Asymptotics of wave propagation and run-up in the framework of shallow water equations.

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QUASILINEAR EQUATIONS, INVERSE PROBLEMS AND THEIR APPLICATIONS

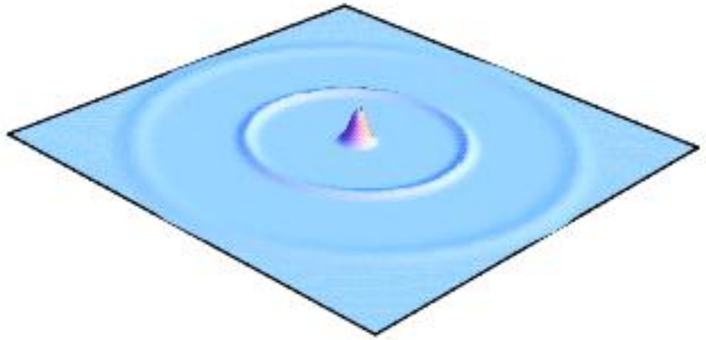
Moscow Institute of Physics and Technology, 12 -15 September 2016

PISTON MODEL for TSUNAMI PROBLEM

2-D SHALLOW WATER EQUATION

$$\frac{\partial \boldsymbol{\eta}}{\partial t} + div\big((\boldsymbol{\eta} + D(x))\mathbf{u}\big) = 0, \quad \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}, \nabla)\mathbf{u} + g\nabla\boldsymbol{\eta} = 0.$$

$$u|_{t=0} = u^0(\frac{x}{\mu}) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \eta|_{t=0} = \eta^0(\frac{x}{\mu})$$



LINEARIZATION in the OPEN OCEAN:

$\boldsymbol{\eta} << D(x)$, $|\mathbf{u}| << \sqrt{gD(x)} \Longrightarrow$

$$\frac{\partial \boldsymbol{\eta}}{\partial t} + div\big(D(x)\mathbf{u}\big) = 0, \quad \frac{\partial \mathbf{u}}{\partial t} + g\nabla\boldsymbol{\eta} = 0.$$

The main mathematical problems: focal points and profile metamorphosis

Linearization of free-boundary problem for the water waves $\Longrightarrow$

Wave equation with localized source (tsunami waves)

$\Omega \subset \mathbf{R}^2$ a domain;

$\partial\Omega$ smooth

wave propagation in Ω
from a source localized near $x_0 \in \Omega$:

$$\eta_{tt} - \langle \nabla, c^2(x)\nabla \rangle \eta = 0$$

$$\eta|_{t=0} = V(\mu^{-1}(x - x_0)), \quad \eta_t|_{t=0} = 0$$

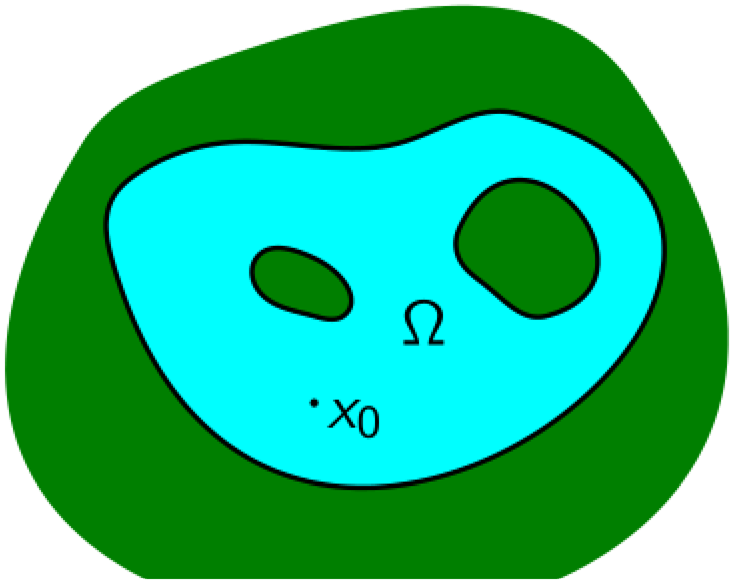
$V(y) \in C^\infty$ decays at ∞; $\mu \to 0$

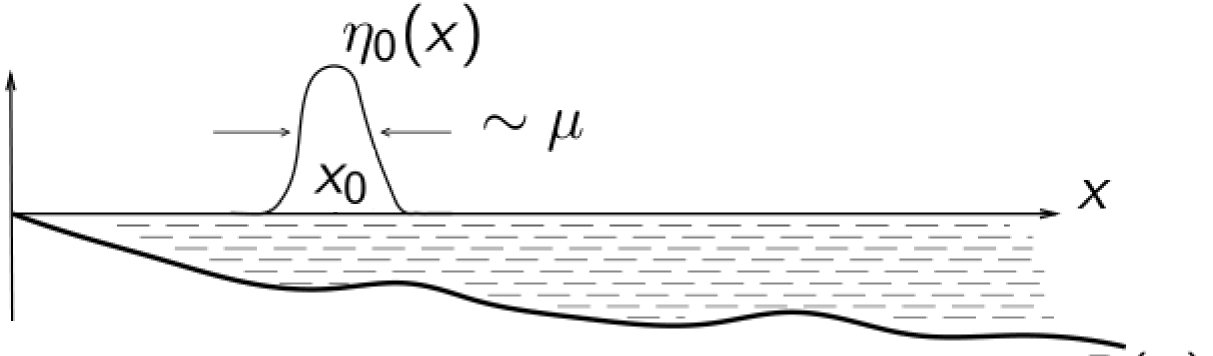
$c^2(x) \in C^\infty$; $c^2(x)|_{\partial\Omega} = 0$;

$\nabla c^2(x)|_{\partial\Omega}$ vanishes nowhere

$$\mu = \frac{\text{characteristic size of the source}}{\text{characteristic size of the basin}} \quad \text{is small}$$

Task: find asymptotic solution as $\mu \to 0$





$c(x) = \sqrt{gD(x)}$

"Missing" boundary conditions

No "classical" boundary conditions needed on $\partial\Omega$ owing to degeneration.

$\boxed{\text{Oleinik \& Radkevich 1969}}$

Instead: Friedrichs extension of $-\langle \nabla, c^2(x)\nabla \rangle$

Energy integral

$$J^2(t) = \frac{1}{2}\big(\|\eta_t\|_{L^2}^2 + \|c(x)\nabla\eta\|_{L^2}^2\big)$$

$\boxed{\text{well-posedness}}$ (standard theorem)

General

E. N. Pelinovskii, *Hydrodynamics of Tsunami Waves* (Nizhnii Novgorod, 1996)

Yu. I. Shokin, L. B. Chubarov, A. G. Marchuk, and K. V. Simonov, *Numerical Experiment in Tsunami Problem* (Nauka, Siberian Division, Novosibirsk, 1989).

V.P. Maslov, *Perturbation Theory and Asymptotic Methods* (Moscow University, Moscow, 1965; Dunod, Paris, 1972).

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Global asymptotics without run-up

S. Yu. Dobrokhotov, S. Ya. Sekerzh-Zenkovich, B. Tirozzi, and T. Ya. Tudorovskiy, "Description of a Tsunami Wave Propagation on the Ground of the Maslov Canonical Operator," Dokl. Akad. Nauk **409** (2), 171–175 (2006) [Doklady Mathematics **74** (1) 592–596 (2006)].

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M. V. Berry, "Focused Tsunami Waves," Proc. R. Soc. A **463** (2087), 3055–3071 (2007).

S.I.Kabanikhin, O.I.Krivorot'ko, Numerical algorithm of calculation of the amplitude of a tsunami wave, Sib.J.Comp.Math., 19:2 (2016), 153-165.

Run-up

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T. Vukašinac and P. Zhevandrov, "Geometric Asymptotics for a Degenerate Hyperbolic Equation," Russ. J. Math. Phys. **9** (3), 371–381 (2002).

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V.E. Nazaikinskii, "Phase Space Geometry for a Wave Equation Degenerating on the Boundary of the Domain," Mat. Zametki, **92** (1), 153–156 (2012).

