# On a model of Josephson effect, dynamical systems on two-torus and double confluent Heun equations

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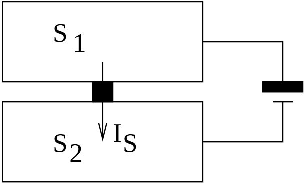
Occurs in some metals at temperature $\mathbb{T} < \mathbb{T}_{crit}$.
The critical temperature $\mathbb{T}_{crit}$ depends on the metal.
Carried by coherent **Cooper pairs** of electrons.

### Josephson effect (B.Josephson, 1962)

*Let two superconductors $S_1$, $S_2$ be separated by a very narrow dielectric, thickness $\leq 10^{-5}cm$ ($<<$ distance in Cooper pair).*
*There exists a* **supercurrent $I_S$ through the dielectric.**




**Quantum mechanics.** State of $S_j$: wave function $\Psi_j = |\Psi_j| e^{i\chi_j}$;

$$\chi_j \text{ is the } \textit{phase}, \quad \phi := \chi_1 - \chi_2.$$

**Josephson relation:** $I_S = I_c \sin\phi$, $I_c \equiv const$.

# Josephson effect

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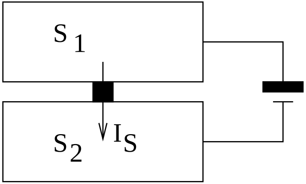


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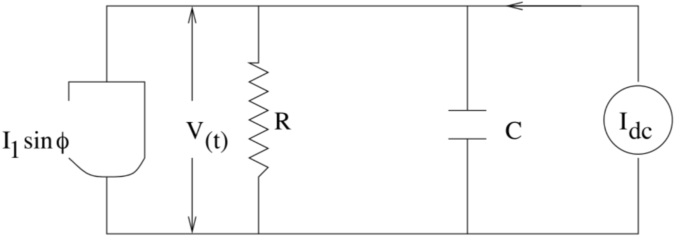
## Josephson relation

$I_S = I_c \sin\phi$, $I_c \equiv const$.

## RSJ model

$\mathbb{T} < \mathbb{T}_{crit}$, but $\frac{\mathbb{T}_{crit} - \mathbb{T}}{\mathbb{T}} << 1$.

# Equivalent circuit of real Josephson junction




See Barone, A. Paterno G. Physics and applications of the Josephson effect 1982, Figure 6.2.

This scheme is described by the equation

$$\frac{\hbar}{2e} C \frac{d^2\varphi}{dt^2} + \frac{\hbar}{2e} \frac{1}{R} \frac{d\varphi}{dt} + I_c \sin\varphi = I_{dc}$$

# Overdamped case

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$$\frac{\hbar}{2e} C \frac{d^2\varphi}{dt^2} + \frac{\hbar}{2e} \frac{1}{R} \frac{d\varphi}{dt} + I_c \sin\varphi = I_{dc}$$

$$\text{Set} \quad \tau_1 = \Omega t = \frac{2e}{\hbar} R\, I_c t$$

$$\epsilon = \frac{\hbar}{2e} \frac{C}{I_c} \left( \frac{2e}{\hbar} R\, I_c \right)^2 = \frac{2e}{\hbar} (C\, R)(R\, I_c)$$

$$\epsilon \frac{d^2\varphi}{d\tau_1^2} + \frac{d\varphi}{d\tau_1} + \sin\varphi = I_c^{-1} I_{dc}$$

**Overdamped case:** $|\varepsilon| << 1$.

In the case, when $I_c^{-1} I_{dc} = B + A\cos\omega\tau_1$, we obtain

$$\frac{d\phi}{d\tau_1} = -\sin\phi + B + A\cos\omega\tau_1 \qquad (1)$$

# Equation (1) in other domains of mathematics

In the case, when $I_c^{-1} I_{dc} = B + A \cos \omega \tau_1$, we obtain

$$\frac{d\phi}{d\tau_1} = -\sin \phi + B + A \cos \omega \tau_1. \qquad (1)$$

Equation (1) occurs in other domains of mathematics.
It occurs, e.g.,

in the investigation of some systems with non-holonomic connections by geometric methods.

It describes a model of the so-called Prytz planimeter.

Analogous equation describes the observed direction to a given point at infinity while moving along a geodesic in the hyperbolic plane.

