Holmgren theorems for the Radon transform

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Holmgren's uniqueness theorem (1901):

Unique continuation across a non-characteristic hypersurface for (distribution) solutions of general linear PDE:s with analytic coefficients.

Hörmander's proof of Holmgren's theorem

Part 1. Microlocal regularity theorem for solutions of PDE:s with analytic coefficients:

$$WF_A(f) \subset WF_A(Pf) \cup \mathrm{char}(P),$$

where $\mathrm{char}(P) = \{(x,\xi);\, p_{\mathrm{pr}}(x,\xi) = 0\}$.

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where $\mathrm{char}(P) = \{(x,\xi);\, p_{\mathrm{pr}}(x,\xi) = 0\}$.

In particular, if $P(x,D)f = 0$, then

$$WF_A(f) \subset \mathrm{char}(P).$$

Hörmander's proof of Holmgren's theorem, cont.

Part 2. Unique continuation theorem for distributions satisfying an analytic wave front condition (microlocally real analytic distributions):

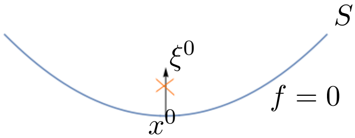
Hörmander's proof of Holmgren's theorem, cont.

Part 2. Unique continuation theorem for distributions satisfying an analytic wave front condition (microlocally real analytic distributions):

Let S be a C^2 hypersurface in $\mathbf{R}^n$. Assume that $f = 0$ on one side of S near $x^0 \in S$, and that

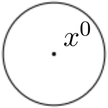
$$(x^0, \xi^0) \notin WF_A(f),$$

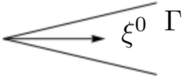
where ξ^0 is conormal to S at x^0.



Then $f = 0$ in some neighborhood of x^0.

The wave front set





$(x^0, \xi^0) \notin WF(f)$ if and only if

$\exists\, \psi \in C_c^\infty$ with $\psi(x^0) \neq 0$ and open cone $\Gamma \ni \xi^0$ such that

$$|\widehat{\psi f}(\xi)| \leq C_m (1 + |\xi|)^{-m}, \quad m = 1, 2, \ldots, \quad \xi \in \Gamma.$$

The analytic wave front set

$$(x^0, \xi^0) \notin WF_A(f) \iff$$

$\exists \psi_m \in C_c^\infty(U)$, $\psi_m = 1$ in $U_0 \ni x^0$ and open cone $\Gamma \ni \xi^0$ such that

$$|\widehat{\psi_m f}(\xi)| \leq \frac{(Cm)^k}{(1+|\xi|)^k}, \quad k \leq m,\ m = 1, 2, \ldots, \quad \xi \in \Gamma.$$

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Equivalent concept was defined for hyperfunctions with completely different methods (Sato, Kawai, Kashiwara, etc.)

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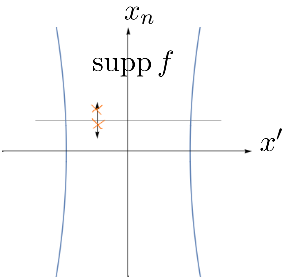
If $x' \mapsto f(x', x_n)$ is compactly supported and

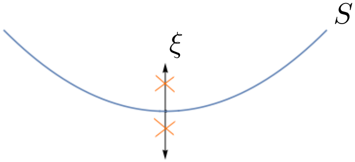
$(x, \pm e_n) \notin WF(f)$ for all x then $x_n \mapsto \int_{\mathbf{R}^{n-1}} f(x', x_n) dx'$ is C^∞.



