Holmgren theorems for the Radon transform

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MIPT, September 14, 2016
Holmgren’s uniqueness theorem (1901):
Unique continuation across a non-characteristic hypersurface for (distribution) solutions of general linear PDE:s with analytic coefficients.
Hörmander’s proof of Holmgren’s theorem

Part 1. Microlocal regularity theorem for solutions of PDE:s with analytic coefficients:

\[ WF_A(f) \subset WF_A(Pf) \cup \text{char}(P), \]

where \( \text{char}(P) = \{(x, \xi); p_{pr}(x, \xi) = 0\} \).
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where \( \text{char}(P) = \{(x, \xi); p_{pr}(x, \xi) = 0\} \).

In particular, if \( P(x, D)f = 0 \), then

\[ WF_A(f) \subset \text{char}(P). \]
Part 2. Unique continuation theorem for distributions satisfying an analytic wave front condition (microlocally real analytic distributions):
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Let \( S \) be a \( C^2 \) hypersurface in \( \mathbb{R}^n \). Assume that \( f = 0 \) on one side of \( S \) near \( x^0 \in S \), and that

\[
(x^0, \xi^0) \notin WF_A(f),
\]

where \( \xi^0 \) is conormal to \( S \) at \( x^0 \).

Then \( f = 0 \) in some neighborhood of \( x^0 \).
The wave front set

\[(x^0, \xi^0) \notin WF(f) \quad \text{if and only if} \]

\[\exists \psi \in C^\infty_c \text{ with } \psi(x^0) \neq 0 \text{ and open cone } \Gamma \ni \xi^0 \text{ such that} \]

\[|\widehat{\psi f}(\xi)| \leq C_m (1 + |\xi|)^{-m}, \quad m = 1, 2, \ldots, \quad \xi \in \Gamma.\]
The analytic wave front set

\[
(x^0, \xi^0) \notin WF_A(f) \iff \\
\exists \psi_m \in C^\infty_c(U), \psi_m = 1 \text{ in } U_0 \ni x^0 \text{ and open cone } \Gamma \ni \xi^0 \text{ such that} \\
|\widehat{\psi_m f}(\xi)| \leq \frac{(Cm)^k}{(1 + |\xi|)^k}, \quad k \leq m, \ m = 1, 2, \ldots, \ \xi \in \Gamma.
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The analytic wave front set

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Properties of the wave front set

If \( \varphi \in C^\infty \), then \( WF(\varphi f) \subset WF(f) \).
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If $\varphi$ is real analytic, then $WF_A(\varphi f) \subset WF_A(f)$. 
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If \( \varphi \) is real analytic, then \( \WF_A(\varphi f) \subset \WF_A(f) \).

If \( x' \mapsto f(x', x_n) \) is compactly supported and

\[(x, \pm e_n) \notin \WF(f) \text{ for all } x \text{ then } x_n \mapsto \int_{\mathbb{R}^{n-1}} f(x', x_n) \, dx' \text{ is } C^\infty.\]
Another unique continuation theorem for microlocally real analytic distributions

**Theorem 1** (B. 1992). Let $S$ be a real analytic submanifold of $\mathbb{R}^n$ and let $f$ be a continuous function such that

$$ (x, \xi) \notin WF_A(f) \quad \text{for every } x \in S \text{ and } \xi \text{ conormal to } S \text{ at } x. $$

Assume moreover that $f$ is flat along $S$ in the sense that $f(x) = O(dist(x,S)^m)$ for every $m$ as $dist(x,S) \to 0$.

Then $f = 0$ in some neighborhood of $S$. 

**Notation:** $N^*(S) = \{ (x, \xi); x \in S \text{ and } \xi \text{ conormal to } S \text{ at } x \}$. 
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Here \( N^*(S) = \{(x, \xi); x \in S \text{ and } \xi \text{ conormal to } S \text{ at } x\} \). Assume moreover that \( f \) is flat along \( S \) in the sense that

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**Remark 1.** If \( S \) is a hypersurface, then the flatness assumption is weaker than in Hörmander’s theorem, but the wave front assumption is stronger.
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**Remark 1.** If $S$ is a hypersurface, then the flatness assumption is weaker than in Hörmander’s theorem, but the wave front assumption is stronger.

