# Wave model of the Schroedinger operator on semiaxis (the limit point case)

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## Plan

### 1. Operator $L_0$

• The class of operators. Schroedinger operator $L_0^{\rm Sch}$ on $(0,\infty)$ with defect indexes (1,1).
• A Green System. The Green system of $L_0^{\rm Sch}$.

### 2. Dynamical System with Boundary Control

• An abstract DSBC associated with $L_0$. Reachable sets, controllability.
• Isotony $I_{L_0}$. The system $\alpha_{L_0^{\rm Sch}}$ and isotony $I_{L_0^{\rm Sch}}$.

### 3. Wave model

• Lattices, lattice-valued functions, atoms. A space $\Omega_{L_0}$ (wave spectrum of $L_0$). The space $\Omega_{L_0^{\rm Sch}}$.
• A wave model $\tilde{L}_0^*$. The model $\tilde{L}_0^{\rm Sch\,*}$.
• Applications to inverse problems. The spectral IP for $L_0^{\rm Sch}$.

## 1 Operator $L_0$

Let $\mathcal{H}$ be a separable Gilbert space, $L_0$ acts in $\mathcal{H}$ and

1. $L_0 = \bar{L}_0\,,\quad \overline{{\rm Dom}\, L_0} = \mathcal{H}$
2. $\exists \varkappa = {\rm const} > 0$ s.t. $(L_0 y, y) \geqslant \varkappa \|y\|^2,\ y \in {\rm Dom}\, L_0$

3. $1 \leqslant n_{\pm}(L_0) = \dim \operatorname{Ker} L_0^* \leqslant \infty$.

Let $L$ be the Friedichs extension of $L_0$:

$$L_0 \subset L \subset L_0^*; \quad L^* = L; \quad (Ly, y) \geqslant \varkappa \|y\|^2, \ \ y \in \operatorname{Dom} L\,.$$

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Notation: $\mathbb{R}_+ := (0, \infty)$, $\bar{\mathbb{R}}_+ := [0, \infty)$.
Let $\mathcal{H} = L_2(\mathbb{R}_+)$ and $L_0 = L_0^{\mathrm{Sch}}$,

$$L_0^{\mathrm{Sch}} y := -y'' + qy\,,$$
$$\operatorname{Dom} L_0^{\mathrm{Sch}} = \left\{ y \in \mathcal{H} \cap H^2_{\mathrm{loc}}(\bar{\mathbb{R}}_+) \mid y(0) = y'(0) = 0; \ -y'' + qy \in \mathcal{H} \right\}\,,$$

where $q = q(x)$ is a real-valued function (*potential*) provided

(1) $q \in C_{\mathrm{loc}}\left(\bar{\mathbb{R}}_+\right)$
(2) $n_{\pm}(L_0^{\mathrm{Sch}}) = 1$ (the limit point case)
(3) $(L_0^{\mathrm{Sch}} y, y) \geqslant \varkappa \|y\|^2, \ y \in \operatorname{Dom} L_0^{\mathrm{Sch}}$.

In such a case, the problem

$$-\phi'' + q\phi = 0\,, \ \ x > 0\,; \qquad \phi(0) = 1\,, \qquad \phi \in L_2(\mathbb{R}_+)$$

has a unique solution $\phi(x)$;

$$(L_0^{\mathrm{Sch}})^* y := -y'' + qy\,,$$
$$\operatorname{Dom}\,(L_0^{\mathrm{Sch}})^* = \left\{ y \in \mathcal{H} \cap H^2_{\mathrm{loc}}(\bar{\mathbb{R}}_+) \mid -y'' + qy \in \mathcal{H} \right\}\,,$$
$$\operatorname{Ker}\,(L_0^{\mathrm{Sch}})^* = \{ c\phi \mid c \in \mathbb{C} \};$$

the Friedrichs extension of $L_0^{\mathrm{Sch}}$ is

$$L^{\mathrm{Sch}} y := -y'' + qy\,,$$
$$\operatorname{Dom} L^{\mathrm{Sch}} = \left\{ y \in \mathcal{H} \cap H^2_{\mathrm{loc}}(\bar{\mathbb{R}}_+) \mid y(0) = 0; \ -y'' + qy \in \mathcal{H} \right\}.$$