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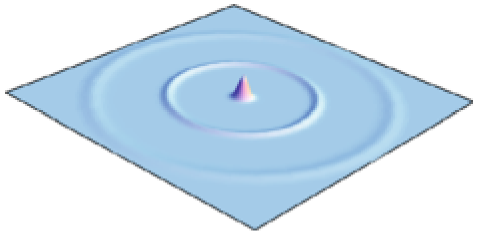
The solution is localized near a front : moving boundary layer

Wave front at time t is the projection onto $\mathbf{R}_x^2$ of the set of endpoints of trajectories of the Hamiltonian system

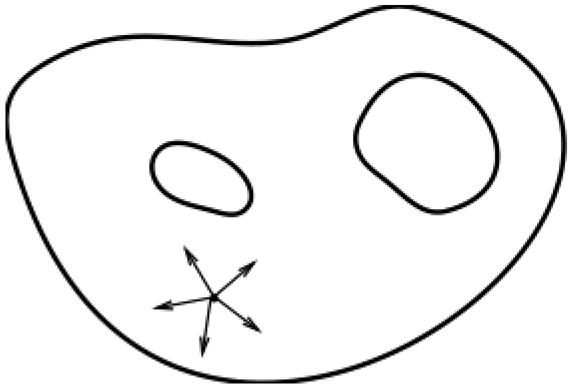
$$\dot{x} = H_p, \quad \dot{p} = -H_x, \qquad H(x,p) = c(x)\sqrt{p_1^2 + p_2^2}$$

with the initial conditions



$$x|_{t=0} = x_0, \quad p|_{t=0} = \begin{pmatrix} \cos\psi \\ \sin\psi \end{pmatrix}$$

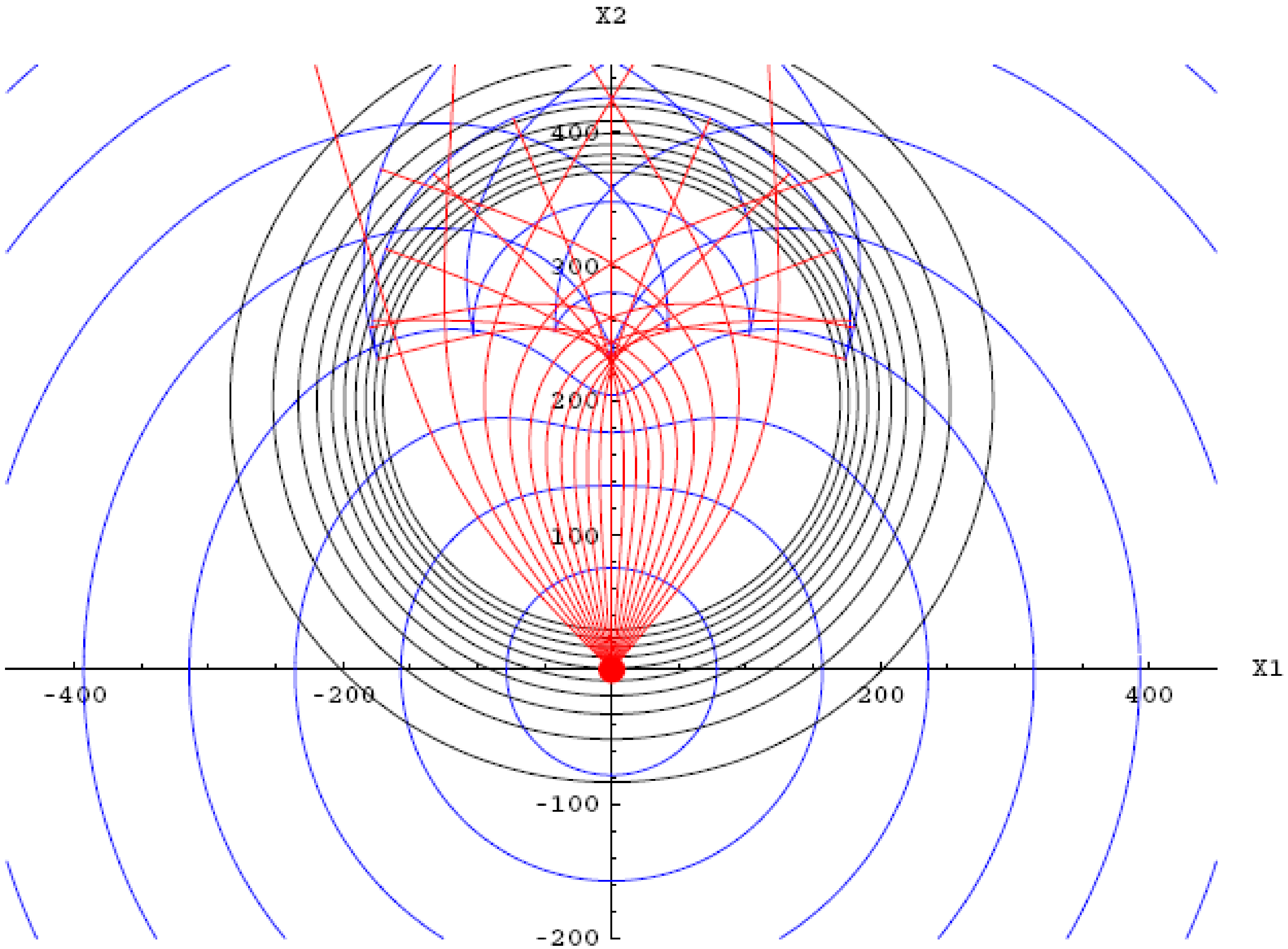
$$\psi \in [0, 2\pi)$$



Solution: $p = P(\psi, t), x = X(\psi, t)$,

For fix t the set $x = X(\psi, t)$ is the front in Ω

“Stoker ???? bottom”



The objects playing role in the description of the propagated wave:

i1) Fronts (bottom topography),

i2) the Green law $\sqrt[4]{\frac{D(x_0)}{D(x)}}$,

i3) the ray divergence $\sqrt{\frac{1}{|X_\psi|}}$,

i4) the form of the initial source $V \iff$ the wave profile

for caustics 1, 2)

i5) characteristic index of caustic

for caustics 3)

i6) angle of the beach $\frac{\partial c}{\partial \mathbf{n}}$

The wave field outside the focal points

$$\eta \approx \sqrt{\frac{\mu}{|X_\psi(\psi,t)|}} \sqrt[4]{\frac{D(x_0)}{D(x\ )}} \mathrm{Re}\Big[\mathrm{e}^{-\mathrm{i}\pi \mathrm{m}(\psi,\mathrm{t})/2} \mathrm{F}\Big(\frac{\mathrm{y}(\mathrm{x},\mathrm{t})}{\mu}\sqrt{\frac{\mathrm{D}(\mathrm{x}_0)}{\mathrm{D}(\mathrm{x}\_)}},\psi\Big)\Big|_{\psi=\psi(\mathrm{x},\mathrm{t})}\Big]$$

Here
$y(x,t)$ is the alternative distance between the point x and the closest point $X(\psi(x,t),t)$ on the front,
$\psi(x,t)$ is the correspondence angle (coordinate) on the front,
$m((\psi,t))$ is the Morse (Maslov) index of this point.