**Remark 2.** The submanifold $S$ can have arbitrary dimension.
We don’t need to assume that $f$ is continuous, because we can formulate the flatness condition for an arbitrary distribution satisfying the wave front condition.
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**Theorem** (B. 1992). Let $S$ be a real analytic submanifold of $\mathbb{R}^n$ and let $f$ be a distribution, defined in some neighborhood of $S$, such that

$$(x, \xi) \notin WF_A(f) \quad \text{for every } (x, \xi) \in N^*(S).$$

Assume moreover that $f$ is flat along $S$ in the sense that

the restriction $\partial^\alpha f \big|_S$ vanishes on $S$ for every derivative of $f$.

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**Remark 3.** The theorem is not true for hyperfunctions (M. Sato).
A non-standard initial value problem for the wave equation.

Assume a wave motion is known with infinite precision at one point for all times. Is the wave motion uniquely determined?

The answer is YES. To prove this, let $S$ be the line in space-time $S = \{(x_0, t); t \in \mathbb{R}\}$. The assumption is that \( \partial^\alpha x u(x_0, t) = 0 \) for all $\alpha$ and $t$, so the flatness condition is fulfilled. What about the wave front condition?
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WF_A(u) \subset \text{char}(P), \quad \text{where } P \text{ is the wave operator.}
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But then we can fill the space-time with a family on non-characteristic surfaces, starting from a cylindrical surface around (finite parts of) \(S\). Hence \(u(x, t) = 0\) for all \((x, t)\).
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This argument can be applied to wave equations with variable analytic coefficients. This was done by Lebeau 1999.
The Radon transform

For continuous $f$, decaying sufficiently fast at infinity, define

$$Rf(L) = \int_L f \, ds,$$

$L$ hyperplane in $\mathbb{R}^n$,

where $ds$ is area measure on $L$. 

Weighted Radon transform:

Define

$$R\rho f(L) = \int_L f(x) \rho(L,x) \, ds,$$

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Helgason’s support theorem

**Theorem** (1965). Let $K \subset \mathbb{R}^n$ be compact and convex. Assume that $f$ is continuous and that

$$Rf(L) = 0 \quad \text{for all hyperplanes } L \text{ that do not intersect } K.$$

Thus $f = 0$ in the complement of $K$. 

[Diagram of $K$ and $L$]
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Assume moreover that

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f(x) = \mathcal{O}(|x|^{-m}) \quad \text{as } |x| \to \infty \text{ for all } m.
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The diagram shows a compact and convex set $K$ and a hyperplane $L$ that does not intersect $K$. The text describes the conditions under which the function $f$ must be zero in the complement of $K$. The notation $\mathcal{O}$ represents a Big O notation used in asymptotic analysis.
Microlocal regularity theorem for $R$

If $Rf(L) = 0$ for all $L$ in some neighborhood of $L_0$, then

$$(x, \xi) \notin WF_A(f) \quad \text{for all } x \in L_0 \text{ and } \xi \text{ conormal to } L_0.$$

In other words $N^* (L_0) \cap WF_A(f) = \emptyset$.

More generally $WF_A(f) \subset \Lambda^{-1} (WF_A(Rf))$, where $\Lambda$ is a $1^{-1}$ map $(x, \xi) \mapsto (L, \eta)$ from $T^* (\mathbb{R}^n)$ to $T^* (\mathbb{R}^n)$.

($H^1$ is the manifold of hyperplanes in $\mathbb{R}^n$.)

Combined with Hörmander's theorem this proves the support theorem for the special case when $f$ is assumed to have compact support.

