Fibration of the periodical eigenfunctions manifold into hypersurfaces

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The space of of self-adjoint periodic eigenvalue and eigenfunction boundary-value problems

$$-y'' + p(x)y = \lambda y, \quad y(0) - y(2\pi) = y'(0) - y'(2\pi) = 0, \qquad (1)$$

$$P := \left\{ p \in C^0(2\pi) \mid \int_0^{2\pi} p(x)dx = 0 \right\}$$

The spectrum consists of real eigenvalues, which have multiplicity at most 2:

$$\lambda_0(p) < \lambda_1^-(p) \le \lambda_1^+(p) < \ldots < \lambda_k^-(p) \le \lambda_k^+(p) < \ldots$$

Eigenfunctions corresponding to eigenvalues with subscript k have precisely $2k$ nondegenerate zeros on the half-open interval $[0, 2\pi)$.

The manifold of eigenfunctions with exactly 2k zeros

$$Y_k := \{y \in C^2(2\pi) : \int_0^{2\pi} y^2 dx = 1,\ (1)\ \textit{with}\ \lambda = \lambda_k^{\pm}(p),\ y \cong -y\}$$

The set Y_k $(k = 0, 1, ...)$ consists of all functions y such that:
1. there exist 2k points $x_i \in [0, 2\pi)$ at which $y(x_i) = y''(x_i) = 0, y'(x_i) \neq 0$;
2. the function y has no other zeros;
3. there exist derivatives $y^{(3)}(x_i) < \infty$
4. Y_k is a manifold which locally C^∞-diffeomorphic to space P.
5. There are mappings which recover the eigenvalue and potential:

$$\Lambda_k : Y_k \to \mathbb{R},\ \Lambda_k(y) = \lambda := -\frac{1}{2\pi} \int_0^{2\pi} \frac{y''}{y} dx;$$

$$f_k : Y_k \to P,\ f_k(y) = p := \frac{y''}{y} + \Lambda_k(y).$$

The degenerate and nondegenerate eigenfunctions

For $k \in \mathbb{N}$ a pair $(y, z) \in Y_k \times Y_k$ is said to be **conjugated** if these functions are generated by the same potential p and $\int_0^{2\pi} yzdx = 0$. Any eigenfunction $y \in Y_k$ has a unique conjugated function $z = I(y)$ and $I^2(y) = y$. If $\lambda^-(p) < \lambda^+(p)$ then $I(y^{\pm}(p)) = y^{\mp}(p)$.

$$\textbf{Lacuna}(y) \textit{ is } \Delta\Lambda_k(y) := \Lambda_k(y) - \Lambda_k(I(y)),$$

$$Y_k(\Delta\Lambda_k = C) := \{y \in Y_k : \ \Delta\Lambda_k(y) = C\}.$$

$$Y_k = \cup_{C\in\mathbb{R}} Y_k(\Delta\Lambda_k = C).$$

The set $Y_k(\Delta\Lambda_k = 0)$ is called degenerate; if $C \neq 0$, $Y_k(\Delta\Lambda_k = C)$ is nondegenerate.

1. For any fixed C, the subset $Y_k(\Delta\Lambda_k = C) \subset Y_k$ is a C^∞-submanifold of codimension 1; for any $C_1 \neq C_2$, $Y_k(\Delta\Lambda_k = C_1) \cong Y_k(\Delta\Lambda_k = C_2)$.
2. $Y_k \cong Y_k(\Delta\Lambda_k = C) \times \mathbb{R} \sim \mathbb{R}P^1$.

The degenerate and nondegenerate potentials

$$|\Delta\lambda_k(p)| := \lambda_k^+(p) - \lambda_k^-(p) \geq 0,$$

$$P(|\Delta\lambda_k| = C) := \{p \in P : \ |\Delta\lambda_k(p)| = C \geq 0\}.$$

$$P = \cup_{C\geq 0} P(|\Delta\lambda_k| = C).$$

1. For any fixed $C > 0$, the nondegenerate subset $P(|\Delta\lambda_k| = C) \subset P$ is a C^∞-submanifold of codimension 1; $P(|\Delta\Lambda_k| = C) \times \mathbb{R}^+ \cong P \setminus P(|\Delta\lambda_k| = 0) \sim \mathbb{R}P^1$.
2. The degenerate subset $P(|\Delta\lambda_k| = 0) \subset P$ is a C^∞-submanifold of codimension 2; $P(|\Delta\lambda_k| = 0) \sim *$.
3. For $C \neq 0$, $f_k|_{\pm C} : Y_k(\Delta\Lambda_k = \pm C) \to P(|\Delta\lambda_k| = |C|)$ is C^∞-diffeomorphism.
4. For $C = 0$, $f_k|_0 : Y_k(\Delta\Lambda_k = 0) \to P(|\Delta\lambda_k| = 0)$ is C^∞-bundle with $\mathbb{R}P^1$ as fiber;
5. For any C, $Y_k(\Delta\Lambda_k = C) \cong P(|\Delta\lambda_k| = 0) \times \mathbb{R}P^1$.

