THE QUESTIONS OF EXISTENCE OF THE PERIODIC AND LIMITED SOLUTIONS OF TRAVELLING WAVE TYPE.

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1 Introduction

Many applied problems lead to studing solutions of travelling waves type for infinite-dimensional dynamic systems. In particular, the theory of a plastic deformation considers the infinite-dimensional dynamic system

$$m\ddot{y}_i = y_{i+1} - 2y_i + y_{i-1} + \phi(y_i), \qquad i \in \mathbb{Z}, \quad t \in \mathbb{R} \tag{1}$$

where the potential $\phi(\cdot)$ is smooth periodic function. The equation (1) is a system with the Fraenkel - Kontorova potential [1]. This system describes a behaviour of a countable number of balls with masses equal to m, located in integer points of the real axis, where any two neighboring balls are connected by a flexible spring. The studying such systems with different potentials is one of the intensively developing directions in the dynamic systems theory. For them, the central problem is the studing of solutions of traveling wave type. Those solutions represent one of observed classes of waves.

Definition 1. *The system solution* (1) *is any sequence* $\{y_i(\cdot)\}_{-\infty}^{+\infty}$ *of continuous functions for which the derivatives sequence* $\{\dot{y}_i(\cdot)\}_{-\infty}^{+\infty}$ *consists of absolutely continuous functions and such sequence satisfies to system of the equations* (1). ■

Definition 2. *Let's say, that the solution* $\{y_i(\cdot)\}_{-\infty}^{+\infty}$ *of the system* (1), *defined for all* $t \in \mathbb{R}$, *is of a traveling wave type, if there exists* $\tau > 0$, *not depending on* t *and* i, *such that the equality*

$$y_i(t+\tau) = y_{i+1}(t) \tag{2}$$

is true for all $i \in \mathbb{Z}$ *and* $t \in \mathbb{R}$. *Such constant* τ *is named of a traveling wave characteristic.* ■

One of methods of investigating such systems is a constructing solutions by using an explicit form of right-hand side of the system, and also its possible periodicity, infinite differentiability, analyticity. The travelling wave type solutions were constucted and for near potentials by the perturbations theory. The review of works on infinite-dimensional systems with potentials of Fraenkel - Kontorova and Fermi - Past - Ulam is presented in two [2]. At the same time, such approach does not allow to describe a space of all solutions of traveling wave type, and also their possible asymptotics. As a rule, the solutions existence is proved in wider space of infinitely differentiated functions, and the solutions uniqueness is proved in narrower space of analytical functions.

In the following, we suggest an approach, by which the solutions of traveling wave type for the system (1) may be realized as those of the one-parameter family of functional-differential equations of pointwise type. In this connection, the system (1) with a potential without singularities can be investigated by more common suppositions for the potential $\phi(\cdot)$, such as a Lipschitz's condition with the constant L. In the framework of the formalism suggested, it is possible to describe the solutions of travelling wave type, and also their asymptotics, connected with of the traveling wave characteristic.

The suggested approach allows to consider the system (1) with unequal of balls masses, too. It is shown, that there exist no solutions of travelling wave type for the systems with unequal masses of balls, different from either stationary stations or rectilinear uniform motions. In this connection, the quasisolutions of travelling wave type are defined. Such quasisolutions are the correct expension of travelling wave type solutions space and ones coincide with the travelling wave type solutions space at the equal masses of balls. Results of this kind have been presented at the same conference in 2015.

At the same time, in many cases (at equal masses) presence of solutions of travelling waves type with special properties such as the periodic and limited solutions of travelling waves type is important. Within of the offered approach also it is possible to receive the conditions of existence both periodic, and the limited solutions of travelling waves type.

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2 Spaces of solutisions.

The infinite-dimensional dynamic system with not local restrictions is considered

$$m\ddot{y}_i = y_{i+1} - 2y_i + y_{i-1} + \phi(y_i), \qquad i \in \mathbb{Z}, \quad t \in \mathbb{R}, \tag{1*}$$
$$y_i(t+\tau) = y_{i+1}(t). \tag{2*}$$

We noticed that the previous authors at study of solutions of travelling wave type used a concrete kind of potential, and also such local properties as infinite differentiability, or analyticity. In our approach we will study solutions with the set global properties in a kind of asymptotics both on time, and on space.