# Reduction to a dynamical system on 2-torus

Set $\tau = \omega\tau_1$, $f(\tau) = \cos\tau$.

$$\begin{cases} \dot{\phi} = -\sin\phi + B + Af(\tau) \\ \dot{\tau} = \omega \end{cases}, \quad (\phi, \tau) \in \mathbb{T}^2 = \mathbb{R}^2/2\pi\mathbb{Z}^2. \tag{2}$$

System (2) also occurred in the work by Yu.S.Ilyashenko and J.Guckenheimer from the slow-fast system point of view.
They have obtained results on its limit cycles, as $\omega \to 0$.

Consider $\phi = \phi(\tau)$. The **rotation number of flow:**

$$\rho(B, A; \omega) = \lim_{n \to +\infty} \frac{\phi(2\pi n)}{n}, \tag{3}$$

## Problem

*Describe the rotation number of flow $\rho(B, A; \omega)$ as a function of the parameters $(B, A, \omega)$.*

# Rotation number of circle diffeomorphism

V. I. Arnold introduced rotation number for circle diffeomorphisms $g : S^1 \to S^1$.
Consider the universal covering $p : \mathbb{R} \to S^1 = \mathbb{R}/2\pi\mathbb{Z}$.
Every circle diffeomorphism $g : S^1 \to S^1$ lifts to a line diffeomorphism $G : \mathbb{R} \to \mathbb{R}$ such that

$$g \circ p = p \circ G.$$

$G$ is uniquely defined up to translations by the group $2\pi\mathbb{Z}$.
The **rotation number** of the diffeomorphism $g$:

$$\rho := \frac{1}{2\pi} \lim_{n \to +\infty} \frac{G^n(x)}{n} \tag{4}$$

It is well-defined, independent on $x$, and $\rho \in S^1 = \mathbb{R}/\mathbb{Z}$.

**Example**

Let $g(x) = x + 2\pi\theta$. Then $\rho \equiv \theta(mod\mathbb{Z})$.

# Arnold Tongues

**Properties in general case:**

$$\rho = 0 <==> g \text{ has at least one fixed point.}$$

$$\rho = \frac{p}{q} \ <==> \ g \text{ has at least one } q - \text{periodic orbit}$$

ordered similarly to an orbit of the rotation $x \mapsto x + 2\pi\frac{p}{q}$.

**Arnold family of circle diffeomorphisms:**

$g_{a,\varepsilon}(x) = x + 2\pi a + \varepsilon \sin x, \ 0 < \varepsilon < 1.$

V.I.Arnold had discovered **Tongues Effect** for given family $g_{a,\varepsilon}$:

for small $\varepsilon$ the **level set** $\{\rho = r\} \subset \mathbb{R}^2_{a,\varepsilon}$ has **non-empty interior,** if and only if $r \in \mathbb{Q}$.

He called these level sets with non-empty interiors **phase-lock areas.** Later they have been named **Arnold tongues.**

**Arnold family of circle diffeomorphisms:** $g_{a,\varepsilon}(x) = x + 2\pi a + \varepsilon \sin x$, $0 < \varepsilon < 1$.

**Arnold Tongues Effect** for given family of diffeomorphisms $g_{a,\varepsilon}$:
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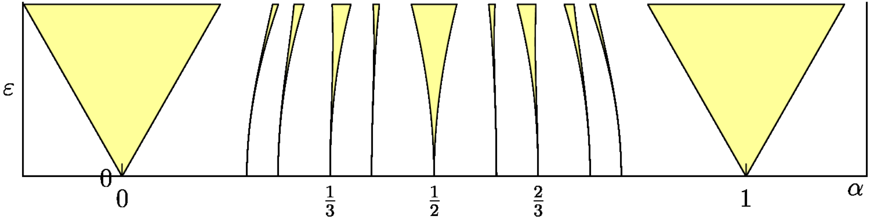
Arnold called these level sets with non-empty interiors **phase-lock areas.**

Later they have been named **Arnold tongues.**

The tongues are connected and start from $(\frac{p}{q}, 0)$.




See V. I. Arnold, Geometrical Methods in the Theory of Ordinary Differential Equations. Grundlehren der mathematischen Wissenschaften, Vol. 250, 1988, page 110, Fig. 80.

