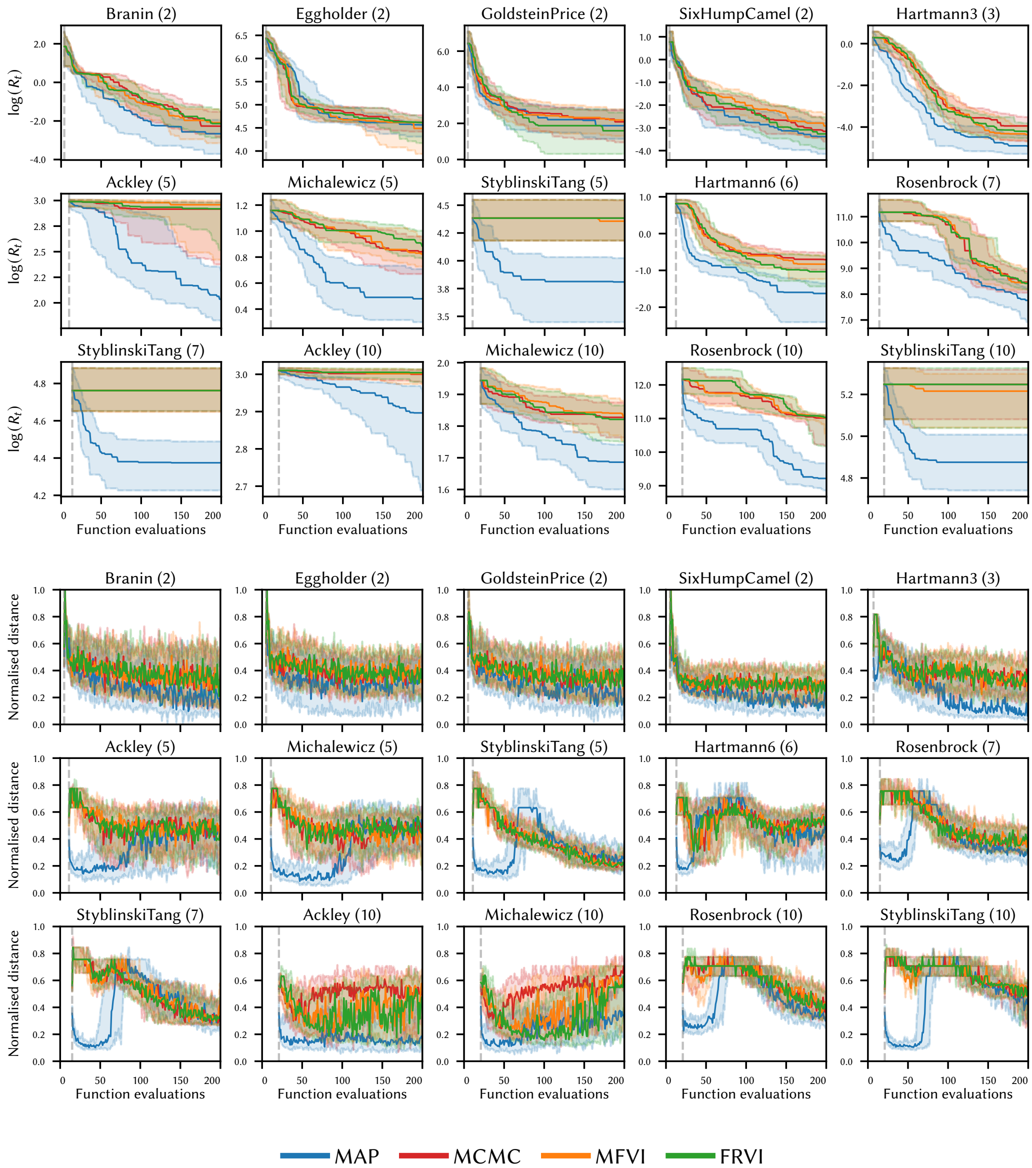

Fig. 14. Convergence (upper) and distance (lower) plots for the UCB acquisition function with an isotropic kernel on the $\sigma_n = 0.1$ problems.

