Loewner Evolution as Itô Diffusion

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Abstract

F. Bracci, M.D. Contreras, S. Díaz Madrigal proved that any evaluation family of order d is described by a generalized Loewner chain. G. Ivanov and A. Vasil'ev considered randomized version of the chain and found a substitution which transforms it to an Itô diffusion.We generalize their result to vector randomized Loewner chain and prove there are no other possibilities to transform such Loewner chains to Itô diffusions.

Keywords: Loewner chain, Loewner equation, Ito diffusion, Herglötz function.

1. Introduction

The Schramm-Loewner evolution (SLE), also known as a stochastic Loewner evolution [8, 12] is a conformaly invariant stochastic process which attracts many researchers during last 16 years. First contributions to this growing theory was discovery by O. Schramm [13] in 2000. This process is a stochastic generalization of the Loewner-Kufarev differential equations. SLE has the domain Markov property which is closely related to the fact that the equations can be represented as time homogeneous diffusion equations.

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The classical Loewner equation was introduced by K. Loewner in 1923. The idea was represent domains by means of family (known as Loewner chains) of univalent functions defined on the unit disk $\mathbb{D} := \{\zeta \in \mathbb{C} : |\zeta| < 1\}$ and satisfying a suitable differential equation.

The classical Loewner equation in the unit disk is the following differential equation

$$\begin{cases} \frac{d\phi_t(z)}{dt} = G(\phi_t(z), t) \\ \phi_0(z) = z \end{cases} \tag{1}$$

for almost every $t \in [0, \infty)$ where $G(w, t) = -wp(w, t)$ with the function $p : \mathbb{D} \times [0, \infty) \to \mathbb{C}$ measurable in t, holomorphic in z, $p(0, t) = 1$ and $\Re(p(z, t)) \geq 0$ for all $z \in \mathbb{D}$ and $t \geq 0$ (such functions p are called Herglötz functions).

Generalization of the Loewner- Kufarev approach was improved by F. Bracci, M.D. Contreras, S. Díaz Madrigal [2]. Section two includes one of the main results of generalized Loewner theory that is an essentially one to one correspondence between evolution families, Hergltz vector fields and functions.

Recently G. Ivanov and A. Vasil'ev [5] considered random version of this Loewner differential equation with $G(w, t) = \frac{(\tau(t)-w)^2 p(w,t)}{\tau(t)}$ for

$$\tau(t) = \tau(t, w) = \exp(ikB_t(w)). \tag{2}$$

They found a substitution which transforms the randomized Loewner equation with $p(w, t) = \tilde{p}\left(\frac{w}{\tau(t)}\right)$ to an Itô diffusion and obtained the infinitesimal generator of the Itô diffusion in this form:

$$A = \left(-\frac{z}{2}k^2 + (1-z)^2\tilde{p}(z)\right)\frac{d}{dz} - \frac{1}{2}k^2 z^2 \frac{d^2}{dz^2}. \tag{3}$$

The main result is an inverse statement. Namely we prove that under rather general suppositions on $\tau(t) = \tau(t, B_t)$, it is possible to find a substitution which transforms (1) to an Itô diffusion if and only if τ is given by (2). We generalize this necessary and sufficient condition for higher dimensions when τ depends on some independent Brownian motions

$$\tau(t) = \tau(\mathbf{B}_t)$$

where $\mathbf{B}_t = (B_t^1, B_t^2, \ldots, B_t^n)$.

We denote by $\check{C}$ the set of functions $f(z, \mathbf{x})$ from $C^n(\mathbb{D} \times \mathbb{R}^n)$ such that these functions have continuous derivatives up to order n, $\frac{\partial f}{\partial x_j}$ don't vanish and $H(\mathbb{D})$ is the set of analytic functions in $\mathbb{D}$.

2. Loewner Evolution

The most prominent contribution for semi groups of conformal maps was given by Loewner in 1923. He introduced the nowadays well known Loewner parametric method and so called Loewner differential equations. We are mainly interested on generalization of the Loewner-Kufarev approach by F. Bracci, M.D. Contreras, S. Díaz Madrigal [2]. We briefly describe the basic notations of the theory.