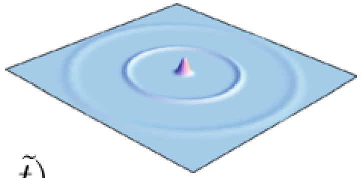
$$F(s,\psi) = \frac{e^{-i\pi/4}}{\sqrt{2\pi}} \int_0^\infty \widetilde{\eta}^{\,0}(\rho \mathbf{n}(\psi))\sqrt{\rho} e^{is\rho}\, d\rho, \qquad \widetilde{\eta}^{\,0}(k) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \eta^0(z) e^{i\langle k,z\rangle}\, dz$$

Asymmetric Dotsenko-Sergievskii-Cherkasov-Wang type source

$$\eta^0(z) = \frac{A}{(1+(z_1/B_1)^2+(z_2/B_2)^2)^{3/2}}, \qquad F(s,\psi) = \frac{Ae^{-i\frac{\pi}{4}}}{2\sqrt{2}\left(\sqrt{B_1^2\cos^2\psi + B_2^2\sin^2\psi} - is\right)^{3/2}}.$$

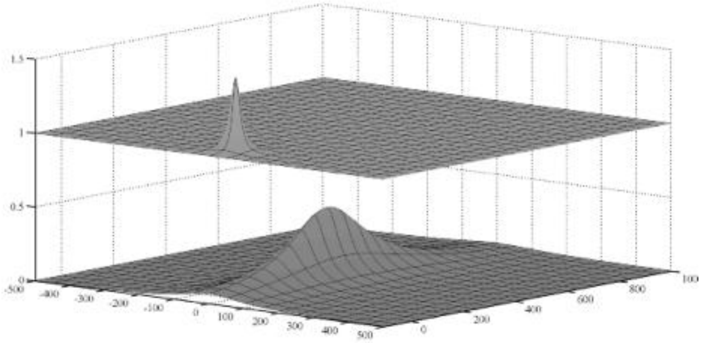
WAVE MODE: DIFFERENT TYPES OF FOCAL POINTS AND CREATION OF TRAPPED MODES BY UNDERWATER BANKS AND RIDGES

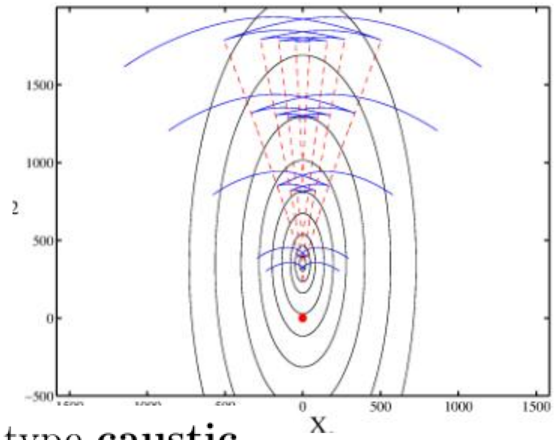
3-types of focal points:

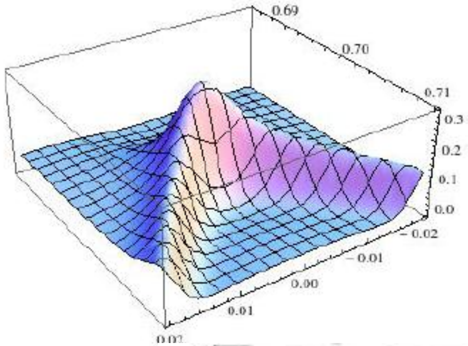


1) For $t = 0$ the front is the point $x = x_0$ ("non general position")

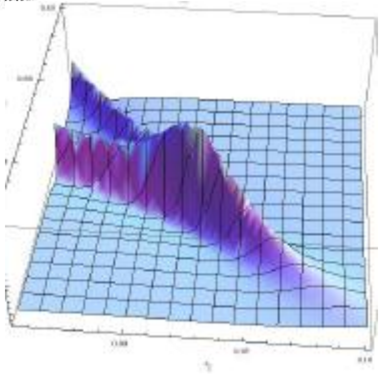
2) For some $\tilde{t}$ and $\tilde{\psi}$ the vector $\frac{\partial X}{\partial \psi}(\tilde{\psi}, \tilde{t}) = 0$, the focal point $x = X(\tilde{\psi}, \tilde{t})$ belong to 2-D space-time caustic







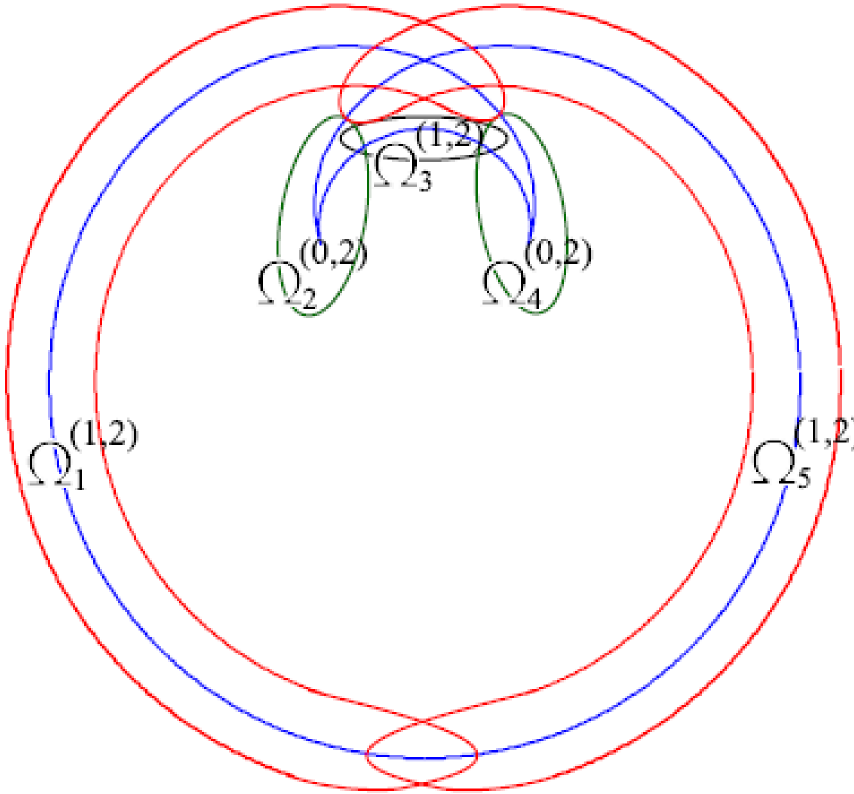
3) The **boundary** $c(x) =$ is a special type **caustic**

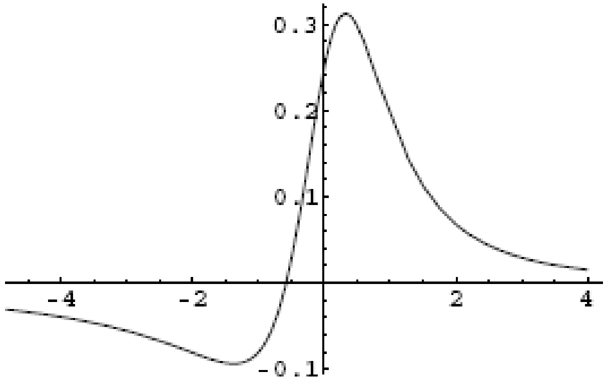


$\Longrightarrow$ The representation of the solution via the modified Maslov canonical operator:

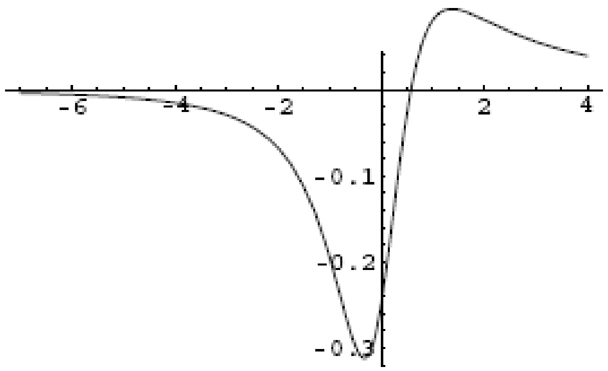
$$\eta \approx \int_0^\infty K_\Lambda^{\mu/\rho} A(t, \alpha, \psi, \rho)\, d\rho$$