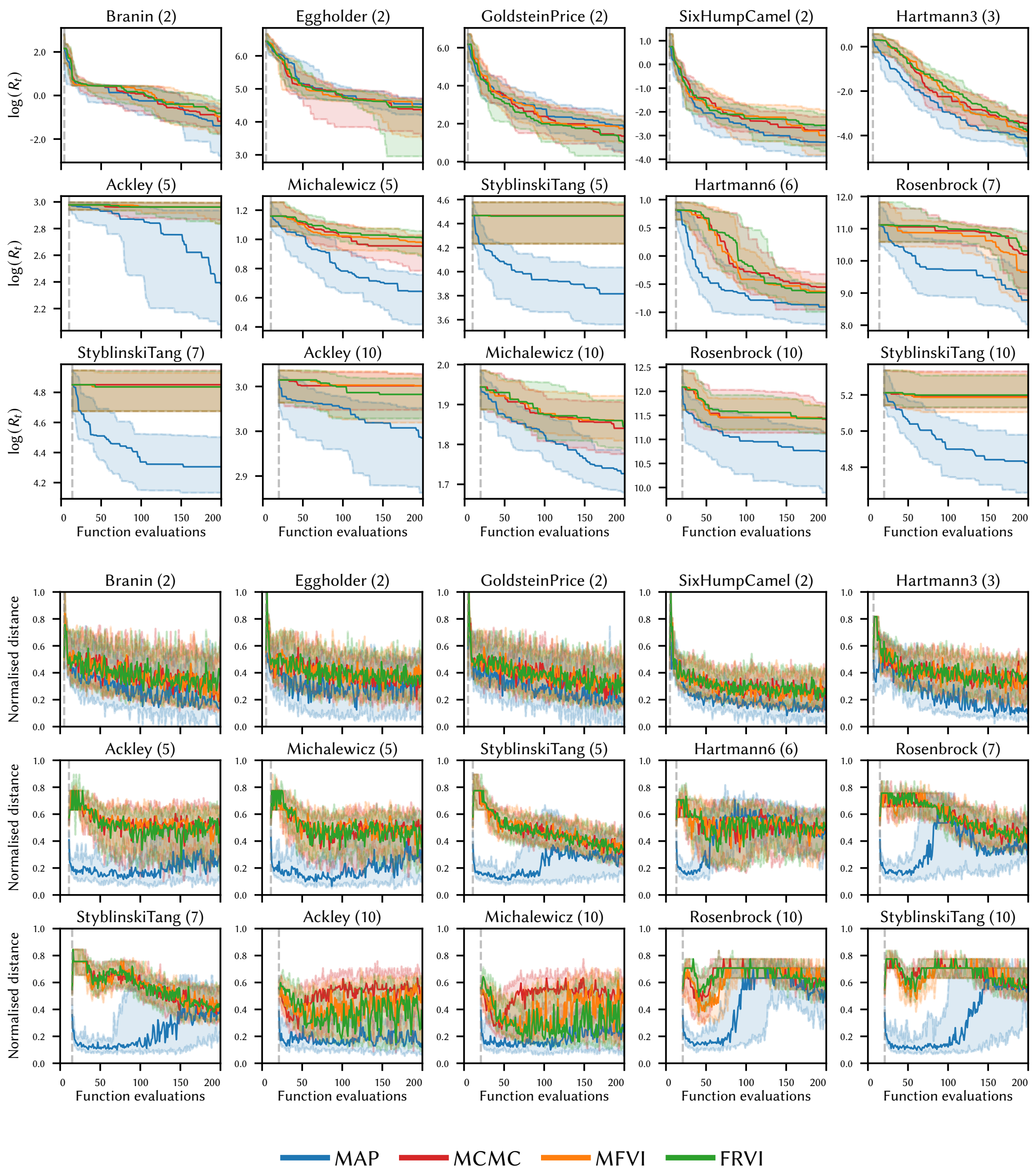

Fig. 15. Convergence (upper) and distance (lower) plots for the UCB acquisition function with an isotropic kernel on the $\sigma_n = 0.2$ problems.