For this purpose, we will define a one-parameter family of the embedded Banach's spaces of functions with weights

$$\mathcal{L}^n_\mu C^{(k)}(\mathbb{R}) = \left\{ x(.): \ x(.) \in C^{(k)}\left(\mathbb{R}, \mathbb{R}^n\right), \qquad \max_{0\le r\le k} \sup_{t\in\mathbb{R}} \|x^{(r)}(t)\mu^{|t|}\|_{\mathbb{R}^n} < +\infty \right\}, \qquad \mu \in]0,1[$$

and such vector space

$$K^n = \overline{\prod_{i\in\mathbb{Z}}} R^n_i, \qquad R^n_i = \mathbb{R}^n, \quad i \in \mathbb{Z}$$
$$(\varkappa \in K^n, \qquad \varkappa = \{x_i\}_{-\infty}^{+\infty}).$$

is definded with the standard topology of the full direct product (the space metrized). Elements of the space K^2 are infinite sequences

$$\varkappa = \{(u_i,\, v_i)'\}_{-\infty}^{+\infty}, \qquad u_i,\, v_i \in \mathbb{R}$$

(a prime denotes the transposition).

In space K^n we will define the Banach subspaces family $K^n_{\infty\mu}$, $\quad \mu \in (0,1]$

$$K^n_{\infty\mu} = \{\varkappa: \ \sup_{i\in Z} \|x_i\|_{R^n}\mu^{|i|} < +\infty\}$$

with norm

$$\|\varkappa\|_{\infty\mu} = \sup_{i\in Z} \|x_i\|_{R^n}\mu^{|i|},$$

and the Hilbert subspaces family $K^n_{2\mu}$, $\quad \mu \in (0,1)$

$$K^n_{2\mu} = \left\{ \varkappa: \ \varkappa \in K^n; \qquad \sum_{i=-\infty}^{+\infty} \|x_i\|^2_{R^n}\mu^{2|i|} < +\infty \right\}$$

with norm

$$\|\varkappa\|_{2\mu} = [\sum_{i=-\infty}^{+\infty} \|x_i\|^2_{R^n}\mu^{2|i|}]^{\frac{1}{2}}.$$

Here μ it is free parameter by which the space of solutions will be choosed.

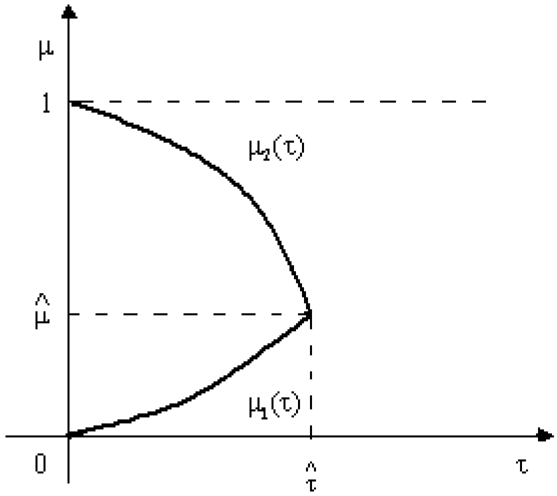

Рис. 1: Graphs of functions $\mu_1(\tau)$, $\mu_2(\tau)$.

Let's consider the equation relatively of two variables $\tau \in (0, +\infty)$ and $\mu \in (0, 1)$

$$C\tau \left[2\mu^{-1} + 1\right] = \ln \mu^{-1}, \tag{3}$$

where

$$C = \max\{1; \quad [L+2] \ m^{-1}\}.$$

The set of solutions of the equation (3) is described by functions $\mu_1(\tau)$, $\mu_2(\tau)$, set in figure 1.

Equality $\hat{\mu} = 2C\hat{\tau}$ takes place. As $0 < \hat{\mu} < 1$, then there is some absolute estimation for magnitude $\hat{\tau}$

$$\hat{\tau} < (2C)^{-1}.$$

3 Solutions of travelling wave type. A case of equal masses.

For a case of equal masses we will formulate the theorem of existence and uniqueness of the solutions of travelling wave type.