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**Green system**

Let $\mathcal{H}$ $\mathcal{B}$ be the Hilbert spaces, $A : \mathcal{H} \to \mathcal{H}$ and $\Gamma_i : \mathcal{H} \to \mathcal{B}$ $(i = 1, 2)$ the operators provided:

$$\overline{\operatorname{Dom} A} = \mathcal{H}, \quad \operatorname{Dom} \Gamma_i \supset \operatorname{Dom} A, \quad \overline{\operatorname{Ran} \Gamma_1 \vee \operatorname{Ran} \Gamma_2} = \mathcal{B}.$$

The collection $\mathfrak{G} = \{\mathcal{H}, \mathcal{B}; A, \Gamma_1, \Gamma_2\}$ is a *Green system* if

$$(Au, v)_{\mathcal{H}} - (u, Av)_{\mathcal{H}} = (\Gamma_1 u, \Gamma_2 v)_{\mathcal{B}} - (\Gamma_2 u, \Gamma_1 v)_{\mathcal{B}}$$

for $u, v \in \operatorname{Dom} A$. $\mathcal{H}$ is an *inner space*, $\mathcal{B}$ is a *boundary values space*, $A$ is a *basic operator*, $\Gamma_{1,2}$ are the *boundary operators*.

**System $\mathfrak{G}_{L_0}$**

Operator $L_0$ determines a Green system in a canonical way. Put

$$\mathcal{K} := \operatorname{Ker} L_0^*, \quad \Gamma_1 := L^{-1} L_0^* - \mathbb{I}, \quad \Gamma_2 := P_{\mathcal{K}} L_0^*,$$

where $P_{\mathcal{K}}$ projects in $\mathcal{H}$ onto $\mathcal{K}$

**Proposition 1.** *The collection* $\mathfrak{G}_{L_0} := \{\mathcal{H}, \mathcal{K};\ L_0^*, \Gamma_1, \Gamma_2\}$ *is a Green system.*

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Let $L_0 = L_0^{\mathrm{Sch}}$, $\mathcal{H} = L_2(\mathbb{R}_+)$, $\mathcal{K} = \{c\phi \mid c \in \mathbb{C}\}$;

$$\Gamma_1 y = -\, y(0)\phi\,, \qquad \Gamma_2 y = \left[\frac{y'(0) - y(0)\phi'(0)}{\eta'(0)}\right]\phi\,,$$

where $\eta := (L^{\mathrm{Sch}})^{-1}\phi$. Then $\mathfrak{G}_{L_0^{\mathrm{Sch}}} := \{\mathcal{H}, \mathcal{K};\ (L_0^{\mathrm{Sch}})^*, \Gamma_1, \Gamma_2\}$ is the canonical Green system associated with $L_0^{\mathrm{Sch}}$.

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# 2 Dynamical System with Boundary Control

**System $\alpha_{L_0}$**

The Green system $\mathfrak{G}_{L_0}$ determines a DSBC $\alpha_{L_0}$ of the form

$$\begin{aligned}
& u_{tt} + L_0^* u = 0, && t > 0,\\
& u|_{t=0} = u_t|_{t=0} = 0 && \text{in } \mathcal{H}\\
& \Gamma_1 u = h\,, && t \geqslant 0,
\end{aligned}$$

where $h = h(t)$ is a *boundary control* (a $\mathcal{K}$-valued function of the time), $u = u^h(t)$ is a solution (*wave*, an $\mathcal{H}$-valued function of the time). For a smooth enough class $\mathcal{M} \ni h$, $u^h$ is classical.

