An explicit algorithm for solving the acoustic tomography problem for a moving fluid

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- A moving fluid in a bounded domain $D \subset \mathbb{R}^d$, $d \geq 2$, is characterized by sound speed $c = c(x)$, density $\rho = \rho(x)$, velocity $\mathbf{v} = \mathbf{v}(x)$ and absorption $\alpha = \omega^{\zeta(x)}\alpha_0(x)$
- There are acoustic transducers on ∂D. A transducer produces time-harmonic acoustic waves which are scattered by the fluid. Scattered acoustic waves are recorded by other transducers.

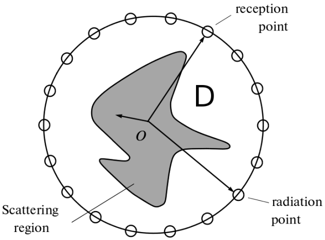


image: (Burov et al. '13)

Acoustic tomography problem. Given this data, recover fluid parameters.

Main applications in ocean tomography (*determine the ocean temperature and heat transferring currents*) and in medical diagnostics (*determine scalar inhomogeneities and the blood flow*)

$$L_\omega = -\Delta - 2i\left(\frac{\omega \mathbf{v}}{c^2} + \frac{i}{2}\nabla \ln \rho\right)\cdot\nabla - \frac{\omega^2}{c^2} - 2i\omega\frac{\alpha}{c} \qquad \text{(AC)}$$

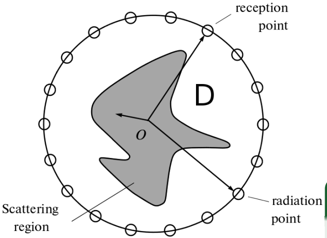


image: (Burov et al. '13)

Data from point sources: $G_\omega|_{X\times Y}$, $\omega \in \Omega$, where $X, Y \subset \partial D$, $\Omega \subset \mathbb{R}_{\geq 0}$,

$$\begin{cases} L_\omega G_\omega(x,y) = -\delta_y(x), & x \in \mathbb{R}^d, \\ G_\omega(\cdot, y) \text{ radiates at } \infty \end{cases}$$

Acoustic tomography problem

Given $G_\omega|_{X\times Y}$ for $\omega \in \Omega$ and c_0, find c, $\mathbf{v}$, $\nabla\rho$ and α in D

$$\boxed{\text{data from point sources}} \xRightarrow{\text{acoustic tomography problem}} \boxed{\text{fluid parameters}}$$

Mathematical framework

- We consider the following operator with smooth coefficients:

$$L_{A,Q} = -\sum_{j=1}^{d}\left(\frac{\partial}{\partial x_j} + iA_j(x)\right)^2 + Q(x), \qquad \text{(OP)}$$

where $x = (x_1, \ldots, x_d) \in D$,

$$A = (A_1, \ldots, A_d), \quad A_j(x) \in M_n(\mathbb{C}), \quad Q(x) \in M_n(\mathbb{C}),$$
$$D \textit{ is an open bounded domain in } \mathbb{R}^d \textit{ with boundary } \partial D$$

- $L_{A,Q}$ acts on $\mathbb{C}^n$-valued functions in D

$$L_{A,Q} = -\sum_{j=1}^{d}\left(\frac{\partial}{\partial x_j} + iA_j(x)\right)^2 + Q(x), \qquad \text{(OP)}$$

- Suppose that $E \in \mathbb{C}$ is not a DE for $L_{A,Q}$ in D:

$$\begin{cases} L_{A,Q}\psi = E\psi & \textit{in } D, \\ \psi|_{\partial D} = f, \end{cases}$$

  is uniquely solvable for any sufficiently regular f on ∂D.

- The Dirichlet-to-Neumann map $\Lambda_{A,Q} = \Lambda_{A,Q}(E)$:

$$\Lambda_{A,Q} f = \sum_{j=1}^{d} \nu_j\left(\tfrac{\partial}{\partial x_j} + iA_j\right)\psi\Big|_{\partial D}, \qquad \text{(DN)}$$

  where $\nu = (\nu_1, \ldots, \nu_d)$ is the unit exterior normal to ∂D.