# Arnold family and dynamical system (2)

$$\begin{cases} \dot{\phi} = -\sin\phi + B + Af(\tau) \\ \dot{\tau} = \omega \end{cases}, \quad (\phi, \tau) \in \mathbb{T}^2 = \mathbb{R}^2 / 2\pi\mathbb{Z}^2. \tag{2}$$

Consider $\phi = \phi(\tau)$. The **rotation number of flow:**

$$\rho(B, A; \omega) = \frac{1}{2\pi} \lim_{n \to +\infty} \frac{\phi(2\pi n)}{n},$$

It is equivalent $(mod 1)$ to the rotation number of the flow map for the period $2\pi$.

## Problem

*How the rotation number of flow depends on $(B, A)$ with fixed $\omega$?*

The $\varepsilon$ from Arnold diffeomorphisms family corresponds to the parameter $A$ in (2).

Arnold family is a family of diffeomorphisms arbitrarily close to rotations.

The time $2\pi$ flow diffeomorphisms of the system (2) for $A = 0$ are not rotations and even **not simultaneously conjugated to rotations:**
for $A = B = 0$ we obtain $\dot{\phi} = -\sin\phi$: the flow map has attractive fixed point 0.

# Phase-lock areas for dynamical system (2)

**Phase-lock areas:** level sets $\{\rho(B,A)=r\}\subset\mathbb{R}^2_{B,A}$ with non-empty interiors. Here $\rho(B,A)=\rho(B,A,\omega)$ with fixed $\omega$.

Their picture is completely different from Arnold tongues picture.

## New effects (V.M.Buchstaber, O.V.Karpov, S.I.Tertychnyi)

1) **quantization:** phase-lock areas exist only for $r\in\mathbb{Z}$.
2) In the **initial Josephson case,** $f(\tau)=\cos\tau$**:**
- infinitely many **adjacencies** in every phase-lock area;
- a big phase-lock area for $r=0$ **based on the segment** $[-1,1]\times\{0\}$.

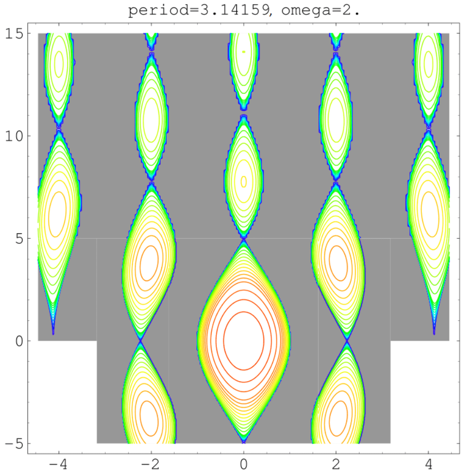
The *Shapiro step* notion is important in the theory and applications of Josephson effect.

The Shapiro steps can be estimated by the intersections of the phase-lock areas for dynamical system (2) with horizontal lines $A=const$.

# Phase-lock areas for $\omega = 2$

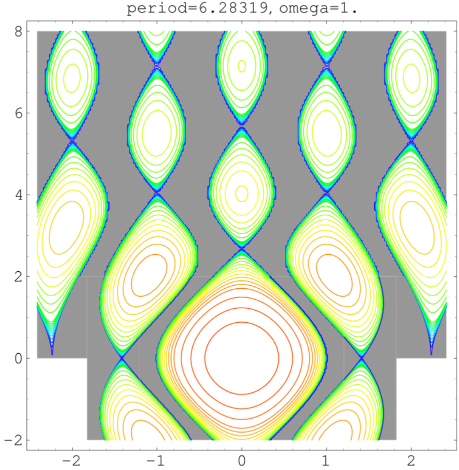
**Phase-lock areas:** level sets $\{\rho(B,A) = r\} \subset \mathbb{R}^2_{B,A}$ with non-empty interiors.

- **quantization:** phase-lock areas exist only for $r \in \mathbb{Z}$.
- for $f(\tau) = \cos\tau$: infinitely many **adjacencies** in each phase-lock area.



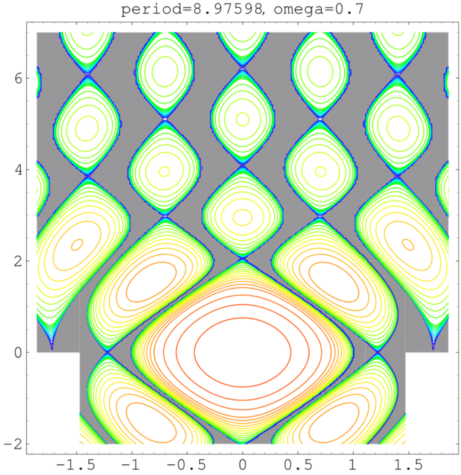
# Phase-lock areas for $\omega = 1$

**Phase-lock areas:** level sets $\{\rho(B, A) = r\} \subset \mathbb{R}^2_{B,A}$ with non-empty interiors.
- **quantization:** phase-lock areas exist only for $r \in \mathbb{Z}$.
- for $f(\tau) = \cos\tau$: infinitely many **adjacencies** in every phase-lock area.