+ simplification near front using boundary layer expansion



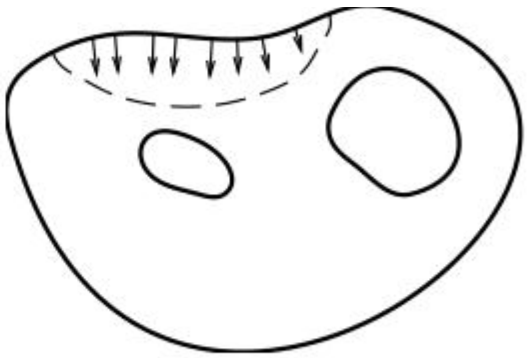


Maslov index=0

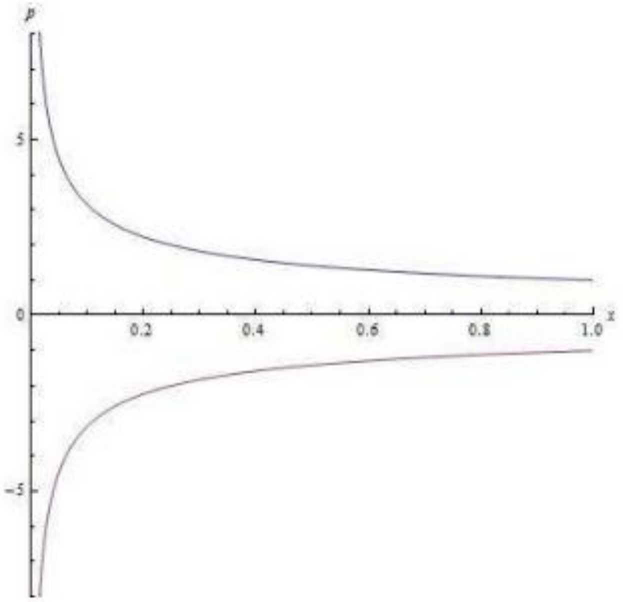


Maslov index=1

3) The **boundary** $c(x) =$ is a special type **caustic**



Change of variables: x =coordinate along the normal

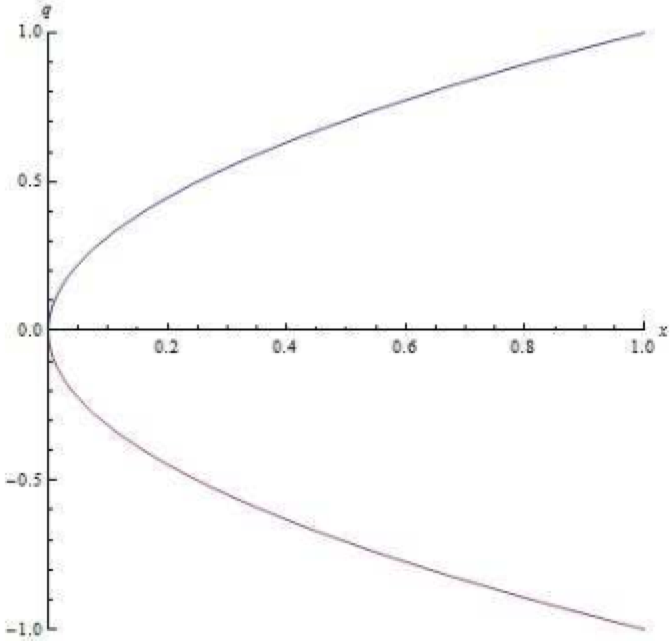


Question:

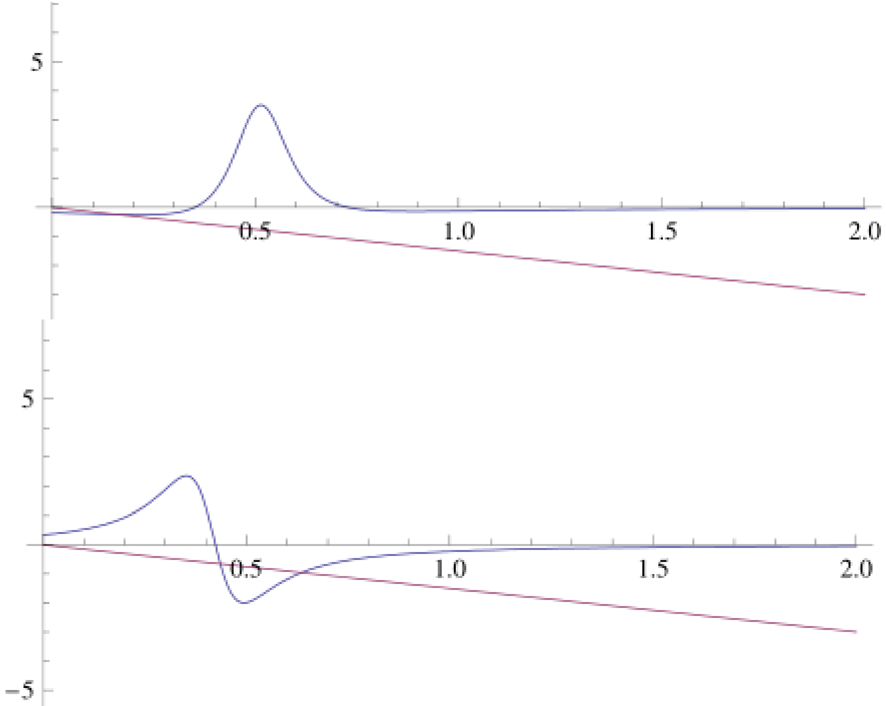
How to extend the trajectory after it hits the shore?

Answer (Vukasinič -Zhevandrov 2002; 1D case, fast oscillating solutions)

- Make the change of variables $q = 1/p$ in the Hamiltonian system
- After that, smooth continuation of the trajectories is possible



A wave before and after reflection



$\boxed{\text{derived from energy finiteness requirement}}$

In the asymptotics, this leads to known effects such as the reflection of an N-wave (Pelinovskii-Mazova, 1996)

The general case

- Make a canonical transformation:

x_1 is the coordinate along the normal
x_2 is the coordinate along the boundary

$$(x_1, p_1) \longmapsto (E, q), \qquad q = \frac{1}{p_1}, \quad E = x_1 p_1^2$$

(x_2, p_2) remain the same
Then

$$dE \wedge dq + dp_2 \wedge dx_2 = dp_1 \wedge dx_1 + dp_2 \wedge dx_2$$

(canonical transformation)

- Add points with $q = 0$

$\Longrightarrow$ new phase space $\Phi = T^*\Omega \sqcup \{q = 0\}$

• The Hamiltonian $H(x,p)$ proves to be a smooth function on Φ

• The trajectories of the Hamiltonian are infinitely extendible in time, both as $t \to \infty$ and $t \to -\infty$.

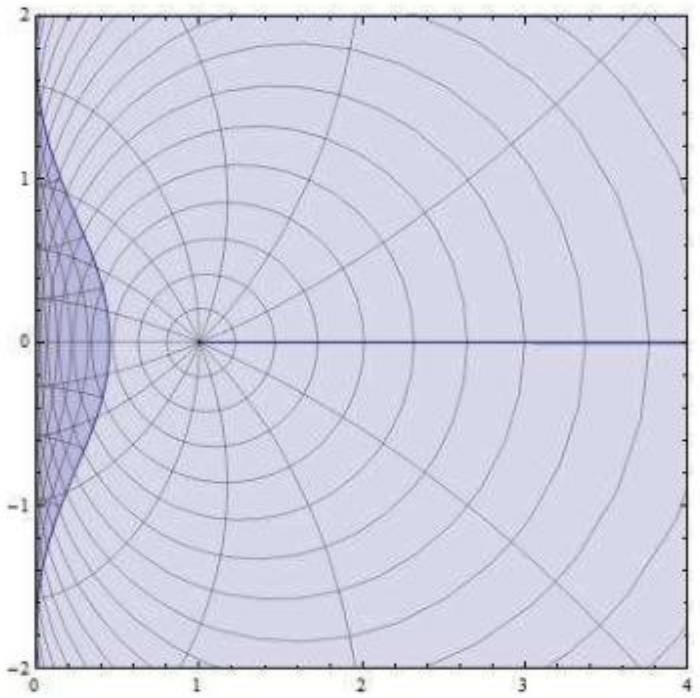
$$\boxed{\text{Reflection from the boundary} \equiv \text{passage through } q = 0}$$

The Hamiltonian system on Φ can be effectively used to construct characteristics, wave fronts, and eventually asymptotic solutions.