Mathematical framework

$$L_{A,Q} = -\sum_{j=1}^{d}\left(\frac{\partial}{\partial x_j} + iA_j(x)\right)^2 + Q(x), \qquad \text{(OP)}$$

$$\Lambda_{A,Q} f = \sum_{j=1}^{d} \nu_j \left(\tfrac{\partial}{\partial x_j} + iA_j\right)\psi\big|_{\partial D}, \qquad \text{(DN)}$$

- Conjugation of $L_{A,Q}$ by a smooth $GL_n(\mathbb{C})$-valued function g:

$$\begin{cases} gL_{A,Q}g^{-1} = L_{A^g,Q^g}, \\ A_j^g = gA_jg^{-1} + i\frac{\partial g}{\partial x_j}g^{-1}, \quad j = 1,\dots,d, \\ Q^g = gQg^{-1}. \end{cases} \qquad \text{(GT)}$$

- The following formula holds:

$$\Lambda_{A^g,Q^g} = g|_{\partial D}\Lambda_{A,Q}(g|_{\partial D})^{-1}.$$

$$L_{A,Q} = -\sum_{j=1}^{d}\left(\frac{\partial}{\partial x_j} + iA_j\right)^2 + Q, \qquad \text{(OP)}$$

$$\Lambda_{A,Q}(\psi|_{\partial D}) = \sum_{j=1}^{d} \nu_j\left(\frac{\partial}{\partial x_j} + iA_j\right)\psi\Big|_{\partial D},\ L_{A,Q}\psi = E\psi, \qquad \text{(DN)}$$

$$\begin{cases} gL_{A,Q}g^{-1} = L_{A^g,Q^g}, \\ \Lambda_{A^g,Q^g} = \Lambda_{A,Q}, \\ g \text{ is smooth } GL_n(\mathbb{C})\text{-valued, } g|_{\partial D} = \mathrm{Id} \end{cases} \qquad \text{(GT)}$$

The inverse Dirichlet-to-Neumann problem

Given $\Lambda_{A,Q}$ at fixed E, find $L_{A,Q}$ modulo (GT).

$$\boxed{\text{D-to-N map}} \xrightarrow[\text{problem}]{\text{inverse Dirichlet-to-Neumann}} \boxed{\text{conjugacy class of Schrödinger operators}}$$

The IDN problem: scalar case

- A_j, Q are scalar functions, $d \in \{2, 3\}$, $A = (A_1, \dots, A_d)$

$$L_{A,Q} = -\left(\nabla + iA\right)^2 + Q, \tag{OP}$$

$$\Lambda_{A,Q} f = \left(\nu \cdot (\nabla + iA)\right)\psi\big|_{\partial D}, \tag{DN}$$

$$\begin{cases} e^{i\varphi} L_{A,Q} e^{-i\varphi} = L_{A^\varphi, Q^\varphi}, \\ A^\varphi = A + \nabla\varphi, \\ Q^\varphi = Q \end{cases} \tag{GT}$$

- $F = \operatorname{curl} A$ and Q are gauge invariant and are uniquely determined by $\Lambda_{A,V}(E)$, see [10] ($d \geq 3$) and [9] ($d = 2$)
- $(A - (A \cdot \nu)\nu)|_{\partial D}$ is uniquely determined by $\Lambda_{A,V}(E)$, see [6]

Acoustic scattering: reduction to the IDN problem

Use the second Green formula (Nachman '88):

$$G_\omega(x,y) - G_\omega^0(x,y) = \int\limits_{\partial D}\int\limits_{\partial D} G_\omega^0(x,z)(\Lambda_\omega - \Lambda_\omega^0)(z,w)G_\omega(w,y)\,dy\,dw$$

where G_ω^0, Λ_ω^0 correspond to $\mathbf{v} = 0$, $\nabla\rho = 0$, $c = c_0$, $\alpha = 0$.