**Proposition 2.** *For an $h \in \mathcal{M}$, one has*

$$u^h(t) \;=\; -h(t) + \int_0^t L^{-\frac{1}{2}}\, \sin\left[(t-s)L^{\frac{1}{2}}\right]\, h_{tt}(s)\, ds\,, \qquad t \geqslant 0\,, \tag{2.1}$$

*where $L$ is the Friedrichs extension of $L_0$.*

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For $L_0 = L_0^{\rm Sch}$, system $\alpha_{L_0^{\rm Sch}}$ is

$$\begin{aligned} & u_{tt} - u_{xx} + qu = 0, && x > 0,\; t > 0, \\ & u|_{t=0} = u_t|_{t=0} = 0, && x \geqslant 0, \\ & u|_{x=0} = f\,, && t \geqslant 0\,. \end{aligned}$$

For $\mathcal{M} := \{f \in C^\infty[0,\infty) \,|\, \operatorname{supp} f \subset (0,\infty)\}$, the solution $u^f$ is classical.

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**Controllability**

For the DSBC $\alpha_{L_0}$, the set

$$\mathcal{U}^T_{L_0} \;:=\; \{u^h(T) \,|\, h \in \mathcal{M}\}$$

is called *reachable* (at the moment $t = T$);

$$\mathcal{U}_{L_0} \;:=\; \bigcup_{T>0} \mathcal{U}^T_{L_0}$$

is a *total* reachable set.

System $\alpha_{L_0}$ is *controllable*, if

$$\overline{\mathcal{U}}_{L_0} \;=\; \mathcal{H}\,.$$

**Proposition 3.** *System $\alpha_{L_0}$ is controllable iff $L_0$ is a completely nonself-adjoint operator (i.e., $L_0$ induces a self-adjoint part in no subspace of $\mathcal{H}$).*

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For $L_0 = L_0^{\rm Sch}$, one has

$$\mathcal{U}^T_{L_0^{\rm Sch}} \;=\; \{y \in C^\infty(\bar{\mathbb{R}}_+) \,|\, \operatorname{supp} y \subset [0,T]\}\,, \quad \overline{\mathcal{U}}_{L_0^{\rm Sch}} = L_2(\mathbb{R}_+) \;=\; \mathcal{H}\,,$$

so that $\alpha_{L_0^{\rm Sch}}$ is controllable.

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**Isotony**

Introduce a dynamical system $\beta_{L_0}$:

$$\begin{aligned} & v_{tt} + Lv = g\,, && t > 0, \\ & v|_{t=0} = v_t|_{t=0} = 0\,, \end{aligned}$$

where $g = g(t)$ is an $\mathcal{H}$-valued function. For a smooth enough $g$, the solution $v = v^g(t)$ is

$$v^g(t) = \int_0^t L^{-\frac{1}{2}} \sin\left[(t-s)L^{\frac{1}{2}}\right] g(s)\,ds\,, \qquad t \geqslant 0$$

holds.

Fix a subspace $\mathcal{G} \subset \mathcal{H}$; let $g$ be a $\mathcal{G}$-valued function. A set

$$\mathcal{V}^t_{\mathcal{G}} := \left\{ v^g(t) \mid g \in L_2^{\mathrm{loc}}\left([0,\infty);\mathcal{G}\right)\right\}$$

is a reachable set of system $\beta_{L_0}$.

**Proposition 4.** *If $\mathcal{G} \subset \mathcal{G}'$ and $t \leqslant t'$ then $\mathcal{V}^t_{\mathcal{G}} \subset \mathcal{V}^{t'}_{\mathcal{G}'}$.*

Let $\mathfrak{L}(\mathcal{H})$ be the lattice of subspaces of $\mathcal{H}$ with the standard operations

$$\mathcal{A} \vee \mathcal{B} = \overline{\{a + b \mid a \in \mathcal{A},\, b \in \mathcal{B}\}}\,, \mathcal{A} \wedge \mathcal{B} = \mathcal{A} \cap \mathcal{B},\ \mathcal{A}^\perp = \mathcal{H} \ominus \mathcal{A}\,,$$

the partial order $\subseteq$, the least and greatest elements $\{0\}$ and $\mathcal{H}$, and the relevant topology.