Another unique continuation theorem for microlocally real analytic distributions

Theorem 1 (B. 1992). Let S be a real analytic submanifold of $\mathbf{R}^n$ and let f be a continuous function such that

$$(x, \xi) \notin WF_A(f) \quad \text{for every } x \in S \text{ and } \xi \text{ conormal to } S \text{ at } x.$$



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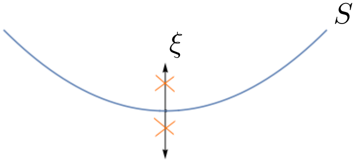

Assume moreover that f is flat along S in the sense that

$$f(x) = \mathcal{O}\big(\operatorname{dist}(x,S)^m\big) \quad \text{for every } m \text{ as } \operatorname{dist}(x,S) \to 0.$$

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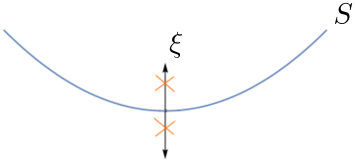

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Notation: $N^*(S) = \{(x,\xi);\, x \in S \text{ and } \xi \text{ conormal to } S \text{ at } x\}$.

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Remark 2. The submanifold S can have arbitrary dimension.

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Assume moreover that f is flat along S in the sense that

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Remark 3. The theorem is not true for hyperfunctions (M. Sato).

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The answer is YES. To prove this, let S be the line in space-time

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$$\partial_x^\alpha u(x^0, t) = 0 \quad \text{for all } \alpha \text{ and } t,$$

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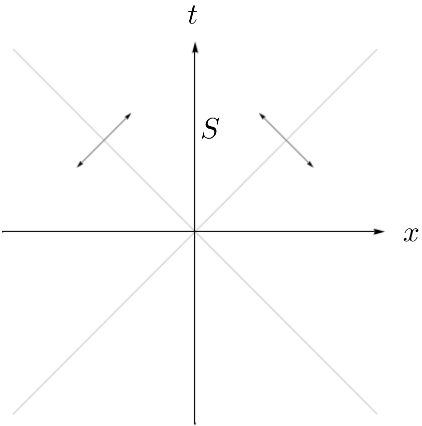
What about the wave front condition?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This argument can be applied to wave equations with variable analytic coefficients. This was done by Lebeau 1999.

The Radon transform

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Weighted Radon transform:

Define

$$R_\rho f(L) = \int_L f(x)\rho(L,x)ds, \quad L \text{ hyperplane in } \mathbf{R}^n,$$

where $\rho(L,x)$ is a smooth, positive function defined for all pairs (L,x) where $x \in L$.

Helgason's support theorem

Theorem (1965). Let $K \subset \mathbf{R}^n$ be compact and convex. Assume that f is continuous and that

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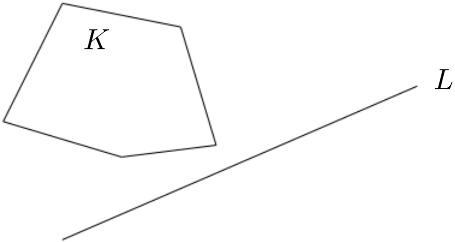

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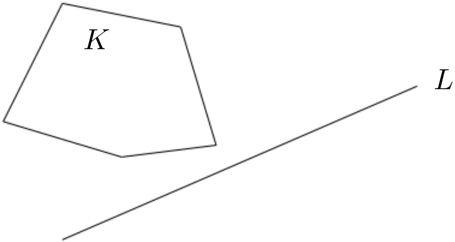

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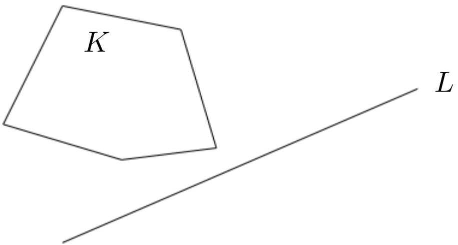

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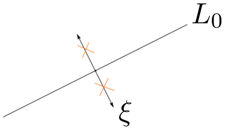
$$N^*(L_0) \cap WF_A(f) = \emptyset.$$



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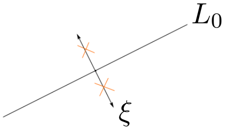

More generally

$$WF_A(f) \subset \Lambda^{-1}\big(WF_A(Rf)\big),$$

where Λ is a $1-1$ map $(x, \xi) \mapsto (L, \eta)$ from $T^*(\mathbf{R}^n)$ to $T^*(\mathcal{H}_n)$. ($\mathcal{H}_n$ is the manifold of hyperplanes in $\mathbf{R}^n$.)