$$\begin{array}{ccc} \boxed{\text{data from point sources}} & \xrightarrow{\text{acoustic tomography problem}} & \boxed{\text{fluid parameters}} \\ \Big\downarrow{\scriptstyle \text{second Green formula}} & & \\ \boxed{\text{D-to-N map(s)}} & & \end{array}$$

Gauge fixing

- *Question.* Suppose that we know how to solve the IDN problem. How to complete the following diagram?

$$\begin{tikzcd}
\boxed{\text{data from point sources}} \arrow[r, Rightarrow, "\text{acoustic tomography problem}"] \arrow[d, "\text{second Green formula}"] & \boxed{\text{fluid parameters}} \\
\boxed{\text{D-to-N map(s)}} \arrow[r, "\text{inverse Dirichlet-to-Neumann problem}"] & \boxed{\text{conjugacy class(es) of Schrödinger operators}} \arrow[u, dashed, "?"]
\end{tikzcd}$$

Gauge fixing

$$L_\omega = -\Delta - 2i\left(\frac{\omega \mathbf{v}}{c^2} + \frac{i}{2}\nabla \ln \rho\right)\cdot\nabla - \frac{\omega^2}{c^2} - 2i\omega\frac{\alpha}{c} \qquad \text{(AC)}$$

- functions F and q_ω are invariants of the conjugacy class:

$$\begin{aligned} F &= \operatorname{curl}\tfrac{\mathbf{v}}{c^2}, \\ q_\omega &= f_1 - \omega^2 f_2 + i\omega f_3 - 2i\omega^{1+\zeta}\alpha_0, \end{aligned}$$

$$f_1 = \rho^{\frac{1}{2}}\Delta\rho^{-\frac{1}{2}}, \quad f_2 = \tfrac{1}{c^2} + \tfrac{\mathbf{v}}{c^2}\tfrac{\mathbf{v}}{c^2}, \quad f_3 = \nabla\cdot\left(\tfrac{\mathbf{v}}{c^2}\right) - \tfrac{\mathbf{v}}{c^2}\cdot\nabla\ln\rho$$

- The fluid parameters can be recovered as follows:

$$\begin{tikzcd}
\boxed{\Lambda_{\omega_1},\dots,\Lambda_{\omega_N}} \arrow[r] \arrow[d, Rightarrow] & \boxed{\begin{array}{c}\text{conjugacy classes}\\ \text{of } L_{\omega_1},\ \dots,\ L_{\omega_N}\end{array}} \arrow[r] & \boxed{F,\ q_{\omega_1},\ \dots,\ q_{\omega_N}} \arrow[d] \\
\boxed{\mathbf{v},\ c,\ \rho,\ \zeta,\ \alpha_0} & & \boxed{F,\ f_1,\ f_2,\ f_3,\ \zeta,\ \alpha_0} \arrow[ll]
\end{tikzcd}$$

Gauge fixing: summary

$$L_\omega = -\Delta - 2i\left(\frac{\omega \mathbf{v}}{c^2} + \frac{i}{2}\nabla \ln \rho\right)\cdot\nabla - \frac{\omega^2}{c^2} - 2i\omega^{1+\zeta}\frac{\alpha_0}{c}, \qquad \text{(AC)}$$