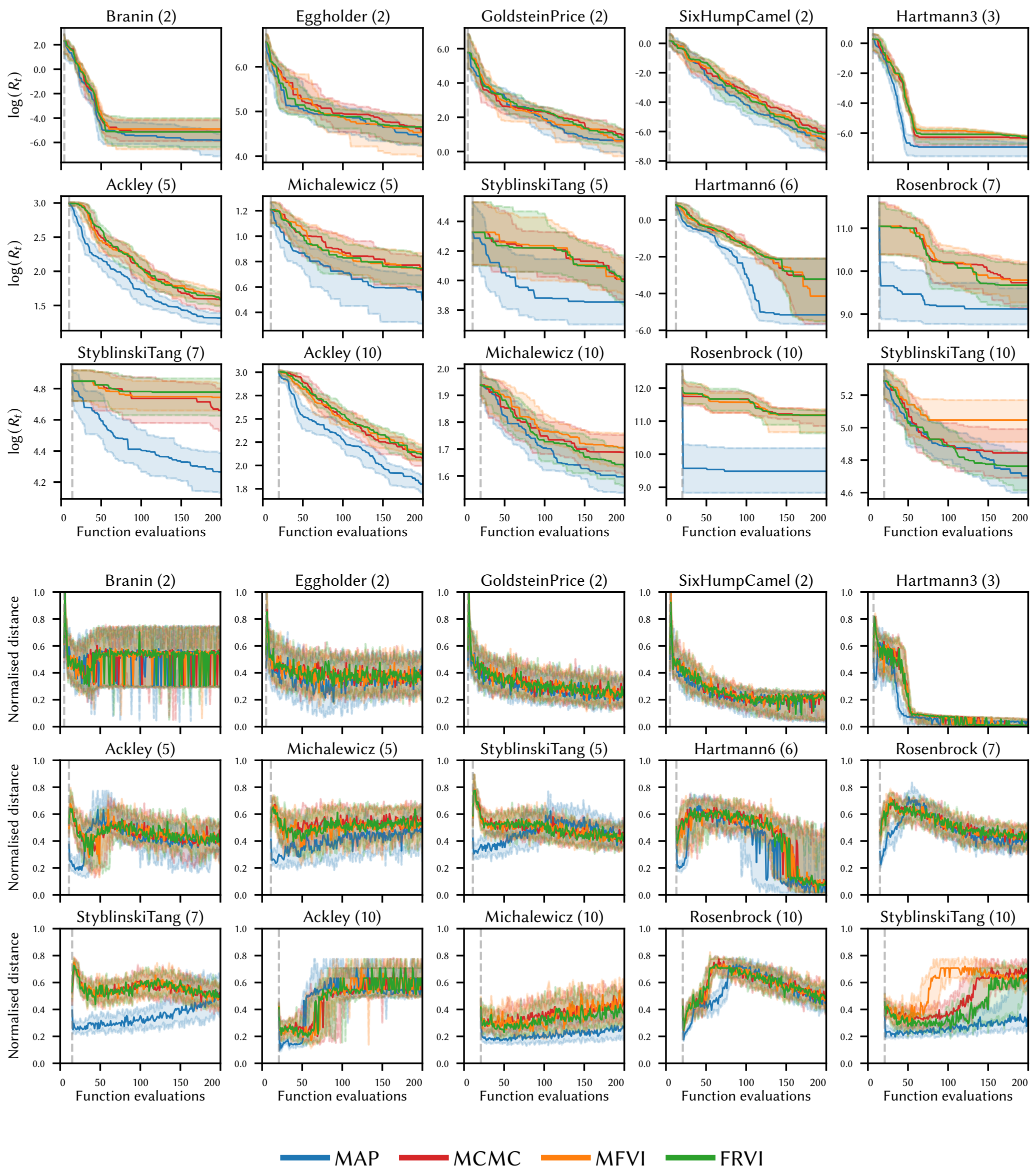

Fig. 16. Convergence (upper) and distance (lower) plots for the UCB acquisition function with an ARD kernel on the noise-free problems.

