An analog of Chang inversion formula for weighted Radon transforms in multidimensions

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Weighted Radon transforms

Let $f \in C_0(\mathbb{R}^n)$, $n \geq 2$ then Radon Rf and weighted Radon $R_W f$ transforms are defined correspondingly:

$$Rf(s,\theta) \overset{def}{=} \int_{x\theta=s} f(x)dx_H, \tag{1}$$

$$R_W f(s,\theta) \overset{def}{=} \int_{x\theta=s} W(x,\theta)f(x)dx_H,\ (s,\theta) \in \mathbb{R}^n \times \mathbb{S}^{n-1}, \tag{2}$$

where W is complex-valued, $W \in C(\mathbb{R}^n \times \mathbb{S}^{n-1}) \cap L^\infty(\mathbb{R}^n \times \mathbb{S}^{n-1})$.

Typical question: $Ker(R), Ker(R_W) = \{0\}$?

Classical Results on weighted Radon transforms

- [L.Chang, 1978] A method for attenuation correction in radionuclide computed tomography. *IEEE Transactions on Nuclear Science.*
- [J.Boman, E.T. Quinto, 1987] Support theorems for real-analytic Radon transforms. *Duke Mathematical Journal.*
- [R.G. Novikov, 2002] An inversion formula for the attenuated X-ray transformation. *Arkiv für Matematik.*
- [L.A. Kunyansky, 1992] Generalized and attenuated Radon transforms: restorative approach to the numerical inversion. *Inverse problems.*
- [S. Gindikin, 2010] A remark on the weighted Radon transform on the plane. *Inverse Problems and Imaging.*
- ...

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1. Introduction
   - Weighted Radon transforms
   - Chang inversion formula in 2D
2. Main result
   - Novikov's result for Chang formula in 2D
   - Analog of Chang inversion formula for ND
   - Possible tomographical applications in 3D
3. Summary

Simple example: SPECT

Weighted Radon transform ($n = 2, 3$):

$$W_a(y, \theta) = \exp\left(- \int_0^{+\infty} a(y + s\theta) ds \right), \; y \in \mathbb{R}^n, \; \theta \in \mathbb{S}^{n-1}. \tag{3}$$

$$R_{W_a} f(l) = \int_{y \in l} W_a(y, \theta) f(y) dy, \; l \in T\mathbb{S}^{n-1}, \; a \in \mathcal{S}(\mathbb{R}^n). \tag{4}$$

where $T\mathbb{S}^{n-1} = \{(x, \theta) \in \mathbb{R}^n \times \mathbb{S}^{n-1} : (x \cdot \theta) = 0\}$ – manifold of all oriented lines in $\mathbb{R}^n$, $\S(\mathbb{R}^n)$ – Schwartz class.

From SPECT to Chang inversion formula in 2D

Let $f \in C_0(\mathbb{R}^2)$, $W \in C(\mathbb{R}^2 \times \mathbb{S}^1) \cap L^\infty(\mathbb{R}^2 \times \mathbb{S}^1)$, then

$$f_{appr} \overset{def}{=} \frac{1}{4\pi w_0(x)} \int_{\mathbb{S}^1} h'_W(x\theta^\perp, \theta) d\theta, \; h' = \frac{d}{ds} h(s, \theta), \tag{5}$$

$$h_W(s, \theta) = \frac{1}{\pi} p.v. \int_{\mathbb{R}} \frac{R_W(t, \theta)}{s - t} dt, \; (s, \theta) \in T\mathbb{S}^1 \simeq \mathbb{R} \times \mathbb{S}^1, \tag{6}$$

$$w_0(x) = \frac{1}{2\pi} \int_{\mathbb{S}^1} W(x, \theta) d\theta, \; w_0(x) \neq 0. \tag{7}$$

- [L.Chang, 1978] A method for attenuation correction in radionuclide computed tomography. *IEEE Transactions on Nuclear Science.*
- [R.G. Novikov, 2002] An inversion formula for the attenuated X-ray transformation. *Arkiv für Matematik.* – exists exact inversion formula for R_{W_a}.

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Inversion with Chang formula in 2D

Let $W(x,\theta)$, $(x,\theta) \in \mathbb{R}^2 \times \mathbb{S}^1$ is complex valued,
$W \in C(\mathbb{R}^2 \times \mathbb{S}^1) \cap L^\infty(\mathbb{R}^2 \times \mathbb{S}^1)$ and

$$w_0(x) \overset{def}{=} \frac{1}{2\pi} \int_{\mathbb{S}^1} W(x,\theta) d\theta,\ w_0(x) \neq 0. \tag{8}$$

Theorem (R.G. Novikov, 2011)

Let W satisfies conditions above, $f \in C_0(\mathbb{R}^2)$ and f_{appr} is defined by the Chang inversion formula. Then $f = f_{appr}$ (in terms of distributions) if and only if

$$W(x,\theta) - w_0(x) \equiv w_0(x) - W(x,-\theta). \tag{9}$$

R.G. Novikov, Weighted Radon transforms for which Chang's approximate inversion formula is exact. *Russian Mathematical Surveys*, 66(2): 442-443, 2011.