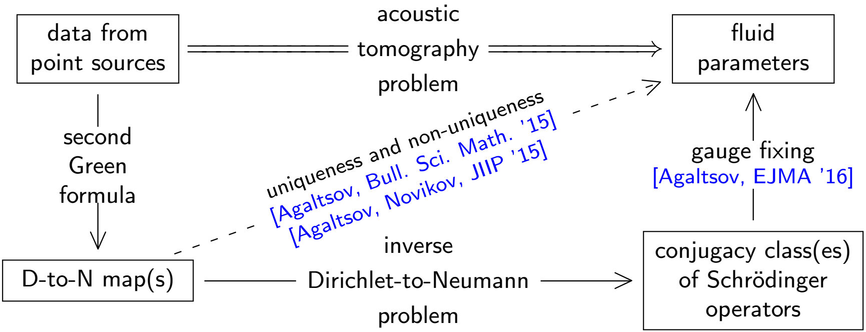
$$\Lambda_\omega(\psi|_{\partial D}) = \frac{\partial\psi}{\partial\nu}\Big|_{\partial D}, \quad L_\omega\psi = 0.$$

- $\rho \equiv \rho_0,\ \alpha_0 \equiv 0 \implies \Lambda_\omega$ at fixed ω determines $\mathbf{v}$, c
- $\alpha_0 \equiv 0 \implies \Lambda_\omega$ at 2 ω's determines $\mathbf{v}$, c, ρ
- $\zeta \neq 0 \implies \Lambda_\omega$ at 3 ω's determines $\mathbf{v}$, c, ρ, ζ, α_0
- Explicit examples of non-uniquenes when $\zeta \equiv 0$

[Agaltsov, Bull. Sci. Math. '15]: uniqueness
[Agaltsov, Novikov, JIIP '15] : uniqueness and invisible fluids
[Agaltsov, EJMA '16]: algorithms

Solving the acoustic tomography problem

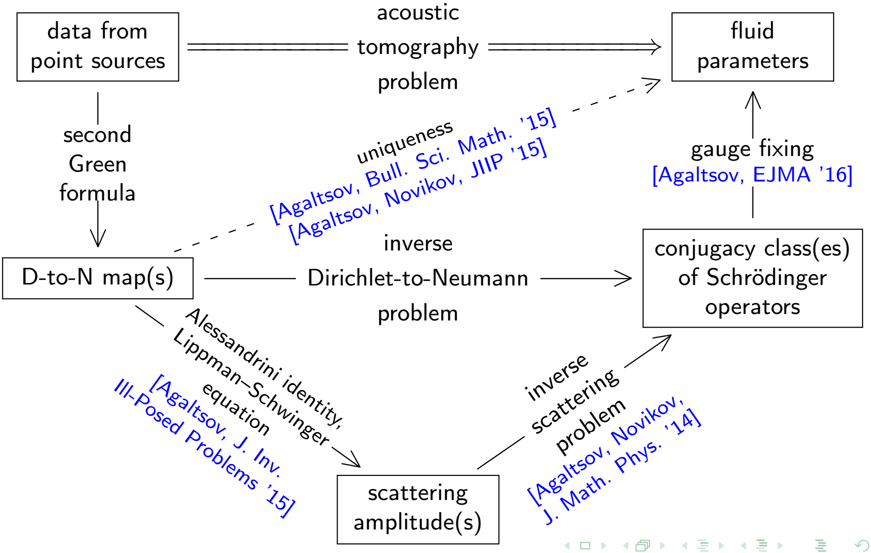
- So far we have the following scheme with vertical arrows being explicit algorithms:



- *Question.* How to solve constructively the inverse Dirichlet-to-Neumann problem?

Solving the acoustic tomography problem

A common project with Moscow University Acoustical Physics group (Burov et al.)



The inverse scattering problem

$$L_{A,Q} = -\sum_{j=1}^{d}\left(\frac{\partial}{\partial x_j} + iA_j\right)^2 + Q, \tag{OP}$$

A_j, Q are smooth $M_n(\mathbb{C})$-valued in D

- Set A, Q equal to zero outside of D
- Consider functions $\psi^+(\cdot, k)$, $k \in S^{d-1}_{\sqrt{E}} = \{\kappa \in \mathbb{R}^d \mid \kappa^2 = E\}$:

$$\begin{cases} L_{A,Q}\psi^+(x,k) = E\psi^+(x,k), & x \in \mathbb{R}^d, \\ \psi^+(x,k) = e^{ikx}\mathrm{Id}_n + \psi^+_{\mathrm{sc}}(x,k), \\ \psi^+_{\mathrm{sc}} \text{ radiates at } \infty \end{cases}$$

