Inverse potential problems in divergence form: some uniqueness, separation and recovery issues

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Quasilinear equations, inverse problems and their applications,
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based in part on joint work with:
S. Chevillard, J. Leblond (INRIA), D. Pei, Q. Tao (Macau).

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- In particular, we have that

$$f(\xi) = \mathcal{C}^{+} - \mathcal{C}^{-}.$$

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- Also, the Cauchy theorem holds:

$$0 = \frac{1}{2i\pi} \int_\Gamma \frac{f(\xi)}{\xi - z} d\xi, \qquad f \in H^p(D^\pm), \qquad z \in D^\mp.$$

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Theorem

For Γ a smooth Jordan curve and $1 < p < \infty$ there holds a topological sum:

$$L^p(\Gamma) = H^p(D^+) \oplus H^p(D^-).$$

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Corollary

A $\mathbb{R}^2$-valued vector field of L^p-class on Γ is uniquely the sum of the trace of the gradient of a harmonic function in D^+ and the trace of the gradient of a harmonic function in D^-, where both gradients have nontangential maximal function in L^p.

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$\mathrm{div}\, m$ *has null potential in* D^- *if, and only if* $(m_1 + im_2)/v \in H^p(D^+)$.

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- They occur frequently when modeling electro-magnetic phenomena in the quasi-static approximation to Maxwell's equations.

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- Likewise, the inverse magnetization problem is to recover the magnetization $\mathbf{M}$ on a given object, from measurements of the field $H = -\nabla \phi$ near the object.
- Today, inverse magnetization problems are a hot topic in Earth and Planetary Sciences.

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- Let us look at the elementary case where V is supported on the horizontal plane with L^p density there, $1 < p < \infty$.
- This geometry is in fact realistic in scanning microscopy of rocks which are typically sanded down to thin slabs.

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- and $A_j(X) = \mathcal{P}_X(R_j m_j)/2$ for $j = 1, 2$, where

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  is $\operatorname{sgn} x_3$ times half the harmonic (Poisson) extension of m_3: $A_3(X) = \operatorname{sgn} x_3 \mathcal{P}_X(m_3)/2$,
- and $A_j(X) = \mathcal{P}_X(R_j m_j)/2$ for $j = 1, 2$, where

$$R_j(f)(Y) := \lim_{\epsilon \to 0} \frac{1}{2\pi} \iint_{\mathbb{R}^2 \setminus B(Y,\epsilon)} f(X') \frac{(y_j - x_j')}{|Y - X'|^3} \, dX', \qquad j = 1, 2,$$

are the Riesz transforms.

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Question

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To approach it, we introduce some classical function spaces.

Hardy space of harmonic gradients

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- Let $\mathfrak{H}^p_+$ consist of ∇u, u harmonic in $\{x_3 > 0\}$, such that

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- We put $\mathcal{D}^p$ for divergence-free vector fields in $L^p(\mathbb{R}^2, \mathbb{R}^2)$.

The Hardy-Hodge decomposition on $\mathbb{R}^2$

Theorem (L.B., D. Hardin, E. Lima, E.B. Saff, B. Weiss)

For $1 < p < \infty$ one has the direct sum:

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- Easily checked using $R_1^2 + R_2^2 = -\mathrm{Id}$.

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- M is silent iff it is tangent and divergence-free.
- Transparent if we observe the orthogonality:

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- If $M \in (L^2(\mathbb{R}^n))^3$ then $P_{\mathfrak{H}^2_-} M$ yields the magnetization of least $(L^2(\mathbb{R}^n))^3$-norm which is equivalent to M from above.

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If $\mathcal{M}$ is smooth, this coincides with the usual notion of divergence free tangent vector field.

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- Note the above nontangential limits are not tangent to $\mathcal{M}$.

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Let $\mathcal{M}$ be a compact simply connected Lipschitz hypersurface in $\mathbb{R}^n$ and $p_0(\mathcal{M}) < p < \infty$. Then, there is a direct sum

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- When $\mathcal{M}$ is smooth the decomposition holds in more general spaces of functions or distributional currents.

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- Thus, we are left to decompose $V - D - \nabla u$ which is a normal vector field on $\mathcal{M}$.

Sketch of proof

- Let $V \in L^p(\mathcal{M})^n$. Write $V = V_n + V_t$ according to the normal and tangential components.
- By Hodge decomposition, $V_t = G + D$ where $D \in \mathcal{D}^p$ and $G \in \mathcal{G}^p$. If $\mathcal{M}$ is smooth, the Hodge decomposition is a byproduct of L^p Hodge theory on complete manifolds [X-D Li,2009]. In the Lipschitz case, it can be proved by solving an extremal problem.
- We save D which is the last summand in the decomposition.
- G is the tangential gradient of some function $\psi \in W^{1,p}(\mathcal{M})$.
- Let u be harmonic in Ω^+ and solve the Dirichlet problem $u_{|\mathcal{M}} = \psi$. Then $\nabla u \in \mathcal{H}^p_+$ [Verchota,1984] and the tangential component of its nontangential limit on $\mathcal{M}$ is G.
- Thus, we are left to decompose $V - D - \nabla u$ which is a normal vector field on $\mathcal{M}$. For this we need preliminaries in Clifford analysis. We restrict to $n = 3$ for simplicity.