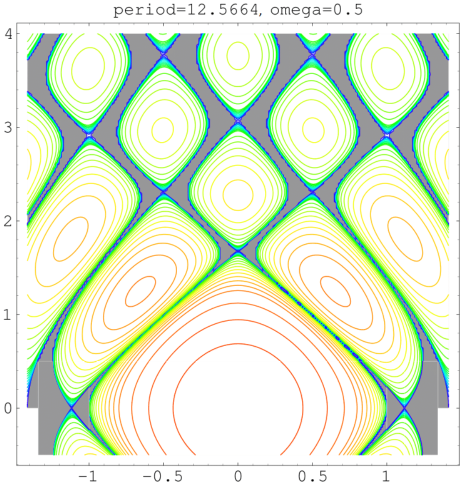
# Phase-lock areas for $\omega = 0.7$

**Phase-lock areas:** level sets $\{\rho(B, A) = r\} \subset \mathbb{R}^2_{B,A}$ with non-empty interiors.
- **quantization:** phase-lock areas exist only for $r \in \mathbb{Z}$.
- for $f(\tau) = \cos\tau$: infinitely many **adjacencies** in every phase-lock areas.



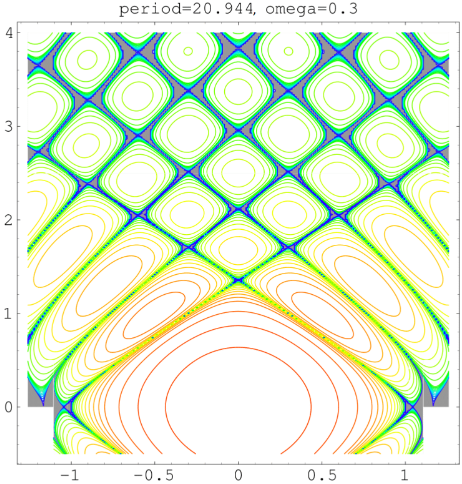
# Phase-lock areas for $\omega = 0.5$

- **quantization:** phase-lock areas exist only for $r \in \mathbb{Z}$.
- for $f(\tau) = \cos\tau$: infinitely many **adjacencies** in every phase-lock areas.



# Phase-lock areas for $\omega = 0.3$

- **quantization:** phase-lock areas exist only for $r \in \mathbb{Z}$.
- for $f(\tau) = \cos\tau$: infinitely many **adjacencies** in every phase-lock areas.



## Effect

Phase-lock areas exist only for $r \in \mathbb{Z}$.

**Proof by Riccati equation method.** Set

$$\Phi = e^{i\phi},$$

$$\frac{d\Phi}{d\tau} = \frac{1-\Phi^2}{2\omega} + \frac{i}{\omega}(B + Af(\tau))\Phi. \qquad (5)$$

It is **quadratic** in $\Phi$. This is a **projectivization of a rank 2 linear differential equation** on vector function $(u(\tau), v(\tau))$, $\Phi = \frac{v}{u}$.
**Monodromy mapping of Riccati equation (5):**

$$\overline{\mathbb{C}} \to \overline{\mathbb{C}}; \Phi(0) \mapsto \Phi(2\pi).$$

It is a **fractional-linear (Möbius) transformation** $\overline{\mathbb{C}} \to \overline{\mathbb{C}}$.
The unit circle $S^1 = \{|\Phi| = 1\}$ is invariant.

Poincaré mapping $S^1 \to S^1$ of dynamical system on torus = the **monodromy mapping** of Riccati equation (5) restricted to $S^1$.

**Main alternative for Möbius circle transformation $g$ with a periodic orbit:**

- either it is periodic: $g^n = Id$;
- or it has a fixed point.