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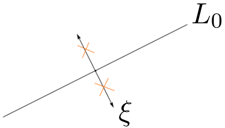

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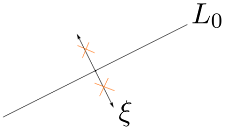

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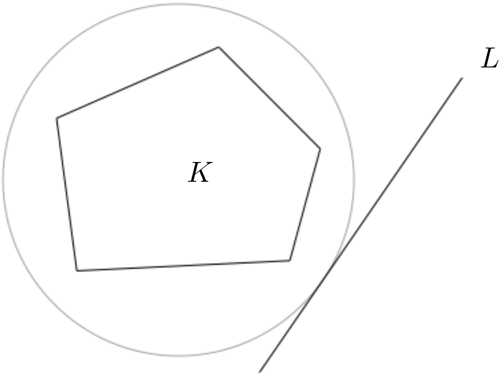
These assertions are also true for R_ρ, if $(L,x) \mapsto \rho(L,x)$ is real analytic an positive. (B. and Quinto 1987.)



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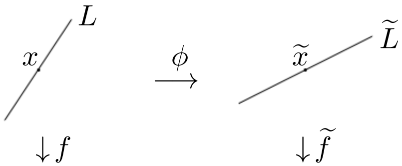
Factorable mappings



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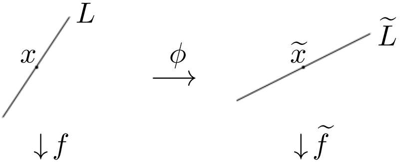

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$$R\widetilde{f}(\widetilde{L}) = \int_{\widetilde{L}} \widetilde{f}(\widetilde{x})\widetilde{ds} = \int_L f(x)J(L,x)ds = J_0(L)\int_L f(x)J_1(x)ds,$$

because the Jacobian $J(L,x)$ factors

$$J(L,x) = J_0(L)J_1(x),$$

where $J_0(L)$ and $J_1(L)$ are positive and real analytic.

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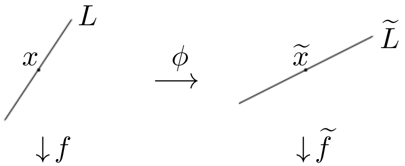

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$$R\widetilde{f}(\widetilde{L}) = \int_{\widetilde{L}} \widetilde{f}(\widetilde{x})\widetilde{ds} = \int_L f(x)J(L,x)ds = J_0(L)\int_L f(x)J_1(x)ds,$$

because the Jacobian $J(L,x)$ factors

$$J(L,x) = J_0(L)J_1(x),$$

where $J_0(L)$ and $J_1(L)$ are positive and real analytic.

See *Reconstructive integral geometry* by V. Palamodov, Section 3.1: Factorable mappings.

An extension of Helgason's theorem

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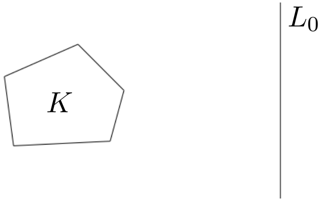

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$$\widetilde{f}(x) = \mathcal{O}\big(\operatorname{dist}(x, L_0)^m\big) \quad \text{for every } m \text{ as } \operatorname{dist}(x, L_0) \to 0,$$

Hence Theorem 1 implies that $\widetilde{f}$ must vanish in some neighborhood of L_0.

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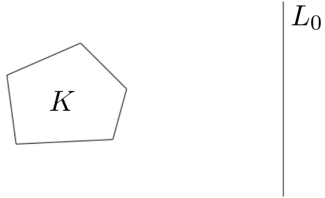
Hence Theorem 1 implies that $\widetilde{f}$ must vanish in some neighborhood of L_0. So our original function f must vanish in some neighborhood of the plane at infinity, which menas that it must have compact support.

And then we know that it must vanish in the complement of K.