A family of maps $I = \{I^t\}_{t \leqslant 0}$, $I^t : \mathfrak{L}(\mathcal{H}) \to \mathfrak{L}(\mathcal{H})$ is said to be an *isotony* if

$$I^0 := \mathrm{id} \qquad \text{and} \qquad \mathcal{G} \subset \mathcal{G}',\ t \leqslant t' \quad \text{implies} \quad I^t\mathcal{G} \subset I^{t'}\mathcal{G}'\,,$$

i.e., $I$ respects the natural order on $\mathfrak{L}(\mathcal{H}) \times [0,T)$.

By Proposition 4, the family

$$I^0_{L_0} := \mathrm{id}\,; \qquad I^t_{L_0}\mathcal{G} := \overline{\mathcal{V}^t_{\mathcal{G}}}\,, \quad t > 0$$

is the isotony determined by $L_0$.

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For $E \subset \bar{\mathbb{R}}_+$, let

$$E^t := \{x \in \bar{\mathbb{R}}_+ \mid \mathrm{dist}\,(x,E) := \inf_{e \in E} |x - e| < t\}\,, \qquad t > 0$$

(a metric neighborhood). Denote $\Delta_{a,b} := [a,b] \subset \bar{\mathbb{R}}_+$ and

$$L_2(\Delta_{a,b}) := \{y \in L_2(\mathbb{R}_+) \mid \mathrm{supp}\, y \subset [a,b]\}\,.$$

**Proposition 5.** *For any* $0 \leqslant a < b \leqslant \infty$, *the relation*

$$I^t_{L_0^{\rm Sch}} L_2(\Delta_{a,b}) = L_2(\Delta^t_{a,b})\,, \qquad t > 0$$

*holds.*

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# 3 Wave model

### Lattices

Let $\mathfrak{L}_{L_0} \subset \mathfrak{L}(\mathcal{H})$ be a *minimal* (sub)lattice s.t.

$$\overline{\mathcal{U}}^T_{L_0} \subset \mathfrak{L}_{L_0}\,, \;\; T \geqslant 0 \qquad \text{and} \qquad I^t_{L_0}\mathfrak{L}_{L_0} \subset \mathfrak{L}_{L_0}\,, \;\; t \geqslant 0$$

(i.e., $\mathfrak{L}_{L_0}$ is *invariant* w.r.t. the isotony $I_{L_0}$).

The space of the growing $\mathfrak{L}_{L_0}$-valued functions

$$\mathcal{F}_{I_{L_0}}([0,\infty);\mathfrak{L}_{L_0}) := \left\{F(\cdot)\,|\; F(t) = I^t_{L_0}\mathcal{A},\; t \geqslant 0\,,\;\; \mathcal{A} \in \mathfrak{L}_{L_0}\right\}$$

is a lattice w.r.t. the point-wise operations, order, and topology.

### Wave spectrum

**Reminder.** $\mathcal{P}$ is a partially ordered set, 0 is the least element. An $a \in \mathcal{P}$ is an *atom* (*minimal element*) if

$$0 \neq p \preceq a \quad \text{implies} \quad p = a\,.$$

Let ${\rm At}\,\mathcal{P}$ be the set of atoms in $\mathcal{P}$.

**Basic definition.** The set

$$\Omega_{L_0} \;:=\; {\rm At}\,\overline{\mathcal{F}_{I_{L_0}}([0,\infty);\mathfrak{L}_{L_0})}$$

endowed with a relevant topology is a *wave spectrum* of operator $L_0$.

Each $\omega = \omega(\cdot) \in \Omega_{L_0}$ is a growing $\mathfrak{L}_{L_0}$-valued function of $t$.