Some Clifford analysis

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- $\mathfrak{C}$ is the skew unital algebra generated over $\mathbb{R}$ by $\{e_1, e_2, e_3\}$ with:

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- A typical element of $\mathfrak{C}$ is of the form

$$z = x_0 + x_1 e_1 + x_2 e_2 + x_3 e_3 + x_{1,2} e_1 e_2 + x_{2,3} e_2 e_3 + x_{1,3} e_1 e_3 + x_{123} e_1 e_2 e_3$$

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- The norm of z is $|z| = \left(\sum_{0 \le k \le 3} x_k^2 + \sum_{i<j} x_{i,j}^2 + x_{123}^2\right)^{1/2}$.

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Lemma

A vector-valued function is left monogenic if and only if it is monogenic, if and only if it is the gradient of a harmonic function.

Cauchy-Clifford formula

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- If f is left monogenic in Ω^+ and its nontangential maximal function lies in $L^p(\mathcal{M})$, then f has a nontangential limit $f^+ \in L^p(\mathcal{M})$ a.e. on $\mathcal{M}$ (Verchota), and by the Green formula (see *e.g.* "Clifford Algebras and Dirac Operators in Analysis" by Gilbert and Murray):

$$f(z) = \mathcal{C}f^+(z) := \frac{1}{4\pi} \int_{\mathcal{M}} \frac{\overline{y-z}}{|y-z|^3} n(y) f^+(y) d\sigma(y), \qquad z \in \Omega^+.$$

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Plemelj-Clifford formulas

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- For a $\mathfrak{C}$-valued $h \in L^p(\mathcal{M})$, $\mathcal{C}h$ is left monogenic on $\mathbb{R}^3 \setminus \mathcal{M}$ and its nontangential maximal function lies in $L^p(\mathcal{M})$ [Coifman-McIntosh-Meyer, 1982].

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where $\mathcal{S}\mathcal{C}h$ is the *singular Cauchy integral operator:*

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- This gives us an analog of the Plemelj formula:

$$\mathcal{C}^{+}h(y) - \mathcal{C}^{-}h(y) = h(y).$$

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- Uniqueness follows from uniqueness of the Hodge decomposition and the Liouville theorem for harmonic functions.

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Theorem

The distribution is silent from outside if and only if

$$2\pi\psi(y) = -\lim_{\varepsilon\to 0}\int_{\mathcal{M}\setminus B(y,\varepsilon)} \frac{\xi - y}{|\xi - y|^3}.(\psi n + R(D))(\xi)d\sigma(\xi) \quad y \in \mathcal{M}.$$

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This is a more complicated singular integral equation involving the curvature.

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Corollary

Let $\mathcal{M}$ be a sphere in $\mathbb{R}^3$ and $m \in (L^p(\mathcal{M}))^3$ with $1 < p < \infty$. Then m is silent from outside (resp. inside) if and only if

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- If $\operatorname{supp} m$ is a strict subset of $\mathcal{M}$, things get simple:

Corollary

If $\operatorname{supp} m \neq \mathcal{M}$, then m is silent from outside iff it is silent from inside, which is iff

$$m \in \mathcal{D}^p(\mathcal{M}).$$

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Let $m \in (L^2(\mathcal{M}))^3$ *with* $\operatorname{supp} m \subset \Gamma_0 \neq \mathcal{M}$*, and assume we know the field* $B = \nabla P_{\operatorname{div} m}$ *on a surface patch* Σ *disjoint from* $\mathcal{M}$*.*

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Let $m \in (L^2(\mathcal{M}))^3$ with $\mathrm{supp}\, m \subset \Gamma_0 \neq \mathcal{M}$, and assume we know the field $B = \nabla P_{\mathrm{div}\, m}$ on a surface patch Σ disjoint from $\mathcal{M}$. Then, to each $\varepsilon > 0$ and $\psi \in L^2(\mathcal{M})$, there is $\phi \in (L^2(\Sigma))^3$ depending on $\mathrm{supp}\, m$ but not on m such that

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- Can be used to estimate moments or spherical harmonics expansions.

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- In $\mathbb{R}^n$, let rational approximation mean approximation by gradients of discrete harmonic potentials with finitely many masses. The Hardy-Hodge decomposition implies:

Theorem

Let S be a Lipschitz regular surface patch on a compact connected smooth hypersurface $\mathcal{M} \subset \mathbb{R}^n$. Let v be $\mathbb{R}^n$-valued in $L^p(S)$, $1 < p < \infty$. Then, v can be approximated arbitrarily close by rationals in $L^p(S)$ iff the tangential component of v is a gradient.

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- A fortiori then, a compact set containing no such arc is a grad-set for L^p (any field is approximable y a gradient). The Hardy-Hodge decomposition now implies:

Theorem

Let K be a closed set in a compact connected smooth hypersurface $\mathcal{M} \subset \mathbb{R}^n$, and assume that K contains no simple rectifiable arc of positive length. Then, each $\mathbb{R}^n$-valued v in $L^p(K)$ can be approximated arbitrary close by rationals in $L^p(K)$, $1 < p < \infty$.