**Main alternative implies quantization:**

Indeed, consider the rotation number $\rho(B, A)$ of the dynamical system

$$\begin{cases} \dot{\phi} = -\sin\phi + B + Af(\tau) \\ \dot{\tau} = \omega \end{cases}, \quad (\phi, \tau) \in \mathbb{T}^2 = \mathbb{R}^2/2\pi\mathbb{Z}^2. \qquad (2)$$

If $B_2 > B_1$, then $\rho(B_2, A) \geq \rho(B_1, A)$;

**strict inequality,** if either $\rho(B_1, A) \notin \mathbb{Q}$, or the time $2\pi$ flow map $g$ is periodic.

Therefore, a level set $\{\rho(B, A) = r\}$ has non-empty interior ==>

$r = \frac{p}{q}$, the time $2\pi$ flow map $g$ has a $q$-periodic orbit and is not $q$-periodic: $g^q \neq Id$.

**Main alternative** => the flow map $g$ has fixed point: $r \in \mathbb{Z}$. => **Quantization.**

# Facts on phase-lock areas

## Phase-lock areas

level sets $L_r = \{\rho(B, A) = r\} \subset \mathbb{R}^2_{B,A}$ with non-empty interiors: $r \in \mathbb{Z}$.

**Known facts on phase-lock areas for $f(\tau) = \cos\tau$.**

- boundary of phase-lock area $L_r = \{\rho = r\}$:
two graphs of functions $B = \psi_{r,\pm}(A)$,

- $\psi_{r,\pm}(A)$ have **Bessel asymptotics,** as $A \to \infty$.

Observed by Shapiro, Janus, Holly. Proved by A.V.Klimenko and O.L.Romaskevich.

- each $L_r$ is an infinite chain (garland) of domains going to infinity,
separated by points.

The separation points with $A \neq 0$ are called **adjacency points (adjacencies).** They are ordered by their $A$-coordinates:

$$\mathcal{A}_{r,1}, \mathcal{A}_{r,2}, \mathcal{A}_{r,3}, \dots .$$

- quantization of adjacencies: *all the adjacencies $\mathcal{A}_{r,k}$ lie in the line $\{B = r\omega\}$.*

Now it is conjecture based on numberical simulations (Tertychnyi, Filimonov, Kleptsyn, Schurov).
At the moment it is proved that each adjacency $\mathcal{A}_{r,k}$ lies in a line $\{B = l\omega\}$, where $0 \leq l \leq r$ and $l \equiv r(mod 2)$ (Filimonov, Glutsyuk, Kleptsyn, Schurov).

- zero phase-lock area $L_0$: for every $\omega$ its intersection with the $B$-axis is the segment $[-1, 1] \times \{0\}$;

- the picture of phase-lock areas is symmetric up-down and left-right.

# Main open questions based on numerical simulations

## Conjecture 1

Phase-lock area $L_r$, $r \in \mathbb{N}$ lies to the right from the line $\{B = \omega(r-1)\}$.

Conjecture 1 implies:

## Conjecture 2

*All adjacencies $\mathcal{A}_{r,k}$ lie in the line $\{B = r\omega\}$.*

## Question

What happens with the phase-lock area picture, as $\omega \to 0$?

# Conjectures (Buchstaber–Tertychnyi) based on numerical simulations

- The phase-lock area $L_r$ is a garland of infinitely many connected components separated by adjacencies $\mathcal{A}_{r,1}, \mathcal{A}_{r,2} \dots$ lying in the line $\{B = r\omega\}$ and ordered by their $A$-coordinates.

- For every $k \geq 2$ the $k$-th component in $L_r$ contains the interval $(A_{r,k-1}, A_{r,k})$.

- As $\omega \to 0$, for every $r$ the set $L_{r+} := L_r \cap \{A \geq \mathcal{A}_{r,1}\}$ accumulates to the $A$-axis.

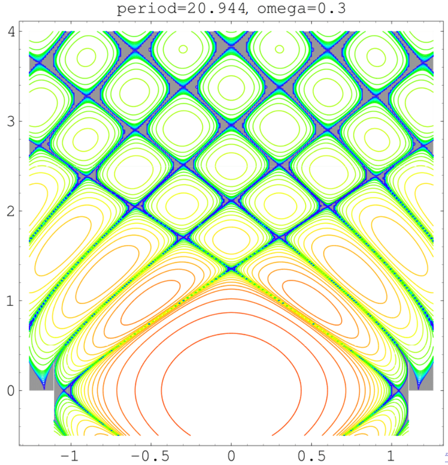
- The first adjacencies $\mathcal{A}_{r,1}$, $r = 1, 2, \dots$ of all the phase-lock areas $L_r$ lie on the same line with azimuth $\frac{\pi}{4}$.