All of those arguments are valid for weighted Radon transforms R_ρ with real analytic weight functions $\rho(L, x)$, provided that the extension of $\rho(L, x)$ to $\mathbf{P}^{n*} \times \mathbf{P}^n$ is real analytic and positive everywhere, that is, also at the hyperplane at infinity.

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Now let L_0 be the plane at infinity and S a subset of the plane at infinity. That f is flat at S then means that f decays in certain directions.

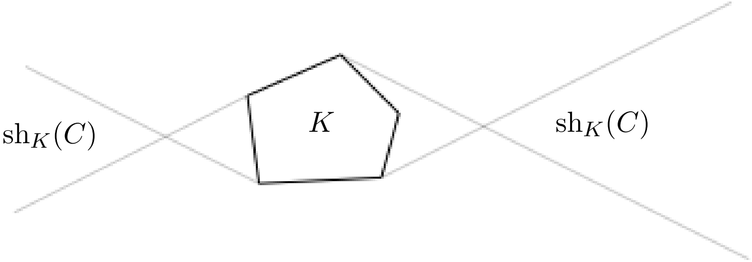
Proposition. Assume $Rf(L) = 0$ for all L that do not intersect K and that there is an open cone $C \subset \mathbf{R}^n$ and

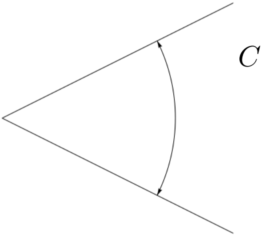
$$f(x) = \mathcal{O}(|x|^{-m}) \quad \text{for all } m \text{ as } |x| \to \infty \text{ for } x \in C.$$

Then $f = 0$ in the set

$$\bigcap_{x \in K} \big(x + C \cup (-C)\big).$$

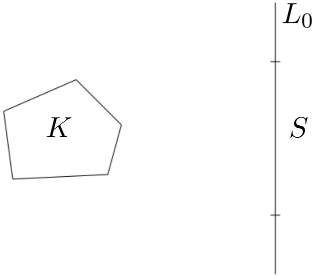
I denote this set by $\mathrm{sh}_K(C)$, the shadow of C (if identified with the corresponding subset of the plane at infinity) with respect to K.





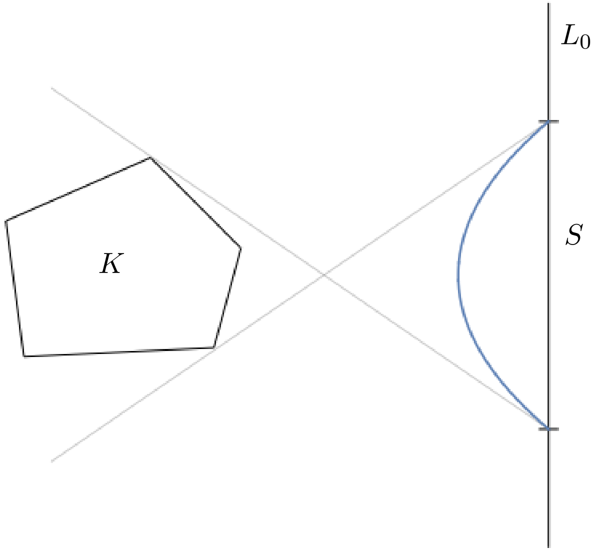
Assume $Rf(L) = 0$ for all L that do not intersect K and that $f(x)$ is rapidly decaying as x approaches a subset S of the hyperplane L_0:

$$f(x) = \mathcal{O}\big(\operatorname{dist}(x,S)^m\big) \quad \text{for every } m \text{ as } \operatorname{dist}(x,S) \to 0.$$