Theorem 1. *There exists a unique solution* $\{y_i(\cdot)\}_{-\infty}^{+\infty}$ *of travelling wave type with the characteristic* τ *at any initial data* $\bar{i} \in \mathbb{Z}$, $a, b \in \mathbb{R}$, $\bar{t} \in \mathbb{R}$ *and the characteristic* τ, *satisfying the condition*

$$0 < \tau < \hat{\tau},$$

for the initial system of the differential equations (1), *such, that the solution satisfies the initial conditions* $y_{\bar{i}}(\bar{t}) = a$, $\dot{y}_{\bar{i}}(\bar{t}) = b$. *Moreover, at any parameter* $\mu \in]\mu_1(\tau), \mu_2(\tau)[$ *each coordinate* $y_i(\cdot)$, $i \in \mathbb{Z}$ *belongs to the space* $\mathcal{L}^1_{\sqrt[\tau]{\mu}} C^{(1)}(\mathbb{R})$ *and the sequence* $\{(y_i(t), \dot{y}_i(t))'\}_{-\infty}^{+\infty}$ *belongs to the space* $K^2_{2\mu}$ *at any* $t \in \mathbb{R}$. *Such solution continuously depends on the initial data* $a, b \in \mathbb{R}$. ■

Here, we take the proximity as the norm of the phase space $K^2_{2\mu}$.

The theorem 1 not only guarantees the existence of a solution, but also sets its asymptotics both on time t and on coordinates $i \in \mathbb{Z}$ (on space).

If the potential $\phi(.)$ is identically equal to zero, then solutions of travelling wave type, guaranteed by the theorem 1, set either the rectilinear uniform motions $\{y_i(t)\}_{-\infty}^{+\infty} = \{bt + bi\tau + \alpha\}_{-\infty}^{+\infty}, (b \neq 0)$ or stationary state $(b = 0)$. If the potential $\phi(.)$ isn't identically equal to zero, then there exists a solution of travelling wave type, guaranteed by the theorem 1, with the characteristic τ, $0 < \tau < \hat{\tau}$ given beforehand and describing neither rectilinear uniform motion $\{y_i(t)\}_{-\infty}^{+\infty} = \{bt + bi\tau + \alpha\}_{-\infty}^{+\infty}$ $(b \neq 0)$ nor stationary state $(b = 0)$ (the existence of nontrivial solutions).

4 Some elements of the used approach

The study of traveling wave type solutions with the set characteristic $\tau > 0$ for system (1^*) means the studying of solutions of system

$$m\ddot{y}_i = y_{i+1} - 2y_i + y_{i-1} + \phi(y_i), \qquad i \in \mathbb{Z}, \quad t \in \mathbb{R}, \tag{4}$$

$$y_i(t+\tau) = y_{i+1}(t). \tag{5}$$

It is obvious that restriction on an interval $[0, \tau]$ any solution of system (4)-(5) with not local restrictions (5) is the solution of the boundary value problem

$$m\ddot{y}_i = y_{i+1} - 2y_i + y_{i-1} + \phi(y_i), \qquad i \in \mathbb{Z}, \quad t \in [0, \tau] \tag{6}$$

$$y_i(\tau) = y_{i+1}(0), \qquad \dot{y}_i(\tau) = \dot{y}_{i+1}(0) \tag{7}$$

with unlocal boundary conditions (7).

The inverse statement also is true: *any solution of the boundary value problem* (6)-(7), *continued by to the differential equation* (4), *is the solution of system* (4)-(5).

On the other hand, the studying of the solutions of the boundary value problem (6)-(7) is equivalent to that of the solutions space of the functional - differential equation

$$\ddot{x}(t) = m^{-1}[x(t+\tau) - 2x(t) + x(t-\tau) + \phi(x(t))], \qquad t \in \mathbb{R}. \tag{8}$$

The solutions of the boundary value problem (6)-(7) and the functional-differential equation (8) are connected by the following way: for any $t \in \mathbb{R}$

$$x(t) = y_{[t\tau^{-1}]}(t - [t\tau^{-1}]),$$

where [.] is the integer part of the number.

Operator representation of travelling wave.

Let's consider the space K^2 with the elements $\varkappa = \{(u_i,\ v_i)'\}_{-\infty}^{+\infty}$ (a prime means transposition). Let's define a linear operator $\mathbb{A}$, the shift operator $\mathbb{T}$ and a nonlinear operator $\mathbb{F}$, acting continuously from space K^2 in itself by the following rule : for any $i \in \mathbb{Z}$, $\quad \varkappa \in K^2$,

$$(\mathbb{A}\varkappa)_i = (v_i,\ m^{-1}[u_{i+1} - 2u_i + u_{i-1}])', \qquad (\mathbb{T}\varkappa)_i = (\varkappa)_{i+1}, \qquad (\mathbb{F}\varkappa)_i = (0, m^{-1}\phi(u_i))'.$$

Let's remark, that the shift operator $\mathbb{T}$ is commutative with operators $\mathbb{A}$ and $\mathbb{F}$.