These assertions are also true for $R\rho$, if $(L, x) \mapsto \rho(L, x)$ is real analytic and positive. (B. and Quinto 1987.)
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Factorable mappings

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$$R\tilde{f}(\tilde{L}) = \int_{\tilde{L}} \tilde{f}(\tilde{x}) \tilde{d}s = \int_{L} f(x) J(L, x) ds = J_0(L) \int_{L} f(x) J_1(x) ds,$$

because the Jacobian $J(L, x)$ factors

$$J(L, x) = J_0(L) J_1(x),$$

where $J_0(L)$ and $J_1(L)$ are positive and real analytic.
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See *Reconstructive integral geometry* by V. Palamodov, Section 3.1: Factorable mappings.
An extension of Helgason’s theorem

Assume again that $f$ is rapidly decaying and that $Rf(L) = 0$ for all $L$ that do not intersect $K$. 
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Assume again that $f$ is rapidly decaying and that $Rf(L) = 0$ for all $L$ that do not intersect $K$.

Make a projective transformation that takes the hyperplane at infinity to a hyperplane $L_0$. 
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Assume again that $f$ is rapidly decaying and that $R_f(L) = 0$ for all $L$ that do not intersect $K$.

Make a projective transformation that takes the hyperplane at infinity to a hyperplane $L_0$.

Since $R(J_1\tilde{f}) = 0$ for all $L$ in a neighborhood of $L_0$, we know that

$$(x, \xi) \notin WF_A(J_1\tilde{f}) \quad \text{for all } x \in L_0 \text{ and } \xi \text{ conormal to } L_0,$$
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$$(x, \xi) \notin WF_A(J_1\tilde{f}) \quad \text{for all } x \in L_0 \text{ and } \xi \text{ conormal to } L_0,$$

and hence

$$(x, \xi) \notin WF_A(\tilde{f}) \quad \text{for all } x \in L_0 \text{ and } \xi \text{ conormal to } L_0.$$
An extension of Helgason’s theorem

Assume again that \( f \) is rapidly decaying and that \( Rf(L) = 0 \) for all \( L \) that do not intersect \( K \).

Make a projective transformation that takes the hyperplane at infinity to a hyperplane \( L_0 \).

Since \( R(J_1\tilde{f}) = 0 \) for all \( L \) in a neighborhood of \( L_0 \), we know that

\[
(x, \xi) \notin WF_A(J_1\tilde{f}) \quad \text{for all } x \in L_0 \text{ and } \xi \text{ conormal to } L_0,
\]

and hence

\[
(x, \xi) \notin WF_A(\tilde{f}) \quad \text{for all } x \in L_0 \text{ and } \xi \text{ conormal to } L_0.
\]
An extension of Helgason’s theorem, cont.

\[(x, \xi) \notin WF_A(\tilde{f}) \quad \text{for all } x \in L_0 \text{ and } \xi \text{ conormal to } L_0.\]
An extension of Helgason’s theorem, cont.

\[(x, \xi) \notin WF_A(\tilde{f}) \text{ for all } x \in L_0 \text{ and } \xi \text{ conormal to } L_0.\]

By the decay assumption we know also that \(\tilde{f}\) decays fast as \(x\) approaches \(L_0\):

\[\tilde{f}(x) = \mathcal{O}(\text{dist}(x, L_0)^m) \text{ for every } m \text{ as } \text{dist}(x, L_0) \to 0,\]

Hence Theorem 1 implies that \(\tilde{f}\) must vanish in some neighborhood of \(L_0\).

So our original function \(f\) must vanish in some neighborhood of the plane at infinity, which means that it must have compact support. And then we know that it must vanish in the complement of \(K\).
An extension of Helgason’s theorem, cont.

\[(x, \xi) \notin WF_A(\tilde{f}) \quad \text{for all } x \in L_0 \text{ and } \xi \text{ conormal to } L_0.\]

By the decay assumption we know also that \(\tilde{f}\) decays fast as \(x\) approaches \(L_0\):

\[\tilde{f}(x) = O\left(\text{dist}(x, L_0)^m\right) \quad \text{for every } m \text{ as } \text{dist}(x, L_0) \to 0,\]

Hence Theorem 1 implies that \(\tilde{f}\) must vanish in some neighborhood of \(L_0\). So our original function \(f\) must vanish in some neighborhood of the plane at infinity, which means that it must have compact support.