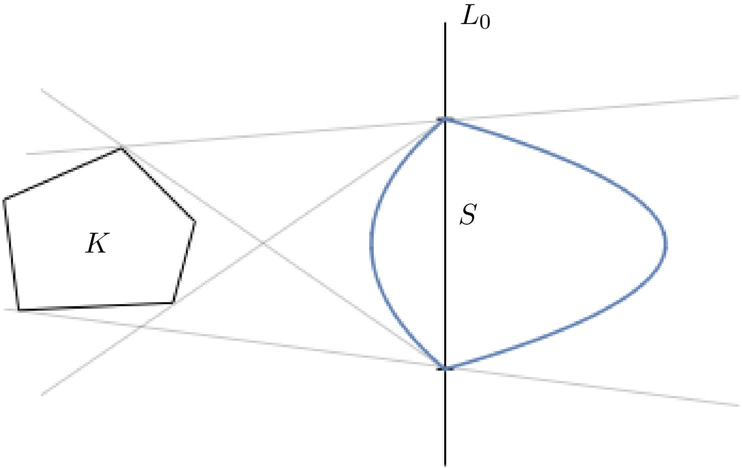


By Theorem 1 it follows that $f = 0$ in some neighborhood of S.

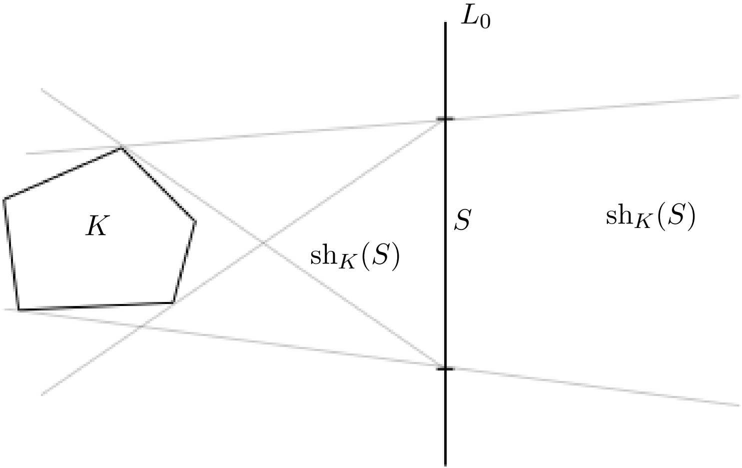
And then we can continue by means of a family of "non-characteristic" surfaces:



Also on the other side of S:



Note that the points of $\mathrm{sh}_K(S)$ are the points that cannot be seen from K, if S serves as a screen and light rays are allowed to go in just one of the directions along the geodesics in P^n.



Let me repeat:

All of those arguments are valid for weighted Radon transforms R_ρ with real analytic weight functions $\rho(L, x)$, provided that the extension of $\rho(L, x)$ to $\mathbf{P}^{n*} \times \mathbf{P}^n$ is real analytic and positive everywhere, that is, also at the hyperplane at infinity.

Unique continuation of CR functions

Let M be a real analytic submanifold of $\mathbf{C}^n$.

A function on M is called a CR function if for every $x \in M$ it satisfies the Cauchy-Riemann equations with respect to all complex directions in $T_x(M)$.

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For $x \in M$ we have two subspaces of the tangent space $T_x(M)$:

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Theorem (Baouendi and Trèves 1988). Let f be a CR function on M and assume that f vanishes together with all its derivatives on the real analytic submanifold $S \subset M$. Assume moreover that for every point $x \in S$

the subspaces $A_x(M)$ and $T_x(S)$ span $T_x(M)$.

Then f must vanish in some neighborhood of S.

Proof.

For a subspace N of $T_x(M)$ we denote by $N^\perp$ the set of its conormals in $T_x^*(M)$. Then

$$\text{the subspaces } A_x(M) \text{ and } T_x(S) \text{ span } T_x(M)$$

$$\text{is equivalent to} \quad A_x(M)^\perp \cap T_x(S)^\perp = \emptyset.$$

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Hence

$$WF_A(f) \cap N^*(S) = \emptyset,$$

so the assumptions of Theorem 1 are fulfilled and the assertion follows.

Proof of Theorem 1

Let us consider the case when f is continuous and S is a hypersurface, which we may assume to be $\{(x', 0);\ |x'| < \gamma\}$ for some $\gamma > 0$.

