On integration of the equations of incompressible fluid flow

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1 The Euler system of equations

The Euler system of equations describing flows of incompressible fluids without viscosity

$$\vec{V}_t + (\vec{V} \cdot \vec{\nabla})\vec{V} + \vec{\nabla} P = 0, \qquad (\nabla \cdot \vec{V}) = 0, \tag{1}$$

in the variables $U(x,y,z,t) = a + U(x-at, y-bt, z-ct)$, $V(x,y,z,t) = b + V(x-at, y-bt, z-ct)$, $W(x-at, y-bt, z-ct)$, $P(x,y,z,t) = P_0 + P(x-at, y-bt, z-ct)$, where a, b, c are the constants, takes the form

$$UU_\xi + VU_\eta + WU_\chi + P_\xi = 0,\; UV_\xi + VV_\eta + WV_\chi + P_\eta = 0,$$

$$UW_\xi + VW_\eta + WW_\chi + P_\chi = 0,$$

$$U_\xi + V_\eta + W_\chi = 0, \tag{2}$$

where $\xi = x - at$, $\eta = y - bt$, $\chi = z - ct$ are the new variables.

Studying of the system (2) allow as to formulate the following theorems

Theorem 1. *Non singular and non stationary periodic solution of the system (1) has the form*

$$U(x,y,z,t) = a + A\sin(z-ct) + C\cos(y-bt),$$

$$V(x,y,z,t) = b + B\sin(x-at) + A\cos(z-ct),$$

$$W(x,y,z,t) = c + C\sin(y-bt) + B\cos(x-at),$$

$$P(x,y,z,t) = 1/2\,CB\sin(-y+bt+x-at) - 1/2\,CB\sin(y-bt+x-at) -$$

$$-1/2\,BA\sin(-z+ct+x-at) - 1/2\,BA\sin(z-ct+x-at) +$$

$$+1/2\,AC\sin(-z+ct+y-bt) - 1/2\,AC\sin(z-ct+y-bt) + \_F2(t), \tag{3}$$

which is is generalization of the famous stationary $ABC - flow$ [1].

Theorem 2. *Non singular and non stationary solution of the system (1) has the form*

$$U(x,y,z,t) = a + 1/2\,C_1\,\sin(y-bt) + 1/2\,E_1\,\cos(y-bt) + 1/2\,F_1\,\cos(z-ct) +$$

$$+1/2\,H_1\,\sin(z-ct),$$

$$V(x,y,z,t) = b + 1/2\,F_1\,\sin(z-ct) - 1/2\,H_1\,\cos(z-ct) - A_1\,\sin(x-at) +$$

$$+B_3\,\cos(x-at),$$

$$W(x,y,z,t) = c + A_3\ \cos(x-at) + B_3\ \sin(x-at) +$$
$$+1/2\,C_1\ \cos(y-bt) - 1/2\,E_1\ \sin(y-bt),$$
$$P(x,y,z,t) =$$
$$-1/4\,A_3\,C_1\ \cos(y-bt+x-at) - 1/4\,A_3\,C_1\ \cos(-y+bt+x-at) -$$
$$-1/4\,A_3\,E_1\ \sin(-y+bt+x-at) + 1/4\,A_3\,E_1\ \sin(y-bt+x-at) -$$
$$-1/4\,B_3\,C_1\ \sin(-y+bt+x-at) - 1/4\,B_3\,C_1\ \sin(y-bt+x-at) -$$
$$-1/4\,A_3\,F_1\ \cos(z-ct+x-at) + 1/4\,A_3\,F_1\ \cos(-z+ct+x-at) -$$
$$-1/4\,B_3\,E_1\ \cos(y-bt+x-at) + 1/4\,B_3\,E_1\ \cos(-y+bt+x-at) -$$
$$-1/4\,A_3\,H_1\ \sin(-z+ct+x-at) - 1/4\,A_3\,H_1\ \sin(z-ct+x-at) + +$$
$$+1/4\,B_3\,F_1\ \sin(-z+ct+x-at) - 1/4\,B_3\,F_1\ \sin(z-ct+x-at) +$$
$$+1/8\,H_1\,E_1\ \sin(-z+ct+y-bt) - 1/8\,H_1\,E_1\ \sin(z-ct+y-bt) +$$
$$+1/4\,B_3\,H_1\ \cos(z-ct+x-at) + 1/4\,B_3\,H_1\ \cos(-z+ct+x-at) +$$
$$+1/8\,H_1\,C_1\ \cos(z-ct+y-bt) - 1/8\,H_1\,C_1\ \cos(-z+ct+y-bt) -$$
$$-1/8\,F_1\,E_1\ \cos(z-ct+y-bt) - 1/8\,F_1\,E_1\ \cos(-z+ct+y-bt) -$$
$$-1/8\,F_1\,C_1\ \sin(-z+ct+y-bt) - 1/8\,F_1\,C_1\ \sin(z-ct+y-bt) + F_2(t),$$