And then we know that it must vanish in the complement of \(K\).
All of those arguments are valid for weighted Radon transforms $R_\rho$ with real analytic weight functions $\rho(L, x)$, provided that the extension of $\rho(L, x)$ to $\mathbb{P}^{n*} \times \mathbb{P}^n$ is real analytic and positive everywhere, that is, also at the hyperplane at infinity.
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Now let \( L_0 \) be the plane at infinity and \( S \) a subset of the plane at infinity. That \( f \) is flat at \( S \) then means that \( f \) decays in certain directions.

\textbf{Proposition.} Assume \( Rf(L) = 0 \) for all \( L \) that do not intersect \( K \) and that there is an open cone \( C \subset \mathbb{R}^n \) and

\[
f(x) = O(|x|^{-m}) \quad \text{for all } m \text{ as } |x| \to \infty \text{ for } x \in C.
\]

Then \( f = 0 \) in the set

\[
\bigcap_{x \in K} (x + C \cup (-C)).
\]
I denote this set by $\text{sh}_K(C)$, the shadow of $C$ (if identified with the corresponding subset of the plane at infinity) with respect to $K$. 
Assume $Rf(L) = 0$ for all $L$ that do not intersect $K$ and that $f(x)$ is rapidly decaying as $x$ approaches a subset $S$ of the hyperplane $L_0$:

$$f(x) = O\left(\text{dist}(x, S)^m\right) \quad \text{for every } m \text{ as } \text{dist}(x, S) \to 0.$$ 

By Theorem 1 it follows that $f = 0$ in some neighborhood of $S$. 
And then we can continue by means of a family of “non-characteristic” surfaces:
Also on the other side of $S$: 

$K$
Note that the points of $\text{sh}_{K}(S')$ are the points that cannot be seen from $K$, if $S$ serves as a screen and light rays are allowed to go in just one of the directions along the geodesics in $P^n$. 
Let me repeat:

All of those arguments are valid for weighted Radon transforms \( R_\rho \) with real analytic weight functions \( \rho(L, x) \), provided that the extension of \( \rho(L, x) \) to \( \mathbb{P}^{n*} \times \mathbb{P}^n \) is real analytic and positive everywhere, that is, also at the hyperplane at infinity.
Unique continuation of CR functions

Let $M$ be a real analytic submanifold of $\mathbb{C}^n$.
A function on $M$ is called a CR function if for every $x \in M$ it satisfies the Cauchy-Riemann equations with respect to all complex directions in $T_x(M)$.

Let $S \subset M$ be a real analytic submanifold of $M$. 

Theorem (Baouendi and Trèves 1988). Let $f$ be a CR function on $M$ and assume that $f$ vanishes together with all its derivatives on the real analytic submanifold $S \subset M$. Assume moreover that for every point $x \in S$ the subspaces $A_x(M)$ and $T_x(S)$ span $T_x(M)$.

Then $f$ must vanish in some neighborhood of $S$. 


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For $x \in M$ we have two subspaces of the tangent space $T_x(M)$:

- $T_x(S)$, the tangent space to $S$, and
- $A_x(M)$, the maximal complex-analytic subspace of $T_x(M)$
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**Theorem** (Baouendi and Trèves 1988). Let \( f \) be a CR function on \( M \) and assume that \( f \) vanishes together with all its derivatives on the real analytic submanifold \( S \subset M \). Assume moreover that for every point \( x \in S \)

\[ \text{the subspaces } A_x(M) \text{ and } T_x(S) \text{ span } T_x(M). \]

Then \( f \) must vanish in some neighborhood of \( S \).
Proof.

For a subspace $N$ of $T_x(M)$ we denote by $N^\perp$ the set of its conormals in $T_x^*(M)$. Then

the subspaces $A_x(M)$ and $T_x(S)$ span $T_x(M)$

is equivalent to $A_x(M)^\perp \cap T_x(S)^\perp = \emptyset$.
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And the fact that \( f \) is a CR function on \( M \) implies that

\[ WF_A(f) \subset A_x(M)^\perp. \]
Proof.