Sketch of the proof in 2D

Let $f \in C_0(\mathbb{R}^2)$, recall:

$$f_{appr} \overset{def}{=} \frac{1}{4\pi w_0(x)} \int_{\mathbb{S}^1} h'_W(x\theta^{\perp}, \theta) d\theta, \; h' = \frac{d}{ds} h(s, \theta),$$

$$h_W(s, \theta) = \frac{1}{\pi} p.v. \int_{\mathbb{R}} \frac{R_W f(t, \theta)}{s - t} dt, \; (s, \theta) \in T\mathbb{S}^1 \simeq \mathbb{R} \times \mathbb{S}^1.$$

Main idea of the proof – "symmetrization" of W:

$$W_s(x, \theta) \overset{def}{=} \frac{1}{2}(W(x, \theta) + W(x, -\theta)), \tag{10}$$

$$R_{W_s} f(s, \theta) = \frac{1}{2}(R_W f(s, \theta) + R_W f(-s, -\theta)), \tag{11}$$

$$h_{W_s}(s, \theta) = \frac{1}{2}(h_W(s, \theta) - h_W(-s, -\theta)), \tag{12}$$

$$h'_{W_s}(s, \theta) = \frac{1}{2}(h'_W(s, \theta) + h'_W(-s, -\theta)), \; (s, \theta) \in \mathbb{R} \times \mathbb{S}^1. \tag{13}$$

Sketch of the proof in 2D

From identities (10)-(13) and definition of f_{appr} it follows:

$$f_{appr} \equiv \frac{1}{4\pi w_0(x)} \int_{\mathbb{S}^1} h'_{W_s}(x\theta^{\perp}, \theta) d\theta, \; h' = \frac{d}{ds} h(s, \theta),$$
$$h_{W_s}(s, \theta) = \frac{1}{\pi} p.v. \int_{\mathbb{R}} \frac{R_{W_s} f(t, \theta)}{s - t} dt, \; (s, \theta) \in T\mathbb{S}^1 \simeq \mathbb{R} \times \mathbb{S}^1.$$

- Sufficiency: $W_s(x, \theta) \equiv w_0(x)$.
- Necessity: From Radon inversion formula and definition f_{appr} it follows:

$$\int_{\mathbb{S}^1} (h'_{w_0}(x\theta^{\perp}, \theta) - h'_{W_s}(x\theta^{\perp}, \theta)) d\theta = 0. \tag{14}$$

From 2D-Fourier transform of (14) it follows:

$$h_{w_0} \equiv h_{W_s} \Rightarrow R_{w_0} f = R_{W_s} f \; (\forall f \in C_0(\mathbb{R}^2)) \Rightarrow w_0 \equiv W_s \tag{15}$$

Outline

Analog of Chang formula in ND

Let f – test function, W – weight, than the following formulas are defined:

- n is odd

$$f_{appr}(x) \overset{def}{=} \frac{(-1)^{(n-1)/2}}{2(2\pi)^{n-1} w_0(x)} \int\limits_{\mathbb{S}^{n-1}} [R_W f]^{(n-1)} (x\theta, \theta)\, d\theta. \tag{16}$$

- n is even

$$f_{appr}(x) \overset{def}{=} \frac{(-1)^{(n-2)/2}}{2(2\pi)^{n-1} w_0(x)} \int\limits_{\mathbb{S}^{n-1}} \mathbb{H} [R_W f]^{(n-1)} (x\theta, \theta)\, d\theta \tag{17}$$

where

$$[R_W f]^{(n-1)} (s, \theta) = \frac{d^{n-1}}{ds^{n-1}} \mathbb{R}_W f(s, \theta), s \in \mathbb{R},\ \theta \in \mathbb{S}^{n-1}, \tag{18}$$

$$\mathbb{H}\phi(s) \overset{def}{=} \frac{1}{\pi} p.v. \int\limits_{\mathbb{R}} \frac{\phi(t)}{s-t} dt,\ s \in \mathbb{R}. \tag{19}$$

[F.Natterer, 1986] The mathematics of computerized tomography, vol.32, *SIAM*.