So, the path is

$$L_0 \;\Rightarrow\; \mathfrak{G}_{L_0} \;\Rightarrow\; \alpha_{L_0}, \mathcal{U}^T_{L_0} \;\Rightarrow\; I_{L_0} \;\Rightarrow\; \mathfrak{L}_{L_0} \;\Rightarrow\; \mathcal{F}_{I_{L_0}}([0,\infty);\mathfrak{L}_{L_0}) \;\Rightarrow\; \Omega_{L_0}\,.$$

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**Theorem 1.** *For $L_0 = L_0^{\rm Sch}$, there is a bijection*

$$\Omega_{L_0^{\rm Sch}} \ni \omega \leftrightarrow x \in \bar{\mathbb{R}}_+$$

*and each atom is of the form*

$$\omega = \omega_x(t) = L_2\left(\{x\}^t\right), \quad t \geqslant 0\,.$$

*Endowing the wave spectrum with the proper metrizable topology, one gets*

$$\Omega_{L_0^{\rm Sch}} \stackrel{\rm isom}{=} \bar{\mathbb{R}}_+\,.$$

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**The model**

**Goal:** the image map $\mathcal{H} \ni y \stackrel{Y}{\mapsto} \tilde{y}$, where $\tilde{y} = \tilde{y}(\cdot)$ is a "function" on $\Omega_{L_0}$.

Fix $\omega \in \Omega_{L_0}$; let $P^t_\omega$ be the projection in $\mathcal{H}$ onto $\omega(t)$. For $u, v \in \mathcal{H}$, a relation

$$u \stackrel{\omega}{=} v \quad \Leftrightarrow \quad \left\{\exists\, t_0 = t_0(\omega, u, v) \text{ s.t. } P^t_\omega u = P^t_\omega v \text{ for all } t < t_0\right\}$$

is an equivalence.

- The equivalence class $[y]_\omega$ is a *germ* of $y \in \mathcal{H}$ at the atom $\omega \in \Omega_{L_0}$.
- The linear space

$$\mathcal{G}_\omega := \{[y]_\omega \mid y \in \mathcal{H}\}$$

is a *germ space*.

- The space

$$\tilde{\mathcal{H}} := \bigsqcup_{\omega \in \Omega_{L_0}} \mathcal{G}_\omega$$

is an *image space*.

- The *image map* is

$$Y : \mathcal{H} \to \tilde{\mathcal{H}}\,, \qquad \tilde{y}(\omega) := [y]_\omega, \quad \omega \in \Omega_{L_0}\,.$$

- The *wave model* of $L_0^*$ is

$$\widetilde{L_0^*} : \tilde{\mathcal{H}} \to \tilde{\mathcal{H}}, \quad \widetilde{L_0^*} := Y L_0^* Y^{-1}\,.$$

Let $y \in \mathcal{H} = L_2(\mathbb{R}_+)$. Take an $e = c\phi \in \operatorname{Ker}(L_0^{\rm Sch})^*$ and fix an $\omega \in \Omega_{L_0}$. Identify $\Omega_{L_0^{\rm Sch}} \equiv \bar{\mathbb{R}}_+$ (by Theorem 1) and

$$\tilde{y}(\omega) \equiv \lim_{t \to 0} \frac{(P^t_\omega y, e)}{(P^t_\omega e, e)}\,, \qquad \omega \in \bar{\mathbb{R}}_+\,.$$

Then $\tilde{\mathcal{H}} \equiv L_{2,\mu}(\mathbb{R}_+)$ with $d\mu = \frac{d\omega}{e^2(\omega)}$.

**Theorem 2.** *The representation*

$$\left(\widetilde{(L_0^{\rm Sch})^*}\tilde{y}\right)(\omega) = -\tilde{y}''(\omega) + p(\omega)\,\tilde{y}'(\omega) + Q(\omega)\tilde{y}(\omega)\,, \qquad \omega > 0$$

*is valid with*

$$p := -2\,\frac{e'(\omega)}{e(\omega)}\,, \quad Q := q(\omega) - \frac{e''(\omega)}{e(\omega)}\,.$$

**Application to IP** Inverse Data: *the spectral function, Weyl function, dynamical response operator*, etc. Then

$$\text{ID} \Rightarrow \text{copy } UL_0^{\rm Sch}U^* \Rightarrow \text{wave model } \widetilde{(L_0^{\rm Sch})^*} \Rightarrow p, Q \Rightarrow q\,.$$

# References


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