- The scattering amplitude $f_{A,Q}$ on $\mathcal{M}_E = S^{d-1}_{\sqrt{E}} \times S^{d-1}_{\sqrt{E}}$:

$$\psi^+_{\mathrm{sc}}(x,k) = \begin{pmatrix}\text{universal}\\ \text{spherical}\\ \text{wave}\end{pmatrix} \cdot f_{A,Q}\left(k, \frac{|k|}{|x|}x\right)\left(1 + o(1)\right), \quad |x| \to \infty.$$

Direct scattering problem

$$L_{A,Q} = -\sum_{j=1}^{d}\left(\frac{\partial}{\partial x_j} + iA_j\right)^2 + Q \qquad \text{(OP)}$$

Direct scattering problem

Given $L_{A,Q}$, find $f_{A,Q}$.

- $\psi^+(\cdot, k)$ satisfies the Lippmann-Schwinger equation:

$$\psi^+(x,k) = e^{ikx}\mathrm{Id}_n + \int_D G^+(x-y,k)\big(L_{A,Q} - L_{0,0}\big)\psi^+(y,k)\,dy, \quad \text{(LS)}$$

$$G^+(x,k) = -(2\pi)^{-d}\int_{\mathbb{R}^d}\frac{e^{i\xi x}\,d\xi}{\xi^2 - k^2 - i0} \text{``}\simeq\text{''} \frac{1}{E - L_{0,0}}$$

- The scattering amplitude $f_{A,Q}$ can be found from:

$$f_{A,Q}(k,l) = (2\pi)^{-d}\int_{\mathbb{R}^d} e^{-ilx}\big(L_{A,Q} - L_{0,0}\big)\psi^+(x,k)\,dx \quad \text{(SA)}$$

Gauge invariance

$$L_{A,Q} = -\sum_{j=1}^{d} \left(\frac{\partial}{\partial x_j} + iA_j\right)^2 + Q, \qquad \text{(OP)}$$

$$f_{A,Q}(k,l) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-ilx}(L_{A,Q} - L_{0,0})\psi^+(x,k)\,dx \qquad \text{(SA)}$$

- Conjugation of $L_{A,Q}$ by a $GL_n(\mathbb{C})$-valued function g:

$$\begin{cases} gL_{A,Q}g^{-1} = L_{A^g,Q^g}, \\ A_j^g = gA_jg^{-1} + i\frac{\partial g}{\partial x_j}g^{-1}, \quad j = 1,\ldots,d, \\ Q^g = gQg^{-1} \end{cases} \qquad \text{(GT)}$$

- The amplitude is gauge invariant:

$$f_{A^g,V^g}(k,l) = f_{A,V}(k,l),$$

if $g \to \mathrm{Id}$ at ∞ sufficiently fast

$$L_{A,Q} = -\sum_{j=1}^{d}\left(\frac{\partial}{\partial x_j} + iA_j\right)^2 + Q, \qquad \text{(OP)}$$

$$f_{A,Q}(k,l) = (2\pi)^{-d}\int_{\mathbb{R}^d} e^{-ilx}(L_{A,Q} - L_{0,0})\psi^+(x,k)\,dx \qquad \text{(SA)}$$

$$\begin{cases} gL_{A,Q}g^{-1} = L_{A^g,Q^g}, \\ f_{A^g,Q^g} = f_{A,Q}, \\ g \text{ is smooth } GL_n(\mathbb{C})\text{-valued}, \\ g \to \mathrm{Id} \text{ at } \infty \text{ sufficiently fast} \end{cases} \qquad \text{(GT)}$$

The inverse scattering problem

Given $f_{A,Q}$ at fixed E, find $L_{A,Q}$ modulo (GT).