The system of equations (4)-(5) may be rewritten in the following equivalent form

$$\dot{\varkappa} = \mathbb{A}\varkappa + \mathbb{F}(\varkappa), \qquad t \in \mathbb{R}. \tag{9}$$

$$\varkappa(t+\tau) = \mathbb{T}\varkappa(t). \tag{10}$$

It is obvious that restriction on an interval $[0,\tau]$ everyone the system solution (9)-(10) with not local restrictions (10) will be the solution of the boundary value problem

$$\dot{\varkappa} = \mathbb{A}\varkappa + \mathbb{F}(\varkappa), \qquad t \in [0,\tau]. \tag{11}$$

$$\varkappa(\tau) = \mathbb{T}\varkappa(0) \tag{12}$$

with not local boundary conditions (12).

In case of equal masses the operators $\mathbb{A}$ and $\mathbb{F}$ are commutated. It follows from the condition of commutativity of operators $\mathbb{A}$ and $\mathbb{F}$, that any solution of the boundary problem (11)-(12), continued on all $\mathbb{R}$ by to the differential equation (9), is the system solution (9)-(10). In it also there is a difference from a case of unequal masses.

The periodic solutions of travelling wave type.

Definition 3. *The solution* $\{y_i(\cdot)\}_{-\infty}^{+\infty}$ *of the systems* (1) *is periodic with the period* $\omega \in \mathbb{R}_+$*, if for any* $i \in \mathbb{Z}$ *the condition* $y_i(t+\omega) = y_i(t),\ \ t \in \mathbb{R}$ *is true.* ■

If $\omega = k\tau,\ \ k \in \mathbb{Z}_+$, then ω-periodic solutions of travelling wave type with the characteristic τ can be presented in the operational form as the solutions of the following system

$$\dot{\varkappa} = \mathbb{A}\varkappa + \mathbb{F}(\varkappa), \qquad t \in \mathbb{R}, \tag{13}$$

$$\varkappa(t+\tau) = \mathbb{T}\varkappa(t), \tag{14}$$

$$\mathbb{T}^{\omega}\varkappa(t) = \varkappa(t). \tag{15}$$

Thus, if the solution of travelling wave type is ω-periodic solution on time, then it is ω-periodic on phase variables $\varkappa$.

It is obvious that restriction on an interval $[0,\tau]$ everyone solution of the system (13)-(15) with not local restrictions (14) will be the solution of the boundary value problem

$$\dot{\varkappa} = \mathbb{A}\varkappa + \mathbb{F}(\varkappa), \qquad t \in [0,\tau]. \tag{16}$$

$$\varkappa(\tau) = \mathbb{T}\varkappa(0) \tag{17}$$

$$\mathbb{T}^{\omega}\varkappa(t) = \varkappa(t) \tag{18}$$

with not local boundary conditions (17).

On the other hand, studying of solution of the boundary value problem (16)-(18) is equivalent studying of the space of ω-periodic solution on time of the functional-differential equation

$$\ddot{x}(t) = m^{-1}[\,x(t+\tau) - 2x(t) + x(t-\tau) + \phi(x(t))\,], \qquad t \in \mathbb{R} \tag{19}$$

and corresponding solution are connected as follows: for any $t \in \mathbb{R}$

$$x(t) = u_{[t\tau^{-1}]}(t - [t\tau^{-1}]),$$

where $\varkappa(t) = \{(u_i(t),\, v_i(t))'\}_{-\infty}^{+\infty}$, and $[.]$ is the whole part of number.

Let's formulate the theorem of existence of the periodic solution of travelling wave type. If the potential has zero value, i.e. for some $b \in \mathbb{R}$ equality $\phi(b) = 0$ is true, then the solution $\{y_i(\cdot)\}_{-\infty}^{+\infty},\quad y_i(t) \equiv b,\quad t \in \mathbb{R},\quad i \in \mathbb{Z}$ is stationary state of system (1), and it is also the solution of travelling wave type with any characteristic τ. It is obvious that such solutions is as the periodic solutions with any period. Therefore it is necessary to consider potentials with a constant sign.

Theorem 2. *Let are set*

$$0 < \tau < \hat{\tau}, \quad \omega = k\tau, \quad k \in \mathbb{Z}_+.$$

There will be such positive constants $\bar{\phi}$ *and* $\bar{L}$ *that under the conditions*

$$|\phi(0)| \leq \bar{\phi}, \quad L \leq \bar{L} \tag{20}$$

for initial system of the differential equations (1) *there is a solution* $\{y_i(\cdot)\}_{-\infty}^{+\infty}$ *of travelling wave type with the characteristic* τ*, being* ω*-periodic solution.* ■