For $y \in Y_k(\Delta\Lambda_k = 0)$ Wronskian
$W(y) := W(y, I(y)) = y \cdot (I(y))' - y' \cdot I(y) = const.$
The mapping

$$F : Y_k(\Delta\Lambda_k = 0) \times \mathbb{R} \to Y,\ F(y, \Delta\lambda) := \frac{exp\left(\frac{\Delta\lambda}{2W(y)} \int_0^x yI(y)dx\right) y}{||...||_{L_2}}$$

is C^∞-diffeomorphism and $F(y, \Delta\lambda) \in Y_k(\Delta\Lambda_k = \Delta\lambda)$.
The inverse mapping is

$$F^{-1} : Y \to Y_k(\Delta\Lambda_k = 0) \times \mathbb{R},$$

$$F^{-1}(y) = \left(\frac{\left(1 + \frac{\Delta\lambda(y)}{W(y(0))} \int_0^x y \cdot I(y)dx\right)^{-1/2} y}{||...||_{L_2}}, \Delta\lambda(y) \right).$$

Levels of functional Λ

1. For any fixed C, the subset $Y_k(\Lambda_k = C) \subset Y_k$ is a C^∞-submanifold of codimension 1; for any $C_1 \neq C_2$, $Y_k(\Lambda_k = C_1) \cong Y_k(\Lambda_k = C_2)$.
2. $Y_k \cong Y_k(\Lambda_k = C) \times \mathbb{R} \sim \mathbb{R}P^1$.

On Y_k consider the vector field

$$\dot{y} = v(y) := \frac{\int_0^{2\pi} y^4 dx - y^2}{4\int_0^{2\pi} (y')^2 dx} y \Rightarrow \dot{\lambda}(v(y)) = 1 \Rightarrow$$

there exists the vector flow $F^t : Y_k \to Y_k \; (-\infty < t < \infty)$

$$F^t(Y_k(\Lambda_k = C)) = Y_k(\Lambda_k = C + t).$$

The parametrization of manifolds Y_k and P

$H_k \subset C^2(2\pi)$ is the set of functions η that satisfy the conditions
1. $\eta(x) \in C^2(2\pi)$,
2. $\eta(x) > 0$,
3. $\int_0^{2\pi} \eta(x)\ dx = 2\pi k$,
4.
$$\int_0^{2\pi} \frac{\sin 2\int_0^x \eta(t)\ dt}{\eta(x)}\ dx = 0, \quad \int_0^{2\pi} \frac{\cos 2\int_0^x \eta(t)\ dt}{\eta(x)}\ dx = 0.$$

The set H_k is homotopy trivial C^∞-manifold.

By definition $\theta(x;\varphi,\eta) := \varphi + \int_0^x \eta(t)dt$, where $\varphi \in \mathbb{R}P^1$

The parametrization of manifold Y_k

$$\Upsilon : H_k \times \mathbb{R} \times \mathbb{R}P^1 \to Y_k,$$

$$\Upsilon(\eta, \Delta\lambda, \varphi) :=$$

$$y^{sign(\Delta\lambda)} = \frac{const}{\eta^{1/2}(x)} \exp\left(\frac{\Delta\lambda}{4}\int_0^x \frac{\sin 2\theta(t;\varphi,\eta)}{\eta(t)}dt\right)\cdot\cos(\theta(x;\varphi,\eta)).$$

Υ is C^∞-diffeomorphism.

The parametrization of manifold P

$$r(x) := \frac{(y_k)''}{y_k} = -\frac{\eta''}{2\eta} + \frac{3(\eta')^2}{4\eta^2} - \eta^2 +$$

$$\frac{\Delta\lambda\eta' \sin 2\theta(x;\varphi,\eta)}{2\eta^2} + \frac{\Delta\lambda^2 \sin^2 2\theta(\dots)}{16\eta^2} - \Delta\lambda \cos 2\theta(\dots) + \frac{\Delta\lambda}{2},$$

$$\lambda_k = -\frac{1}{2\pi}\int_0^{2\pi} r(x)dx,$$

$$\Phi : H_k \times \mathbb{R}^+ \times \mathbb{R}P^1 \to P, \;\; \Phi(\eta, \Delta\lambda, \varphi) = p(x) := r(x) + \lambda_k.$$

Φ is C^∞-diffeomorphism.

Literature

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