- For every $k \in \mathbb{N}$ all the adjacencies $A_{r,k}$, $r = 1, 2, \dots,$ lie on the same line; its azimuth depends on $k$.

- The first component of the zero phase-lock area lies in the square with vertices $(0, \pm 1)$, $(\pm 1, 0)$.

# Conjectures based on numerical simulations

- For every $k \geq 2$ the $k$-th component in $L_r$ contains the interval $(A_{r,k-1}, A_{r,k})$.
- All the first adjacencies $\mathcal{A}_{r,1}$ lie on the same line with azimuth $\frac{\pi}{4}$.
- For every $k \in \mathbb{N}$ all the adjacencies $A_{r,k}$ lie on the same line; its azimuth $= \alpha(k)$.
- The first component of the zero phase-lock area lies in the square with vertices $(0, \pm 1)$, $(\pm 1, 0)$.



# Double confluent Heun equation

**Reduction to double confluent Heun equation.**

$$\frac{d\phi}{d\tau} = \frac{1}{\omega}(-\sin\phi + B + A\cos\tau), \tag{6}$$

$$z = e^{\tau}, \quad \Phi = e^{i\phi}, \ l = \frac{B}{\omega}, \ \mu = \frac{A}{2\omega}, \ \lambda = \frac{1}{4\omega^2} - \mu^2,$$

$$\frac{d\Phi}{dz} = z^{-2}((lz + \mu(z^2+1))\Phi - \frac{z}{2i\omega}(\Phi^2 - 1)).$$

This is the projectivization of system of linear equations in vector function $(u(z), v(z))$ with $\Phi = \frac{v}{u}$:

$$\begin{cases} v' = \frac{1}{2i\omega z} u \\ u' = z^{-2}(-(lz + \mu(1+z^2))u + \frac{z}{2i\omega} v) \end{cases} \tag{7}$$

# Reduction to double confluent Heun equation

Set

$$E(z) = e^{\mu z} v(z)$$

The system

$$\begin{cases} v' = \frac{1}{2i\omega z} u \\ u' = z^{-2}(-(lz + \mu(1 + z^2))u + \frac{z}{2i\omega} v) \end{cases}$$

is equivalent to **double confluent Heun equation:**

$$z^2 E'' + ((l+1)z + \mu(1 - z^2))E' + (\lambda - \mu(l+1)z)E = 0, \tag{8}$$

There exist explicit formulas expressing the solution of the non-linear equation

$$\frac{d\phi}{dt} = -\sin\phi + B + A\cos\omega t$$

via solution of equation (8) (Buchstaber - Tertychnyi).

# Familes of Heun equations

**General 6-parametric family of Heun equations**

$$z(z-1)(z-t)E''+$$
$$(c(z-1)(z-t)+dz(z-t)+(a+b+1-c-d)z(z-1))E'+$$
$$+(abz-\nu)E=0. \quad (9)$$

Four Fuchsian singularities: 0, 1, $t$, $\infty$.
Parameters: $a, b, c, d; t, \nu$.

Double confluent Heun equation

$$z^2E''+((l+1)z+\mu(1-z^2))E'+(\lambda-\mu(l+1)z)E=0,$$

is a limit of appropriate subfamily with pairs of confluenting singularities $(0,1)$, $(t,\infty)$.

# Problems and results on double confluent Heun equations

$$z^2 E'' + ((l+1)z + \mu(1-z^2))E' + (\lambda - \mu(l+1)z)E = 0,$$

This equation has two irregular non-resonant singularities at 0 and $\infty$ of Poincaré rank 1.

**Well-known problems on double confluent Heun equations.**

Find polynomial solutions.

Find entire solutions.

**Results on double confluent Heun equations.**

**adjacency** $<=>$ this equation has **entire solution** (Buchstaber, Tertychnyi). There is explicit **transcendental** equation on parameters for entire solution (Buchstaber-Tertychnyi, Buchstaber-Glutsyuk).

# Equation on parameters

Let $l \geq 0$ (reduction by symmetry).
V.M.Buchstaber, S.I.Tertychnyi: **adjacency** $<=>$ (8) has **entire solution**

$$E(z) = \sum_{k \geq 0} a_k z^k.$$

$<=>$ Explicit **transcendental** equation $\xi_l(\lambda, \mu) = 0$ on parameters (Buchstaber-Tertychnyi, Buchstaber-Glutsyuk),

$\xi_l$ is a holomorphic function on $\mathbb{C}^2$ constructed via an infinite product of explicit linear non-homogeneous matrix functions in $(\lambda, \mu^2)$.