[Agaltsov, Novikov, J. Math. Phys. '14]: algorithm

The inverse scattering problem

- Born approx. ($A=0$, $Q \in C_c^n$), see (Faddeev, '56; R. Novikov '15)

$$\widehat{Q}(k-l) = f(k,l) + O(E^{-\frac{1}{2}}) \Longrightarrow Q = Q^B + O(E^{-\frac{n-2}{2n}})$$

- Generalized scatt. sol. $\psi(x,k) = e^{ikx}\mu(x,k)$ (Faddeev '65), where $k = k(\lambda)$, $\lambda \in \mathbb{C} \setminus S^1$ и $k \in \mathbb{C}^2 \setminus \mathbb{R}^2$, $k^2 = E$
- For $\lambda \in \mathbb{C} \setminus S^1 \cup 0$ (Grinevich, Manakov '86; R. Novikov '99)

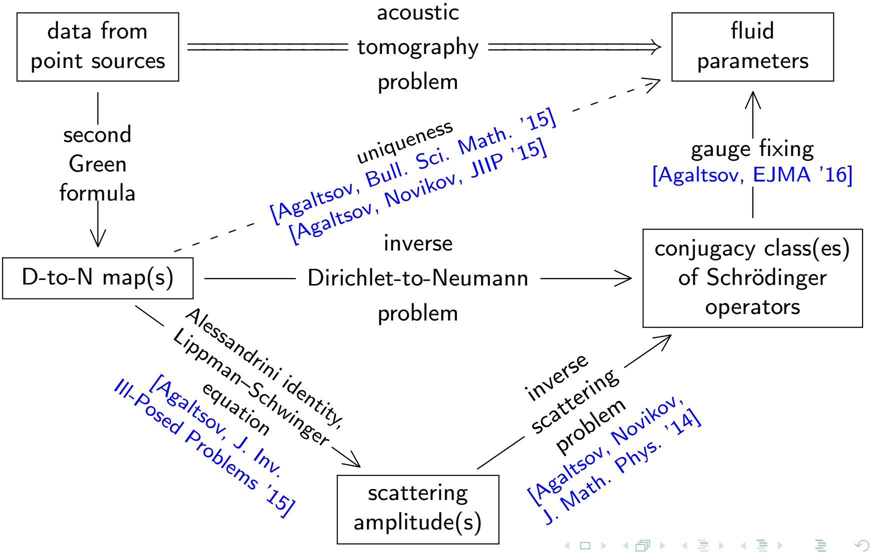
$$\overline{\partial}_\lambda \mu(x,\lambda) = r(x,\lambda)\mu(x,-\tfrac{1}{\bar\lambda}) \approx 0, \quad \text{since } r(x,\cdot) = O(E^{-\frac{n}{2}}),$$
$$\mu_+(x,\lambda) = \mu_-(x,\lambda) + \int_{S^1} \rho(x,\lambda,\lambda')\mu_-(x,\lambda')d\lambda',\ \lambda \in S^1,$$
$$\mu(x,\infty) = 1, \quad \mu(x,\cdot)|_{S^1 \pm 0} = \mu_\mp,$$

  If $r \equiv 0$, we get the nonlocal Riemann-Hilbert problem. Study of such problems goes back to (Manakov '81). Finding ρ from f (R. Novikov '86, '86). Solution for $\rho(x,\lambda,\lambda') = \overline{\rho(x,\lambda',\lambda)}$ (Grinevich, R. Novikov '85, '86)

- $\psi(x,k(\lambda)) = e^{ikx}(\mu_0^\pm + \mu_1^\pm \lambda^{\pm 1} + \cdots)$ collect coefficients of λ^0, $\lambda^{\pm 1}$ in $L_{A,Q}\psi = E\psi$, for obtaining expressions for A, Q in terms of $\mu_k^\pm$ (Grinevich, R. Novikov '85)
- Solution for $A=0$ and estimate $Q = \widetilde{Q} + O(E^{-\frac{n-2}{2}})$ (R. Novikov '98, '99). General solution [Agaltsov, Novikov, JMP '14]

Solving the acoustic tomography problem

A common project with Moscow University Acoustical Physics group (Burov et al.)