For a subspace $N$ of $T_x(M)$ we denote by $N^\perp$ the set of its conormals in $T_x^*(M)$. Then

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$$WF_A(f) \subset A_x(M)^\perp.$$  

Hence

$$WF_A(f) \cap N^*(S) = \emptyset,$$

so the assumptions of Theorem 1 are fulfilled and the assertion follows.
Proof of Theorem 1

Let us consider the case when $f$ is continuous and $S$ is a hypersurface, which we may assume to be $\{(x',0); |x'| < \gamma\}$ for some $\gamma > 0$. 
Proof of Theorem 1

Let us consider the case when $f$ is continuous and $S$ is a hypersurface, which we may assume to be $\{(x', 0); |x'| < \gamma\}$ for some $\gamma > 0$. The assumptions are (write $e_n = (0, \ldots, 0, 1)$)

$$
(x, \pm e_n) \notin WF_A(f) \quad \text{for every } x = (x', 0) \in S, \quad \text{and}
$$
$$
f(x', x_n) = O(x_n^m) \quad \text{as } x_n \to 0 \quad \text{for } (x', 0) \in S \text{ and every } m.
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We have to prove that $f = 0$ in some neighborhood of the origin. Since $(0, \pm e_n) \notin WF_A(f)$ we can choose $\psi_m \in C^\infty$ such that $\text{supp } \psi_m$ is contained in a neighborhood $U$ of the origin, $\psi_m = 1$ in a smaller neighborhood $U_0$ of the origin, and $\varepsilon > 0$, such that

$$|\hat{\psi}_m f(\xi)| \leq (Cm)^k (1 + |\xi|)^{-k}, \quad k \leq m, \quad |\xi'| < \varepsilon |\xi_n|,$$

for all $m$. 
It turns out that it is better to choose $\psi_m$ so that $\psi_m$ tends to and arbitrary test function $\varphi \in C_c^\infty(U_0)$ with convergence in the topology of $C_c^\infty$. One can show that this is possible.
It turns out that it is better to choose $\psi_m$ so that $\psi_m$ tends to and arbitrary test function $\varphi \in C^\infty_c(U_0)$ with convergence in the topology of $C^\infty_c$. One can show that this is possible.

Consider

$$h_m(x_n) = \int_{\mathbb{R}^{n-1}} \psi_m(x', x_n) f(x', x_n) dx',$$

which now depends on $\varphi$. 

One can show that the constant $C$ can be chosen independent of $\varphi$. 

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The Fourier transforms of $h_m$ satisfies

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$$\hat{h}_m(\xi_n) = \hat{\psi}_m f(0, \xi_n).$$

There are good bounds for derivatives of $h_m$, because

$$\sup |\partial^k h_m| \leq \int |\xi_n^k \hat{h}_m(\xi_n)| \, d\xi_n = \int |\xi_n^k \hat{\psi}_m f(0, \xi_n)| \, d\xi_n \leq \int |\xi_n^k| \left(\frac{(Cm)^{k+2}}{(1 + |\xi_n|)^{k+2}}\right) \, d\xi_n \leq 4(Cm)^{k+2} \quad \text{for } k + 2 \leq m.$$
It turns out that it is better to choose $\psi_m$ so that $\psi_m$ tends to and arbitrary test function $\varphi \in C^\infty_c(U_0)$ with convergence in the topology of $C^\infty_c$. One can show that this is possible.

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(1)

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One can show that the constant $C$ can be chosen independent of $\varphi$. 
Since $h_m$ is flat at $x_n = 0$ and its derivatives satisfy (1), Taylor’s formula gives

$$|h_m(x_n)| \leq \frac{\delta^{m-2}}{(m - 2)!} \sup |\partial^{m-2} h_m|$$

$$\leq \frac{\delta^{m-2}}{(m - 2)!} 4(Cm)^m \leq 4C^2 e^2 m^2 (Ce\delta)^{m-2}, \quad |x_n| < \delta.$$

Hence

$$\lim_{m \to \infty} h_m(x_n) = 0, \quad \text{if } |x_n| < \delta < 1/Ce.$$

But $h_m(x_n)$ tends to

$$h(x_n) = \int_{\mathbb R^{n-1}} \varphi(x', x_n) f(x', x_n) dx', \quad \text{as } m \to \infty,$$