Its construction comes from studying recurrent relations on the coefficients $a_k$ equivalent to differential equation (8): $f_k a_k + g_k a_{k-1} + h_k a_{k+1} = 0$, $g_k = k + l$.

# Results (Buchstaber-Tertychnyi)

Equation

$$z^2E'' + ((l+1)z + \mu(1-z^2))E' + (\lambda - \mu(l+1)z)E = 0, \qquad (8)$$

with $l \geq 0$ cannot have polynomial solution.

Indeed, $l \geq 0 => g_k > 0$ for all $k => a_{k-1}$ is uniquely determined by $a_k$ and $a_{k+1}$ $=>$ if $E$ is polynomial, then $E \equiv 0$.

**This equation with $l \geq 0$ replaced by $-l$:**

$$z^2\hat{E}'' + ((-l+1)z + \mu(1-z^2))\hat{E}' + (\lambda + \mu(l-1)z)\hat{E} = 0, \quad l \geq 0. \qquad (10)$$

Obtained from (8) via the transformation $\hat{E}(z) = e^{\mu(z+z^{-1})}E(-z^{-1})$.

Equation (10) has polynomial solution $<=>$ **polynomial** equation $\Delta_l(\lambda,\mu) = 0$, where $\Delta_l(\lambda,\mu)$ is the **determinant** of three-diagonal **Jacobi** ($l \times l$)**-matrix** of three-term recurrent relations equivalent to (10) on coefficients of solutions

$$\hat{E}(z) = \sum_{k\geq 0} a_k z^k, k < l.$$

# Entire and polynomial solutions

**Theorem**

*Main alternative on entire and polynomial solutions.*

(Some its version conjectured and partially studied by Buchstaber–Tertychnyi. Proved by Buchstaber–Glutsyuk).

*Equation (8) has a solution holomorphic on* $\mathbb{C}^*$ $<=>$ *so does (10)* $<=>$ *one of the two following* **incompatible** *statements holds:*

*1) either equation* **(8) has an entire solution:** $\xi_l(\lambda, \mu) = 0$ ($<=>$ **adjacency**);

*2) or equation* **(10) has a polynomial solution:** $\Delta_l(\lambda, \mu) = 0$.

2) $<=>$ *non-adjacency* **intersection** of the line $\{B = l\omega\}$ with **boundary** *of phase-lock area* $L_r$, $0 \leq r \leq l$, **parity effect:** $r \equiv l(mod 2)$**.** (Buchstaber–Glutsyuk).

$$z^2E'' + ((l+1)z + \mu(1-z^2))E' + (\lambda - \mu(l+1)z)E = 0, \quad l \geq 0. \tag{8}$$

$$z^2\hat{E}'' + ((-l+1)z + \mu(1-z^2))\hat{E}' + (\lambda + \mu(l-1)z)\hat{E} = 0, \quad l \geq 0. \tag{10}$$

**Main alternative** (Buchstaber–Tertychnyi, Buchstaber–Glutsyuk). *Equation (8) has solution holomorphic on* $\mathbb{C}^*$ $<=>$ *some of two* **incompatible** *statements holds:*

*1) either equation* **(8) has an entire solution** $<=>$ **adjacency**;

*2) or equation* **(10) has a polynomial solution** $<=>$ *non-adjacency* **point of intersection** $\{B = l\omega\} \cap \partial L_r$, $0 \leq r \leq l$, **parity effect:** $r \equiv l(mod 2)$**.**

**Main part of proof.** (10) has **polynomial** solution $=>$ (8) has **no entire** solution. Uses determinants of **modified Bessel functions** $I_j(x)$ of 1st kind:

$$e^{\frac{x}{2}(z+\frac{1}{z})} = \sum_{j=-\infty}^{+\infty} I_j(x) z^j.$$

Follows from Buchtaber–Tertychnyi results + new result on **Bessel determinants.**