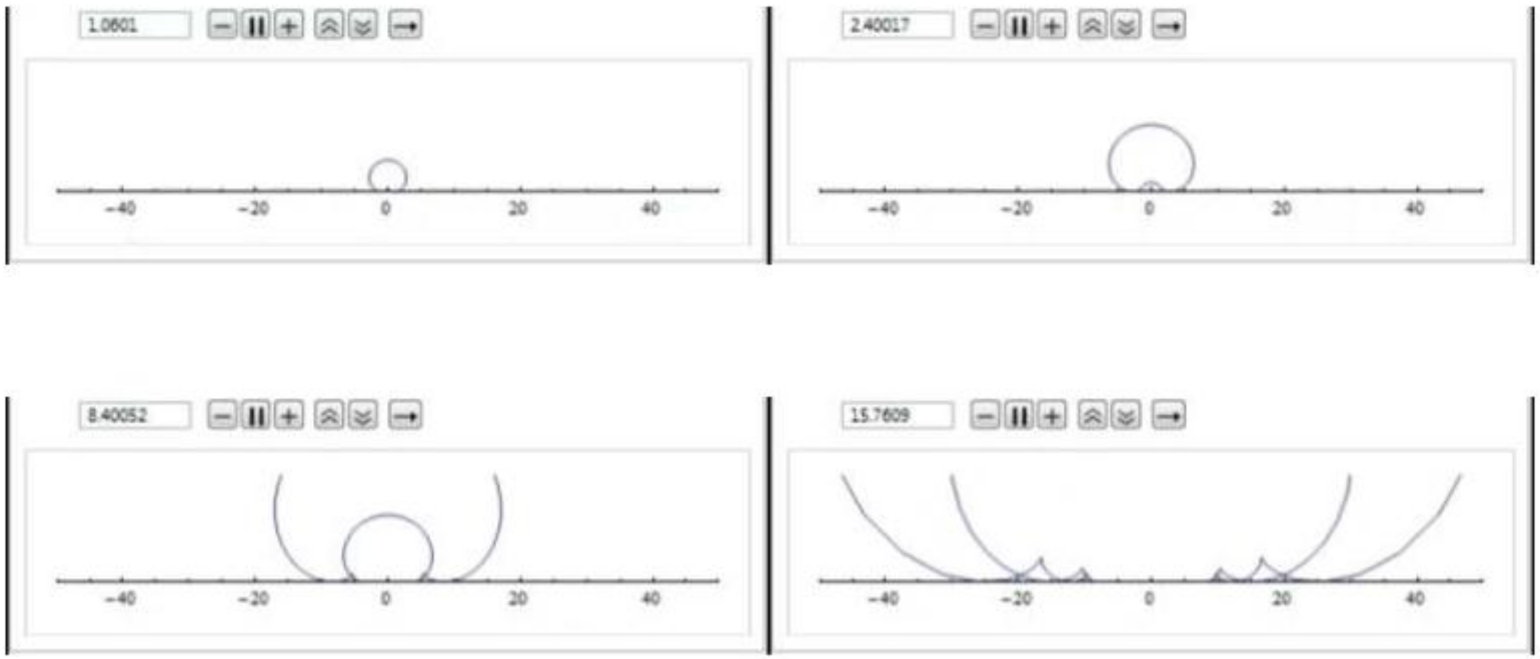
- The Hamiltonian $H(x, p)$ proves to be a smooth function on Φ
- The trajectories of the Hamiltonian are infinitely extendible in time, both as $t \to \infty$ and $t \to -\infty$.

$$\boxed{\text{Reflection from the boundary} \equiv \text{passage through } q = 0}$$

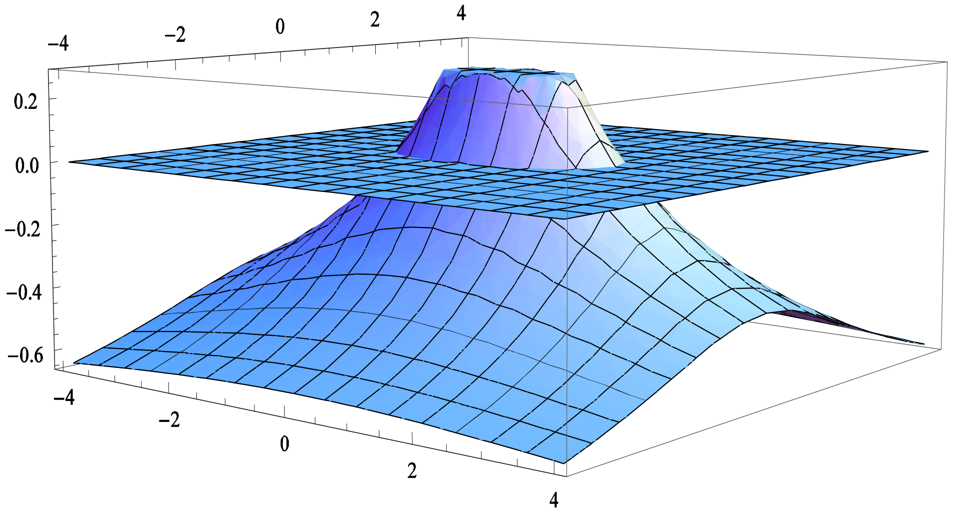
Infinite half-plane (continued): waves trapped by the shore



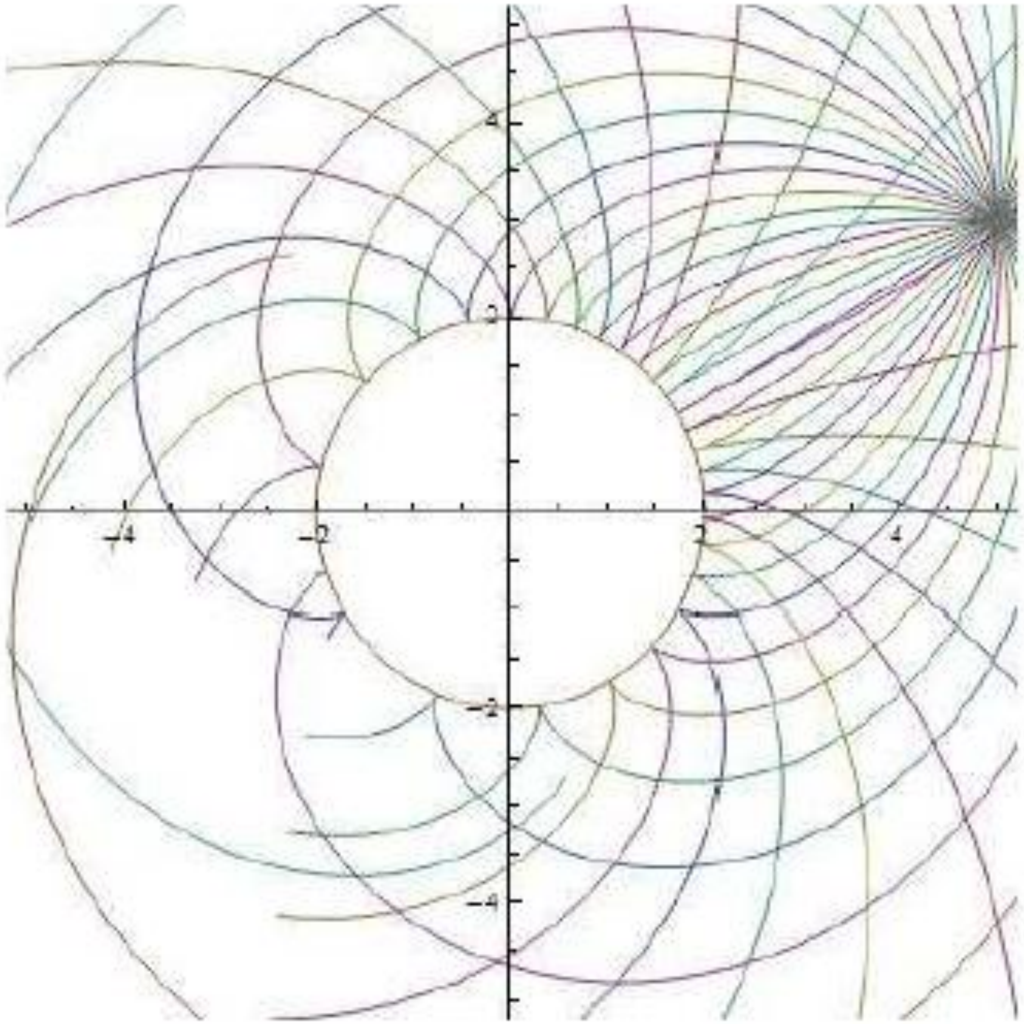
The wave front shortly after reflection from the shore $x_1 = 0$
The reflected part is shown in darker color