where A_i, B_i, C_i, F_i, H_i *are the arbitrary constants.*

Theorem 3. *Non singular one-soliton solution of the system (1) has the form*

$$U(\vec{x},t) = a + \frac{e^{\alpha\,x-\alpha\,at+\beta\,y-\beta\,bt+\delta\,z-\delta\,ct}\,(\beta-\delta)}{1+2\,e^{\alpha\,x-\alpha\,at+\beta\,y-\beta\,bt+\delta\,z-\delta\,ct}+e^{2\,\alpha\,x-2\,\alpha\,at+2\,\beta\,y-2\,\beta\,bt+2\,\delta\,z-2\,\delta\,ct}},$$

$$V(\vec{x},t) = b - \frac{e^{\alpha\,x-\alpha\,at+\beta\,y-\beta\,bt+\delta\,z-\delta\,ct}\left(\delta\,\beta+\delta^2+\alpha^2\right)}{\alpha\,\left(1+2\,e^{\alpha\,x-\alpha\,at+\beta\,y-\beta\,bt+\delta\,z-\delta\,ct}+e^{2\,\alpha\,x-2\,\alpha\,at+2\,\beta\,y-2\,\beta\,bt+2\,\delta\,z-2\,\delta\,ct}\right)},$$

$$W(\vec{x},t) = c + \frac{e^{\alpha\,x-\alpha\,at+\beta\,y-\beta\,bt+\delta\,z-\delta\,ct}\left(\alpha^2+\beta^2+\delta\,\beta\right)}{\alpha\,\left(1+2\,e^{\alpha\,x-\alpha\,at+\beta\,y-\beta\,bt+\delta\,z-\delta\,ct}+e^{2\,\alpha\,x-2\,\alpha\,at+2\,\beta\,y-2\,\beta\,bt+2\,\delta\,z-2\,\delta\,ct}\right)},$$

where (α,β,δ) *are parameters.*

Starting on such type of solution, N- soliton solutions of the system (1) can be constructed with the help of $3D$ -analogue of the Bäcklund transformation.

Theorem 4. *The two-soliton solution of the system (1) has the form*

$$U(\vec{x},t)-a=$$

$$=-\frac{(\delta^2\beta+\delta\,\alpha^2+\beta\,\alpha^2+\beta^3+\delta^3+\delta\,\beta^2)\left(e^{\alpha\,(x-at)+\beta\,(y-bt)+\delta\,(z-ct)}-1\right)e^{\alpha\,(x-at)+\beta\,(y-bt)+\delta\,(z-ct)}}{\left(1+e^{\alpha\,(x-at)+\beta\,(y-bt)+\delta\,(z-ct)}\right)\left(1+2\,e^{\alpha\,(x-at)+\beta\,(y-bt)+\delta\,(z-ct)}+e^{2\,\alpha\,(x-at)+2\,\beta\,(y-bt)+2\,\delta\,(z-ct)}\right)\alpha},$$

$$V(\vec{x},t)-b=$$

$$=\frac{\left(\delta^2+\alpha^2+\beta^2\right)\left(e^{\alpha\,(x-at)+\beta\,(y-bt)+\delta\,(z-ct)}-1\right)e^{\alpha\,(x-at)+\beta\,(y-bt)+\delta\,(z-ct)}}{\left(1+e^{\alpha\,(x-at)+\beta\,(y-bt)+\delta\,(z-ct)}\right)\left(1+2\,e^{\alpha\,(x-at)+\beta\,(y-bt)+\delta\,(z-ct)}+e^{2\,\alpha\,(x-at)+2\,\beta\,(y-bt)+2\,\delta\,(z-ct)}\right)},$$

$$W(x,y,z,t)-c=$$

$$=\frac{\left(\delta^2+\alpha^2+\beta^2\right)\left(e^{\alpha\,(x-at)+\beta\,(y-bt)+\delta\,(z-ct)}-1\right)e^{\alpha\,(x-at)+\beta\,(y-bt)+\delta\,(z-ct)}}{\left(1+e^{\alpha\,(x-at)+\beta\,(y-bt)+\delta\,(z-ct)}\right)\left(1+2\,e^{\alpha\,(x-at)+\beta\,(y-bt)+\delta\,(z-ct)}+e^{2\,\alpha\,(x-at)+2\,\beta\,(y-bt)+2\,\delta\,(z-ct)}\right)}.$$

Three- soliton solution has cumbersome form and we omit it here.

Note that for N-soliton solutions of such form the condition $P(x,y,z,t)=$ *const* fulfilled.

Remark 1. *To construct similar examples of solutions of the Navier-Stokes equations*

$$\vec{V}_t+(\vec{V}\cdot\vec{\nabla})\vec{V}+\vec{\nabla}P=\mu\Delta\vec{V},\qquad(\nabla\cdot\vec{V})=0,$$

instead of the standard condition of incompressibility $(\vec{\nabla}\cdot\vec{V})=0$ *can be used equivalent to him*

$$VWP_x+UWP_y+UVP_z-V^2(WU)_y-W^2(VU)_z-U^2(WV)_x+$$

$$+\mu(WV\Delta U+UW\Delta V+UV\Delta W)=0.$$

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