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hence

$$h(x_n) = 0, \quad \text{if } |x_n| < \delta < 1/Ce.$$ 

Since this is true for all $\varphi$, we can conclude that $f(x', x_n) = 0$ for $(x', x_n) \in U_0 \text{ and } |x_n| < \delta$, which completes the proof.
Lemma. For every $m$ there exists $\phi_m \in C_c^\infty(\mathbb{R})$, even, with $\text{supp} \phi_m \subset [-1, 1]$, $\int \phi_m(x) \, dx = 1$, and

\[(2) \quad \int |\partial^k \phi_m(x)| \, dx \leq (2m)^k, \quad k \leq m.\]
Construction of $\psi_m$

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(2) $\int |\partial^k \phi_m(x)|dx \leq (2m)^k$, $k \leq m$.

**Proof.** Take $\theta(x)$ in $C^\infty$, even, with $\text{supp } \theta \subset [-1, 1]$, $\theta(x) \geq 0$, and $\int \theta(x)dx = 1$. We can find $\theta(x)$ so that $\int |\theta'(x)|dx \leq 2$. 
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\end{equation}

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$$
\phi_m(x) = m\theta(mx) * m\theta(mx) * \ldots * m\theta(mx) \quad (m \text{ factors}).
$$
Construction of $\psi_m$

**Lemma.** For every $m$ there exists $\phi_m \in C^\infty_c(\mathbb{R})$, even, with $
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Then supp $\phi_m \subset [-1, 1]$ and $\int \phi_m(x)dx = 1$. 
Construction of $\psi_m$

**Lemma.** For every $m$ there exists $\phi_m \in C_c^\infty(\mathbb{R})$, even, with $\text{supp } \phi_m \subset [-1, 1]$, $\int \phi_m(x)\,dx = 1$, and

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$$
\phi_m(x) = m\theta(mx) \ast m\theta(mx) \ast \ldots \ast m\theta(mx) \quad (m \text{ factors}).
$$

Then $\text{supp } \phi_m \subset [-1, 1]$ and $\int \phi_m(x)\,dx = 1$. Moreover, if $k \leq m$

$$
\partial^k \phi_m(x) = \underbrace{m^2 \theta'(mx) \ast \ldots \ast m^2 \theta'(mx)}_{k \text{ factors}} \ast \ldots \ast m\theta(mx).
$$

This proves (2).
Construction of $\psi_m$, cont.

**Lemma.** The functions $\phi_m$ satisfy

\[
\phi_m \to \delta_0 \quad \text{in distribution sense, hence}
\]

\[
\psi_m = \phi_m \ast \varphi \to \varphi \quad \text{in } C^\infty_c.
\]
Construction of $\psi_m$, cont.

**Lemma.** The functions $\phi_m$ satisfy

(3) \[ \phi_m \to \delta_0 \quad \text{in distribution sense, hence} \]

\[ \psi_m = \phi_m \ast \varphi \to \varphi \quad \text{in } C_c^\infty. \]

**Proof sketch.** Since \( \int \theta(x)dx = 1 \), \( \theta(x) \geq 0 \), and \( \theta(x) \) is even

\[ \hat{\theta}(\xi) = 1 - c \xi^2 + \ldots \quad \text{as} \quad \xi \to 0, \]

for some $c > 0$. 
Construction of $\psi_m$, cont.

**Lemma.** The functions $\phi_m$ satisfy

\[(3) \quad \phi_m \to \delta_0 \quad \text{in distribution sense, hence} \]
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**Proof sketch.** Since $\int \theta(x) dx = 1$, $\theta(x) \geq 0$, and $\theta(x)$ is even

\[\hat{\theta}(\xi) = 1 - c \xi^2 + \ldots \quad \text{as} \quad \xi \to 0,\]

for some $c > 0$. Hence

\[\hat{\phi_m}(\xi) = \hat{\theta}(\xi/m)^m = (1 - c \frac{\xi^2}{m^2} + \ldots)^m \to 1 \quad \text{as} \quad |\xi| \to \infty\]

uniformly on bounded sets. Since $\hat{\phi_m}$ is uniformly bounded (in fact $|\hat{\phi_m}| \leq 1$) this proves (3).
References


A. Kaneko, *Introduction to hyperfunctions*, Note 3.3.