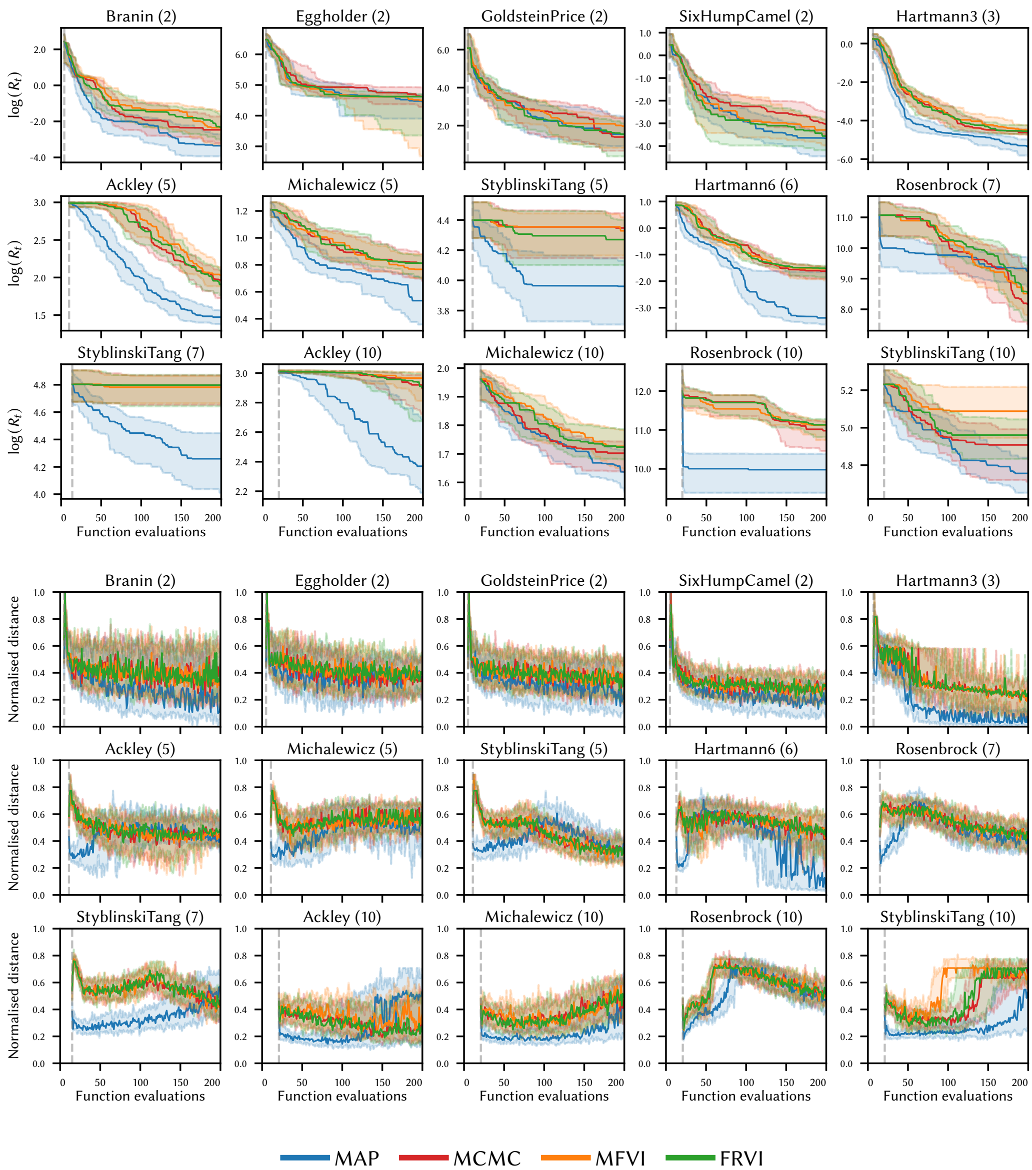

Fig. 17. Convergence (upper) and distance (lower) plots for the UCB acquisition function with an ARD kernel on the $\sigma_n = 0.05$ problems.