Analog of Chang inversion formula for ND

Let $W(x,\theta)$, $(x,\theta) \in \mathbb{R}^n \times \mathbb{S}^{n-1}$ is complex valued, $W \in C(\mathbb{R}^n \times \mathbb{S}^{n-1}) \cap L^\infty(\mathbb{R}^n \times \mathbb{S}^{n-1})$ and

$$w_0(x) \overset{def}{=} \frac{1}{|\mathbb{S}^{n-1}|} \int_{\mathbb{S}^{n-1}} W(x,\theta)\, d\theta,\ w_0(x) \neq 0. \tag{20}$$

Theorem (F.O. Goncharov, R.G. Novikov, 2016)

Let W satisfies conditions above, $f \in C_0(\mathbb{R}^n)$ and f_{appr} is defined by the analog Chang inversion formula in multidimensions. Then $f = f_{appr}$ (in terms of distributions) if and only if

$$W(x,\theta) - w_0(x) \equiv w_0(x) - W(x,-\theta). \tag{21}$$

F.O. Goncharov, R.G. Novikov, An analog of Chang inversion formula for weighted Radon transforms in multidimensions. *EJMCA*, 4(2): 23-32, 2016.

Sketch of the proof in multidimensions

1. Symmetrization:

$$W_s(x,\theta) \stackrel{def}{=} \frac{1}{2}(W(x,\theta) + W(x,-\theta)), \tag{22}$$

$$R_{W_s} f(s,\theta) = \frac{1}{2}(R_W f(s,\theta) + R_W f(-s,-\theta)), \tag{23}$$

then

$$f_{appr}(x) \equiv \frac{(-1)^{(n-1)/2}}{2(2\pi)^{n-1} w_0(x)} \int\limits_{\mathbb{S}^{n-1}} [R_{W_s} f]^{(n-1)}(x\theta,\theta)\, d\theta, \tag{24}$$

$$f_{appr}(x) \equiv \frac{(-1)^{(n-2)/2}}{2(2\pi)^{n-1} w_0(x)} \int\limits_{\mathbb{S}^{n-1}} \mathbb{H}\,[R_{W_s} f]^{(n-1)}(x\theta,\theta) d\theta \tag{25}$$

Sketch of the proof in multidimensions

- Sufficiency: $W_s \equiv w_0$, then f_{appr} coincides with exact Radon inversion formulas.
- Neccesity: Same idea as in 2D case

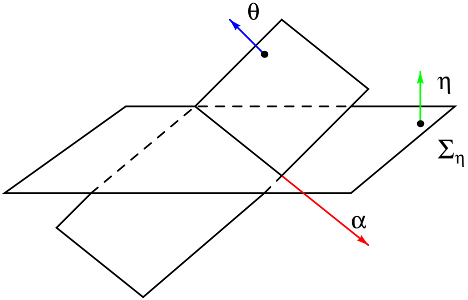
  ND Fourier transform $\rightarrow R_{W_s} f = R_{w_0} f$ $(\mathbb{H}[R_{W_s} f] \equiv \mathbb{H}[R_{w_0} f])$ for all $f \in C_0(\mathbb{R}^n) \rightarrow W_s \equiv w_0$.

Outline

Possible tomographical applications: from 2D to 3D

$$P_\omega f(x,\alpha) = \int_{\mathbb{R}} \omega(x+\alpha t,\alpha) f(x+\alpha t)dt,\ (x,\alpha)\in T\mathbb{S}^2, \tag{26}$$

$$T\mathbb{S}^2 = \{(x,\alpha)\in\mathbb{R}^3\times\mathbb{S}^2 : x\alpha = 0\},\ \alpha\perp\eta, \Sigma_\eta = \{x : x\eta = 0\}.$$



$$R_W f(s,\theta) = \int_{\mathbb{R}} P_\omega f(s\theta + \tau[\theta,\alpha],\alpha)d\tau,\ s\in\mathbb{R},\theta\in\mathbb{S}^2, \tag{27}$$

$$W(x,\theta) = \omega(x,\alpha),\ \alpha = \alpha(\eta,\theta) = \frac{[\eta,\theta]}{|[\eta,\theta]|},\ [\eta,\theta]\neq 0, x\in\mathbb{R}^3. \tag{28}$$

Summary

Achievements:

1. We proposed new approximate inversion formulas analogously to [L.Chang, 1978] for multidimensions.
2. Proposed formulas are explicit in the precise class of weight functions (analogously to the result [Novikov, 2011]).
3. Showed how the weighted Radon transforms in 2D relate to transforms in 3D. Such averaging can drastically reduce the noise impact in the initial data.

Questions for future:

1. Do the numerical tests, maybe it is possible to build iterative algorithms related to the new formulas (e.g. works of [Kunyansky, 1992], [Novikov, 2014] have connections with Chang inversion formula in 2D)?
2. ...