Time evolution of the wave front. Shore is shown by the horizontal axis



−4
−2
0
2
4
0.2
0.0
−0.2
−0.4
−0.6
−4
−2
0
2
4



4
2
−4
−2
2
4
−2
−4

Canonical transformations (Fock-Maslov-Egorov ideas)

G	$\widehat{G}^{-1}$
$(x_1, p_1) \to (p_1, -x_1)$	Inverse Fourier transform
$(p_1, -x_1) \to (q, E)$	Change of variables $q = p^{-1}$ and multiplication by $1/q$
$(q, E) \to (E, -q)$	Inverse Fourier transform
$(x_1, p_1) \to (E, -q)$	Inverse Hankel transform

Let $a = l_2/l_1$ (eccentricity). Then on the coast we have

$$\eta_{\max} = \begin{cases} \dfrac{\sqrt{2D_0}A}{\sqrt{\tan\theta}|X_\psi|^{1/2}} \dfrac{a}{\cos^2(\psi-\beta)+a^2\sin^2(\psi-\beta)} & \text{for } m = 4k, \\ \dfrac{1}{8}\dfrac{\sqrt{2D_0}A}{\sqrt{\tan\theta}|X_\psi|^{1/2}} \dfrac{a}{\cos^2(\psi-\beta)+a^2\sin^2(\psi-\beta)} & \text{for } m = 4k+2, \\ \dfrac{9}{8\sqrt{3}}\dfrac{\sqrt{2D_0}A}{\sqrt{\tan\theta}|X_\psi|^{1/2}} \dfrac{a}{\cos^2(\psi-\beta)+a^2\sin^2(\psi-\beta)} & \text{for odd } m. \end{cases}$$

CARIER-GREENSPAN TRANSFORM

THE LINEAR WAVE EQUATION for $N(\tau, y), U(\tau, y)$:

$$N_\tau + \frac{\partial}{\partial y}(\gamma^2 y U) = 0, \quad U_\tau + g N_y = 0,$$

CONSIDER the SYSTEM

$$x = y + N - \frac{1}{2}U^2, \quad t = \tau - U$$

Let it defines one-to-one map from $\{y \geq 0, \tau \in \mathbb{R}\}$ to the value area of the right hand side

THEN

$$\eta = N - \frac{1}{2}U^2, \quad v = U$$

are the solution to the **ORIGINAL NONLINEAR SYSTEM** in a **PARAMETRIC FORM**

x_1 is the coordinate along the normal to the coast

x_2 is the coordinate along the coast

We fix x_2 and make the Carrier–Greenspan transform

$$\widetilde{\eta} = \eta(x_1, x_2) - \frac{1}{2}u^2(x_1, x_2, t), \quad \widetilde{x}_1 = x_1 + \widetilde{\eta}, \quad \widetilde{t} = t - u^2(x_1, x_2, t),$$

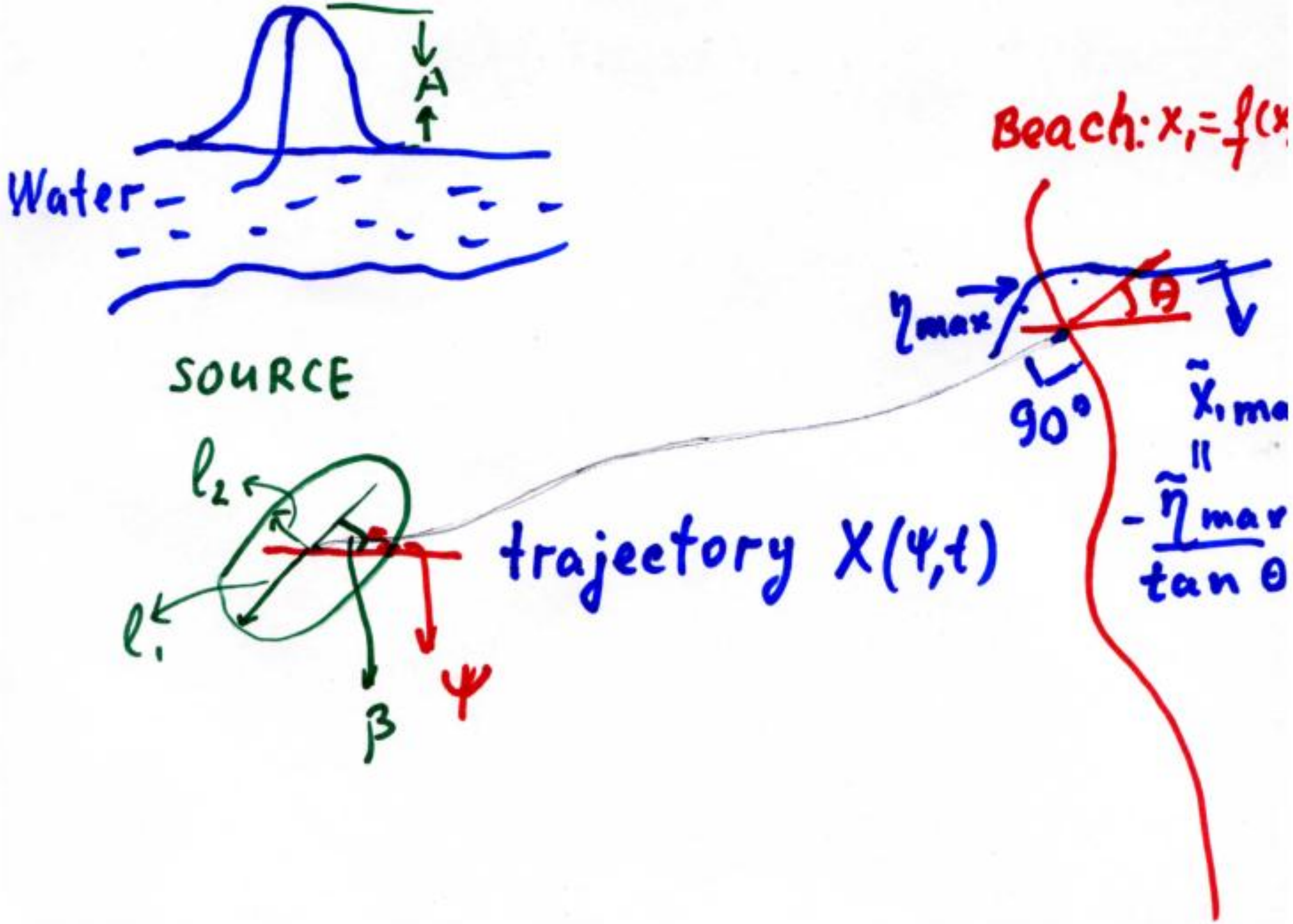
whence we find $\widetilde{\eta} = \widetilde{\eta}(x_1, x_2, t)$ and determine the uprush by solving the equation