From D-to-N map to scattering amplitude

$$L_{A,Q} = -\sum_{j=1}^{d}\left(\frac{\partial}{\partial x_j} + iA_j\right)^2 + Q, \tag{OP}$$

$$\Lambda_{A,Q}(\psi|_{\partial D}) = \sum_{j=1}^{d}\nu_j\left(\frac{\partial}{\partial x_j} + iA_j\right)\psi\Big|_{\partial D},\ L_{A,Q}\psi = E\psi,$$

$$A_j,\ Q \text{ are } M_n(\mathbb{C})\text{-valued with compact support in } D \tag{DN}$$

- Let u_0 satisfy $L_{0,0}u_0 = Eu_0$ and u satisfy $L_{A,Q}u = Eu$ in D. Then the following formula holds (Alessandrini '88):

$$\int_D u_0(x)\left(L_{A,Q} - L_{0,0}\right)u(x)dx = \int_{\partial D} u_0(x)\left(\Lambda_{A,Q} - \Lambda_{0,0}\right)u(x)dx \tag{AI}$$

From D-to-N map to scattering amplitude

$$\psi^+(x,k) = e^{ikx} + \int_D G^+(x-y,k)\big(L_{A,Q} - L_{0,0}\big)\psi^+(y,k)\,dy \tag{LS}$$

$$f_{A,Q}(k,l) = (2\pi)^{-d}\int_D e^{-ilx}(L_{A,Q} - L_{0,0})\psi^+(x,k)\,dx \tag{SA}$$

$$\int_D u_0(x)\big(L_{A,Q} - L_{0,0}\big)u(x)dx = \int\limits_{\partial D} u_0(x)\big(\Lambda_{A,Q} - \Lambda_{0,0}\big)u(x)dx \tag{AI}$$

- (LS) + (AI) imply

$$\psi^+(x,k) = e^{ikx} + \int_{\partial D} G^+(x-y,k)\big(\Lambda_{A,Q} - \Lambda_{0,0}\big)\psi^+(x,k)\,dx$$

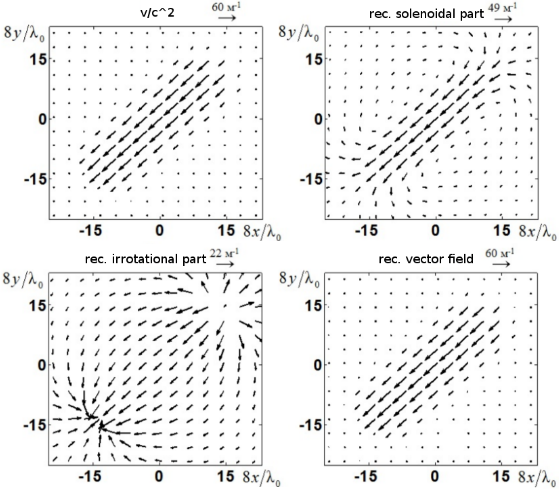
- (SA) + (AI) imply

$$f_{A,Q}(k,l) = (2\pi)^{-d}\int_{\partial D} e^{-ilx}\big(\Lambda_{A,Q} - \Lambda_{0,0}\big)\psi^+(x,k)\,dx$$

(R. Novikov '92): $A = 0$, (Eskin-Ralston '97): $A \neq 0$, (R. Novikov '05): background potential, [Agaltsov, JIIP '15]: general case, matrix coefficients

Acoustic scattering: numerical example of [11]

$$L_\omega = -\Delta - 2i\omega \frac{\mathbf{v}}{c^2} \cdot \nabla - \frac{\omega^2}{c^2} - 2i\omega^2 \frac{\alpha_0}{c}, \quad \omega \in \{\omega_1, \omega_2\}$$



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