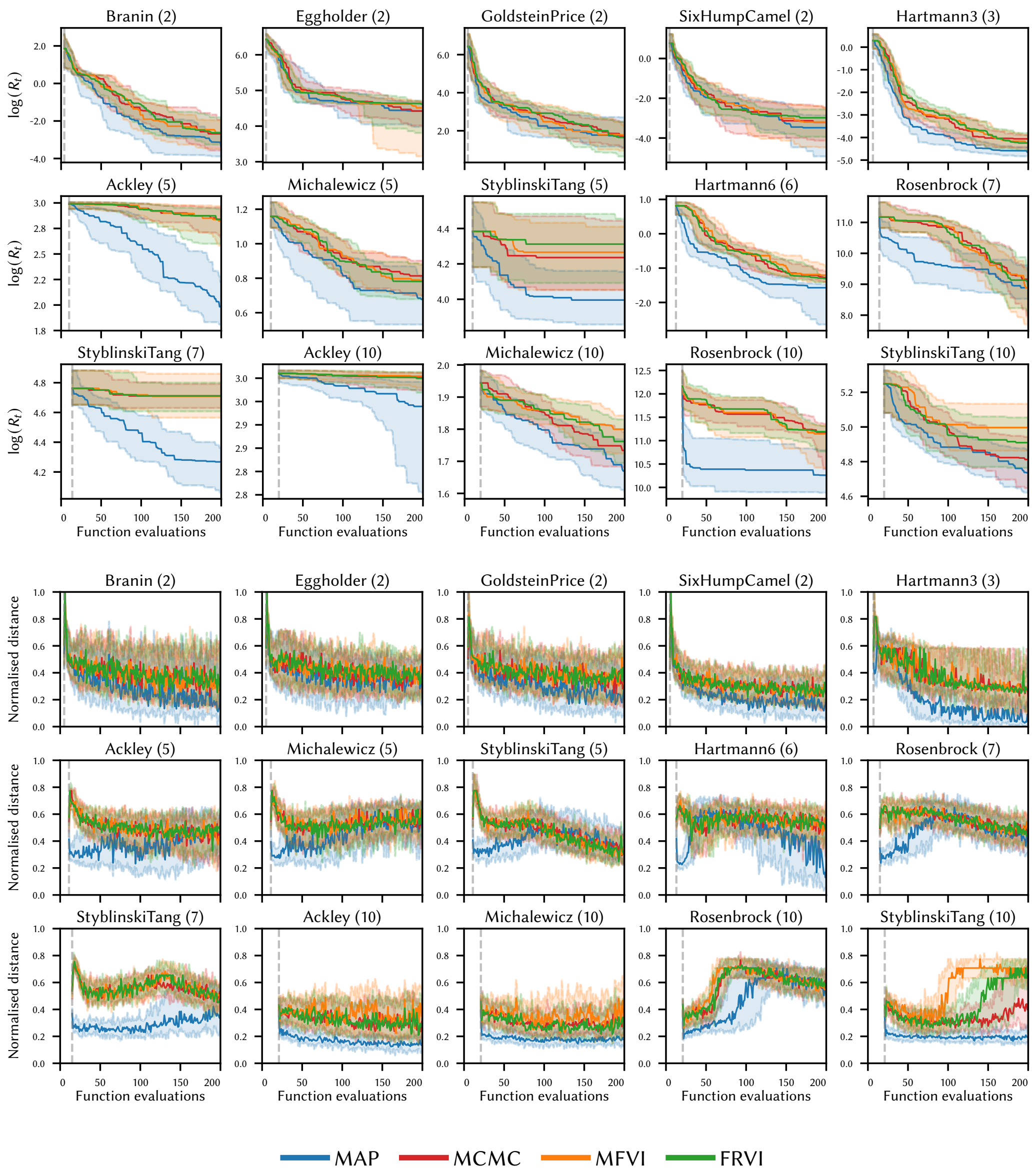

Fig. 18. Convergence (upper) and distance (lower) plots for the UCB acquisition function with an ARD kernel on the $\sigma_n = 0.1$ problems.