$$\widetilde{\eta}(\widetilde{x}_1, x_2, t) + D(\widetilde{x}_1, x_2) = 0.$$

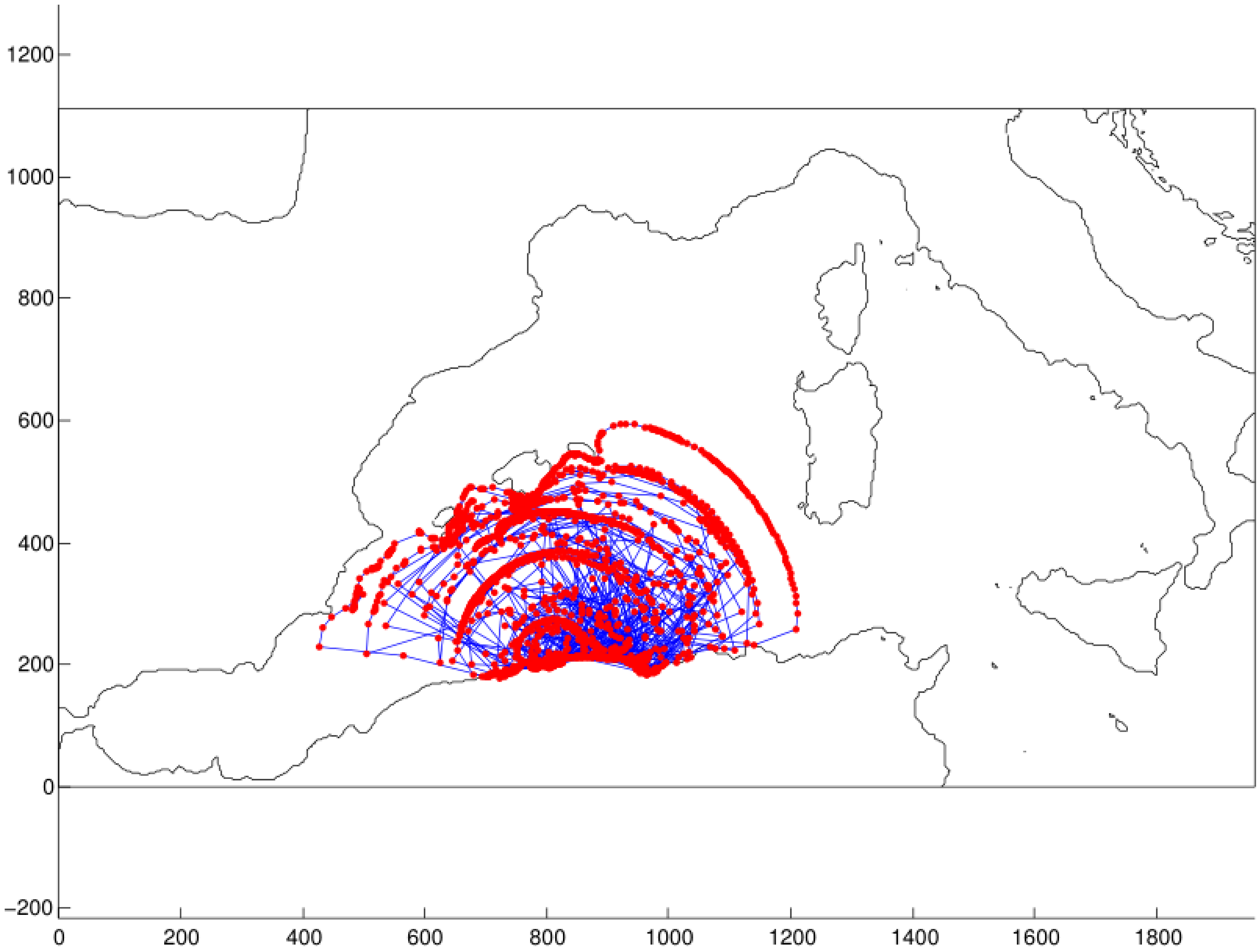
$$D \approx \gamma x_1$$

$$\tilde{x}_1 \approx -\tfrac{1}{\gamma}\tilde{\eta}(\tilde{x}_1, x_2, t)$$

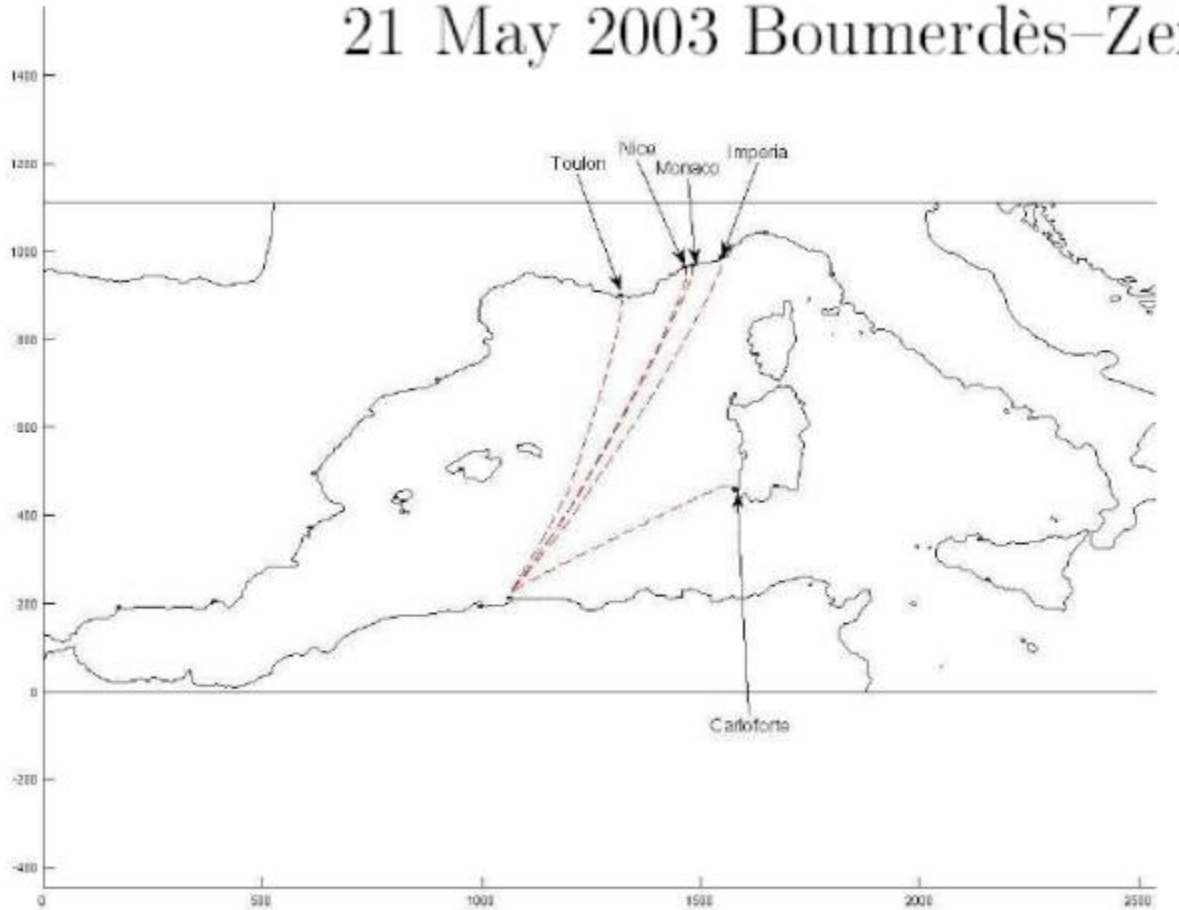
The splash


A
Water
SOURCE
l_2
l_1
β
Ψ
trajectory $X(\Psi,t)$
Beach: $x_1 = f(x$
η_{max}
θ
$90°$
$\tilde{x}_{1\,ma}$
$=$
$-\dfrac{\tilde{\eta}_{max}}{\tan\theta}$

Filtration of the front for Algerian tsunami



21 May 2003 Boumerdès–Zemmouri tsunami (Algeria)



Place	Height	
	actual	computed
Carloforte	0.08	0.08
Imperia	0.08	0.12
Monaco	0.19	0.17
Nice	0.08	0.06
Toulon	0.1	0.09

(The strike and eccentricity of the tsunami source have been taken from [A. Sahal et al., Nat. Hazards Earth Syst. Sci. **9**, 823–1834, 2009]. The tsunami wave height at Carloforte was used to compute the only remaining source parameter, the amplitude A. The actual heights have been taken from the same paper.)