**Two-sided Young diagrams:** $Y(\mathbb{Z}^l) = \{k = (k_1, \dots, k_l) \mid k_1 > \dots > k_l\} \subset \mathbb{Z}^l$.

Let $k$ and $n$ be two two-sided Young diagrams, and

$$a = (\dots, a_{-1}, a_0, a_1, \dots),$$

$$A_{k,n} = (a_{k_j - n_i})_{i,j=1,\dots,l} = \begin{pmatrix} a_{k_1-n_1} & a_{k_2-n_1} & \dots & a_{k_l-n_1} \\ a_{k_1-n_2} & a_{k_2-n_2} & \dots & a_{k_l-n_2} \\ \dots & \dots & \dots & \dots \\ a_{k_1-n_l} & a_{k_2-n_l} & \dots & a_{k_l-n_l} \end{pmatrix}.$$

The determinants $\det A_{k,n}$ form **Plücker coordinates** on the Grassmanian of $l$-subspaces in the Hilbert space $l_2$ of sequences $a$.

$$e^{\frac{x}{2}(z+\frac{1}{z})} = \sum_{j=-\infty}^{+\infty} I_j(x) z^j.$$

**Bessel determinant:** determinant $\det A_{k,n}$, where $a_j$ is the modified Bessel functions $I_j(x)$ of 1st kind.

$$a = (\dots, a_{-1}, a_0, a_1, \dots), \quad A_{k,n} = (a_{k_j - n_i})_{i,j=1,\dots,l}$$

## Theorem (Buchstaber–Glutsyuk)

*Let* $\det A_{k,n}$ *be the Bessel determinant.*
*Then* $\det A_{k,n}(x) > 0$ *for every* $x > 0$ *and for every* $l \geq 1$, $k, n \in Y(\mathbb{Z}^l)$.

**Sketch of proof.** For the Bessel determinant the **sequence function** $f(x) = (f)_k(x) = (\det A_{k,n})_k(x)$ with fixed $n$ and **discrete variable** $k$ satisfies a differential-dirrefence equation with right-hand side containing the **discrete laplacian**:

$$\frac{\partial f}{\partial x} = \Delta_{discr} f + 2lf.$$

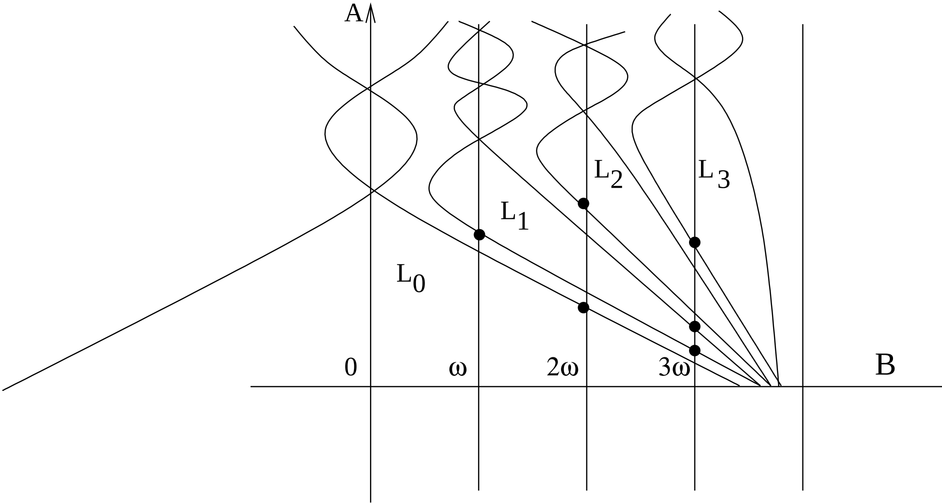
where $\Delta_{discr}$ acts on the space of functions $f = f(k)$ in $k \in \mathbb{Z}^l$ as follows.
Let $T_j$ be the shift operator:

$$(T_j f)(k) = f(k_1, \dots, k_{j-1}, k_j - 1, k_{j+1}, \dots, k_l), \ j = 1, \dots, l,$$

$$\Delta_{discr} := \sum_{j=1}^{l} (T_j + T_j^{-1} - 2). \qquad (11)$$

The positivity of the Bessel determinants is somewhat analogous to positivity of solution of heat equation with positive initial condition.

**A scheme of points corresponding to eq. (10) with polynomial solutions.**



# Simulation of points corresp. to eq. (10) with polynomial solutions

For $l \in \mathbb{N}$ set $\mathcal{P}_l \in \{B = l\omega\}$ = the point corr. to polyn. solution with maximal $A$.

**Conjecture (Buchstaber–Tertychnyi) based on simulation.**

All $\mathcal{P}_l$ lie on the same line.