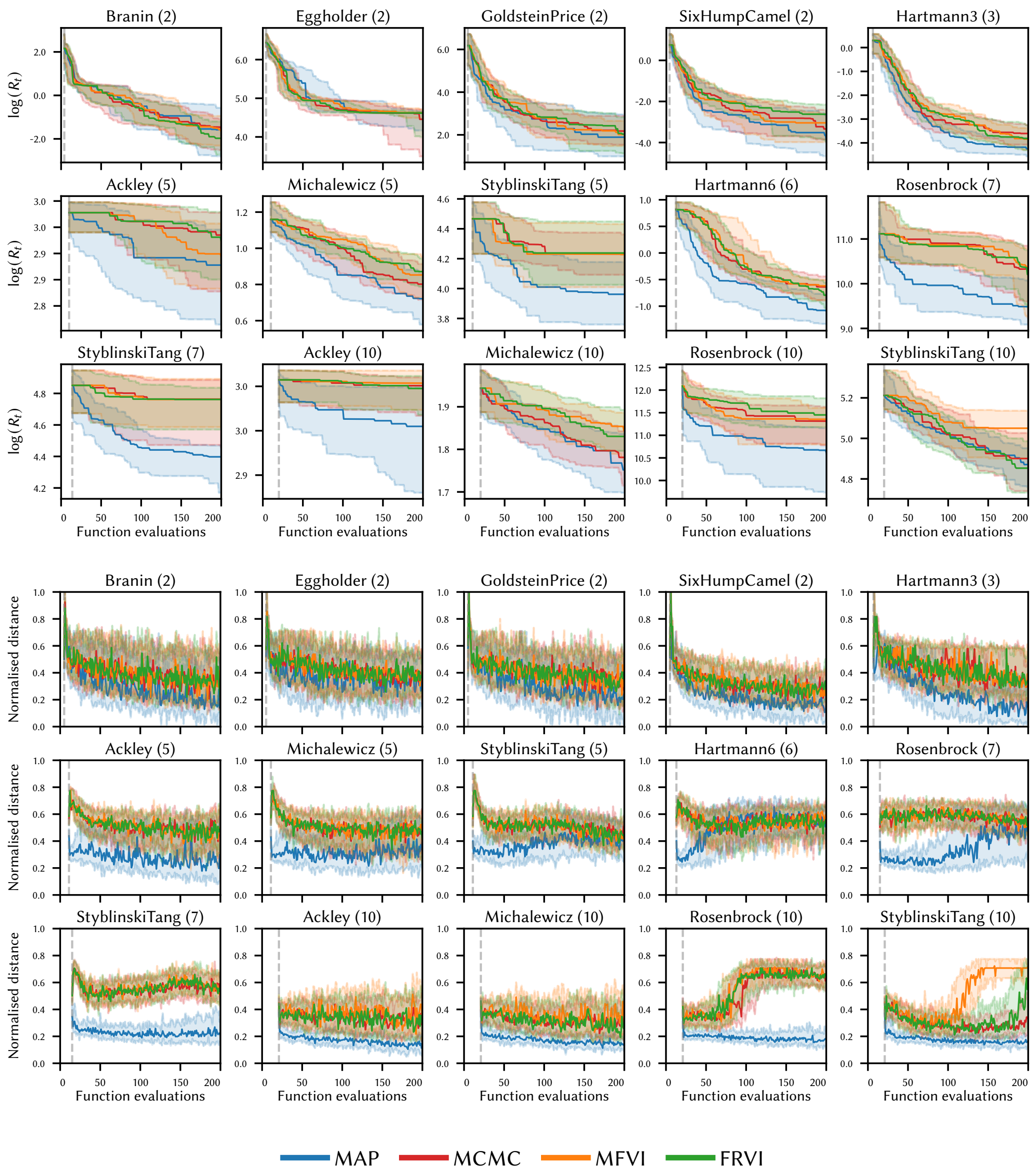

Fig. 19. Convergence (upper) and distance (lower) plots for the UCB acquisition function with an ARD kernel on the $\sigma_n = 0.2$ problems.

REFERENCES

[1] Sture Holm. 1979. A simple sequentially rejective multiple test procedure. *Scandinavian Journal of Statistics* 6, 2 (1979), 65–70.

[2] Joshua D. Knowles, Lothar Thiele, and Eckart Zitzler. 2006. *A Tutorial on the Performance Assessment of Stochastic Multiobjective Optimizers*. Technical Report TIK214. Computer Engineering and Networks Laboratory, ETH Zurich, Zurich, Switzerland.