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$$(x, \pm e_n) \notin WF_A(f) \quad \text{for every } x = (x', 0) \in S, \quad \text{and}$$
$$f(x', x_n) = \mathcal{O}(x_n^m) \quad \text{as } x_n \to 0 \ \ \text{for } (x', 0) \in S \text{ and every } m.$$

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Since $(0, \pm e_n) \notin WF_A(f)$ we can choose $\psi_m \in C^\infty$ such that $\operatorname{supp} \psi_m$ is contained in a neighborhood U of the origin, $\psi_m = 1$ in a smaller neighborhood U_0 of the origin, and $\varepsilon > 0$, such that

$$|\widehat{\psi_m f}(\xi)| \le (Cm)^k (1 + |\xi|)^{-k}, \quad k \le m, \quad |\xi'| < \varepsilon |\xi_n|,$$

for all m.

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There are good bounds for derivatives of h_m, because

$$\begin{aligned} \sup |\partial^k h_m| &\le \int |\xi_n^k \widehat{h_m}(\xi_n)| d\xi_n = \int |\xi_n^k \widehat{\psi_m f}(0, \xi_n)| d\xi_n \\ &\le \int |\xi_n|^k \frac{(Cm)^{k+2}}{(1+|\xi_n|)^{k+2}} d\xi_n \le 4(Cm)^{k+2} \quad \text{for } k+2 \le m. \end{aligned} \tag{1}$$

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\end{aligned}$$

One can show that the constant C can be chosen independent of φ.

Since h_m is flat at $x_n = 0$ and its derivatives satisfy (1), Taylor's formula gives

$$|h_m(x_n)| \leq \frac{\delta^{m-2}}{(m-2)!} \sup |\partial^{m-2} h_m|$$
$$\leq \frac{\delta^{m-2}}{(m-2)!} 4(Cm)^m \leq 4C^2 e^2 m^2 (Ce\delta)^{m-2}, \quad |x_n| < \delta.$$

Hence

$$\lim_{m\to\infty} h_m(x_n) = 0, \quad \text{if } |x_n| < \delta < 1/Ce.$$

But $h_m(x_n)$ tends to

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Since this is true for all φ, we can conclude that $f(x', x_n) = 0$ for $(x', x_n) \in U_0$ and $|x_n| < \delta$, which completes the proof.

Construction of ψ_m

Lemma. For every m there exists $\phi_m \in C_c^\infty(\mathbf{R})$, even, with $\operatorname{supp} \phi_m \subset [-1, 1]$, $\int \phi_m(x)dx = 1$, and

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Proof. Take $\theta(x)$ in C^∞, even, with $\operatorname{supp}\theta \subset [-1,1]$, $\theta(x) \ge 0$, and $\int \theta(x)dx = 1$. We can find $\theta(x)$ so that $\int |\theta'(x)|dx \le 2$.

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Then $\operatorname{supp} \phi_m \subset [-1, 1]$ and $\int \phi_m(x)dx = 1$. Moreover, if $k \le m$

$$\partial^k \phi_m(x) = \underbrace{m^2\theta'(mx) * \ldots * m^2\theta'(mx)}_{k \text{ factors}} * \ldots * m\theta(mx).$$

This proves (2).

Construction of ψ_m, cont.

Lemma. The functions ϕ_m satisfy

$$\begin{aligned} &\phi_m \to \delta_0 \quad \text{in distribution sense, hence} \\ &\psi_m = \phi_m * \varphi \to \varphi \quad \text{in } C_c^\infty \,. \end{aligned} \tag{3}$$

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Proof sketch. Since $\int \theta(x)dx = 1$, $\theta(x) \geq 0$, and $\theta(x)$ is even

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for some $c > 0$. Hence

$$\widehat{\phi_m}(\xi) = \widehat{\theta}(\xi/m)^m = \left(1 - c\frac{\xi^2}{m^2} + \ldots\right)^m \to 1 \quad \text{as } |\xi| \to \infty$$

uniformly on bounded sets. Since $\widehat{\phi_m}$ is uniformly bounded (in fact $|\widehat{\phi_m}| \leq 1$) this proves (3).

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