SOME IDEAS

1-D case: localized solutions and short-wave oscillating solutions

$$u_{tt} = \frac{\partial}{\partial x} c^2(x) \frac{\partial}{\partial x} u, \quad u|_{t=0} = \eta^0(\frac{x}{\mu}), \quad u_t|_{t=0} = 0.$$

Whitham type asymptotics

$$u = f^+\left(\frac{S^+(x,t)}{\mu}, x, t, \mu\right) + f^-\left(\frac{S^-(x,t)}{\mu}, x, t, \mu\right),$$
$$f^\pm\left(z, x, t, \mu\right) = f_0^\pm\left(z, x, t\right) + O(\mu)$$

$$\Downarrow$$

DEFECT:
these formulas cannot be generalized for two-($n-$) D case in the situations when one has "the gradient catastrophe" $\Longleftrightarrow$ THE FOCAL POINTS and CAUSTICS exist

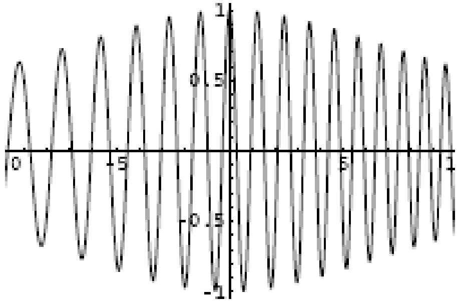
IDEA:
to pass to shortwave (semiclassical) asymptotics where the situation with focal points and caustics is well studied

ELEMENTARY IDEA N 1

The relationship with oscillating WKB-solutions and semiclassical asymptotics

$$A^{\pm}(x,t)V\Big(\frac{S^{\pm}(x,t)}{\mu}\Big) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} A^{\pm}(x,t)e^{i\frac{\rho}{\mu}S^{\pm}(x,t)}\tilde{V}(\rho)\,d\rho =$$

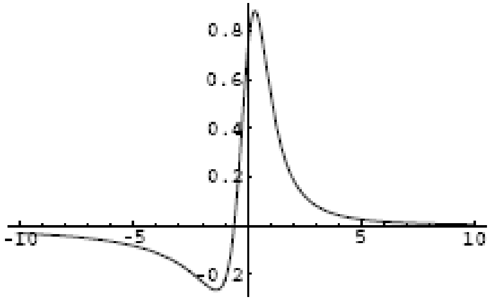
$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathbf{WKB}\big|_{h=\frac{\mu}{\rho}}\tilde{V}(\rho)\,d\rho = \frac{1}{\sqrt{2\pi}}\mathrm{Re}\int_{0}^{\infty}\mathrm{A}^{\pm}(\mathrm{x},\mathrm{t})\mathrm{e}^{\mathrm{i}\frac{\mathrm{S}^{\pm}(\mathrm{x},\mathrm{t})}{\mathrm{h}}}\big|_{\mathrm{h}=\frac{\mu}{\rho}}\tilde{\mathrm{V}}(\rho)\,\mathrm{d}\rho.$$



$$\Longrightarrow$$

semiclassical parameter: $\quad h = \dfrac{\mu}{\rho}$

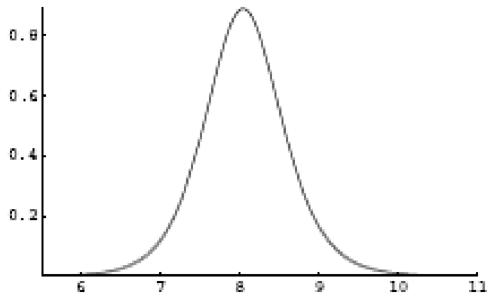
$$\tilde{V}(\rho) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} V(y)e^{i\rho y}\,dy$$



THE INTEGRAL OVER $d\rho$ COMMUTES WITH THE ORIGINAL HYPERBOLIC OPERATOR

ELEMENTARY IDEA N 2:

The "boundary layer" simplification



The solutions are localized near the fronts $x = X^{\pm}(t)$:

$$S^{\pm}(x,t)|_{x=X^{\pm}(t)} = 0$$

$$\Downarrow$$

The Taylor expansion:

$$S^{\pm} \to \frac{\partial S^{\pm}}{\partial x}(X^{\pm}(t))(x - X^{\pm}(t)) \quad \text{and} \quad A^{\pm}(x,t) \to A^{\pm}(X^{\pm}(t),t)$$

The corrections and Petrovskii estimates

h is not small when ρ is small $\Longrightarrow$ one has to *eliminate* "VERY LONG" waves corresponding to small $\rho \Longrightarrow$ **the constructive asymptotic solution exists but the constructive asymptotic expansion** DOES NOT: VERY LONG waves are out of geometrical optics and WKB.

THE CANONICAL MASLOV OPERATOR:

ONE HAS TO CHANGE
1) PHASES S^σ by LAGRANGIAN MANIFOLDS Λ_t^σ IN THE PHASE SPACE $\mathbb{R}^4_{p,x}$, LOCALLY $\Lambda_t^\sigma = \{p = \nabla S^\sigma\}$

2) THE AMPLITUDES SHOULD BE DEFINED ON Λ_t^σ

3)***GENERALIZATION*** : THE "SEMICLASSICAL PARAMETER" h SHOULD BE CHANGED BY μ/ρ and
ONE HAS TO MAKE ADDITION INTEGRATION BY $d\rho$

THUS WE HAVE

$$\eta = \int_0^\infty (\text{WKB-solution}) d\rho \qquad \Longrightarrow \eta = \sum_\sigma \int_0^\infty \{K^h_{\Lambda_t^\sigma} f_\sigma(\rho, \psi)\big|_{h=\frac{\mu}{\rho}}\} d\rho$$

SINGULARITIES OF WAVEFIELDS $\Longleftrightarrow$ SINGULARITIES OF THE PROJECTION OF THE MANIFOLD INTO THE CONFIGURATION SPACE

Local definition of the canonical operator

Canonical coordinates $(E, -q)$:

$$K\varphi = \frac{e^{i\pi/2}}{h} \int \mathbf{J}_0\left(\frac{2}{h}\sqrt{Ex_1}\right) e^{\frac{i}{h}S(E,x_2)} \frac{\varphi}{\sqrt{J_E}} dE$$

$\mathbf{J}_0(z)$ is the Bessel function of order zero

Canonical coordinates (q, E):

$$K\varphi = \frac{e^{i\pi/4}}{h^{3/2}\sqrt{2\pi}} \int \mathbf{J}_0\left(\frac{2}{h}\sqrt{Ex_1}\right) e^{\frac{i}{h}(S(q,x_2)-qE)} \frac{\varphi}{\sqrt{J_q}} dqdE$$

The wave amplitude on the shore $x_1 = f(x_2)$ for the special source

The free surface elevation at the shore $x_1 = f(x_2)$ is given by

$$\eta(f(x_2), x_2, t) \asymp \frac{\sqrt{2D_0}A}{\sqrt{\tan\theta}|X_\psi(T(x_2), \psi(0, x_2))|^{1/2}} \operatorname{re} \frac{e^{-i\pi m/2} l_1 l_2}{(l - ic_0(T(x_2) - t))^2}$$

- l_j $(= b_j\mu)$ are the semiaxes of the source
- $l = \sqrt{l_1^2 \cos^2(\psi - \beta) + l_2^2 \sin^2(\psi - \beta)}$
- D_0 is the source depth, $c_0 = \sqrt{gD_0}$
- A is the amplitude at the source
- m is the Maslov index (the number of focal points on the trajectory arriving at the point $(f(x_2), x_2)$)
- $\tan\theta$ is the bottom slope near the considered point of the beach

Let $a = l_2/l_1$ (eccentricity). The tsunami amplitude on the shore is

$$\eta_{\max} = \begin{cases} \frac{\sqrt{2D_0}A}{\sqrt{\tan\theta}|X_\psi|^{1/2}} \frac{a}{\cos^2(\psi-\beta)+a^2\sin^2(\psi-\beta)} & \text{for even } m, \\ \frac{9}{8\sqrt{3}} \frac{\sqrt{2D_0}A}{\sqrt{\tan\theta}|X_\psi|^{1/2}} \frac{a}{\cos^2(\psi-\beta)+a^2\sin^2(\psi-\beta)} & \text{for odd } m. \end{cases}$$

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THANK YOU FOR YOUR ATTENTION!