2.1. Semigroups and Infinitesimal Generator

By the Schwarz-Pick lemma, every holomorphic self map φ of the unit disk $\mathbb{D}$ may have at most one fixed point τ in $\mathbb{D}$. If such a point τ exists, then the point τ is called the Denjoy-Wolf point of φ. Otherwise there exist a point τ on the unit circle $\mathbb{T} := \{\zeta \in \mathbb{C} : |\zeta| = 1\}$ such that angular limit $\angle \lim_{z \to \tau} \varphi(z) = \tau$. The point τ is called again Denjoy-Wolf point of φ. This case is also known as the Denjoy-Wolf theorem.

Definition 1. *A family $\{\phi_t\}_{t \geq 0}$ of holomorphic self maps of the unit disk $\mathbb{D}$ is called an one parameter continuous semi group if*

1. $\phi_0 = id_{\mathbb{D}}$,
2. *For* $s, t \geq 0$ $\phi_{t+s} = \phi_t \circ \phi_s$,
3. *For all* $s \geq 0$ *and* $z \in \mathbb{D}$ $\lim_{t \to s} \phi_t(z) = \phi_s(z)$,
4. $\lim_{t \to +\infty} \phi_t(z) = z$ *locally uniformly in* $\mathbb{D}$.

A very important contribution to the theory of semi groups of holomorphic self maps of unit disk $\mathbb{D}$ is due to E. Berkson and H. Porta [1]. They proved that a semi group of holomorphic self maps of unit disk $\{\phi_t\}_{t \geq 0}$ is in fact real analytic in the variable t and is the solution of the Cauchy problem

$$\begin{cases} \frac{d\phi_t(z)}{dt} = G(\phi_t(z)) \\ \phi_0(z) = z \end{cases} \tag{4}$$

where the map G the infinitesimal generator of the semi groups has the form

$$G(z) = (z - \tau)(\bar{\tau} z - 1)p(z), \quad z \in \mathbb{D}. \tag{5}$$

for some $\tau \in \bar{\mathbb{D}}$ and a holomorphic function $p : \mathbb{D} \to \mathbb{C}$ with $\Re p \geq 0$. We will use the term Hergltz function for the function $p(z)$. Representation of (5) is unique (if $G(z) \neq 0$) and is known Berksan-Porta representation of G. The point τ turns out to be Denjoy-Wolf point of all functions in $\{\phi_t\}_{t \geq 0}$.

2.2. Evolution Families

We continue with the definition of evolution family.

Definition 2. *A two parameter family* $\{\phi_{s,t}\}_{0 \leq s \leq t < +\infty}$ *of holomorphic self-maps of the unit disk* $\mathbb{D}$ *is called an evolution family of order* $d \in [1, \infty]$ *if*

1. $\phi_{s,s} = id_{\mathbb{D}}$,
2. $\phi_{s,t} = \phi_{u,t} \circ \phi_{s,u}$ *for all* $0 \leq s \leq t < +\infty$,
3. *for any* $z \in \mathbb{D}$ *and* $T > 0$ *there is a non negative function* $k_{z,t} \in L^d([0, T], \mathbb{R})$, *such that*

$$|\phi_{s,u}(z) - \phi_{s,t}(z)| \leq \int_u^t k_{z,T}(\xi) d\xi, \quad z \in \mathbb{D}$$

for all $0 \leq s \leq u \leq t \leq T$.

The problem of differentiability of an evolution family is much more difficult than the case of one parameter semi group. Firstly K. Loewner considered these types of problems for the semi group $\mathfrak{L}$ of functions f holomorphic and univalent in $\mathbb{D}$ such that $f(0) = 0$, $f'(0) \geq 0$ and $|f(z)| \leq 1$ for $z \in \mathbb{D}$. The infinitesimal generator of $\mathfrak{L}$ are described by the formula $G(z) = -zp(z)$ where p is holomorphic function in $\mathbb{D}$ with negative real part. For general case an infinitesimal generator of an evolution family is given in terms of Herglötz vector field [2].

Definition 3. *A function* $G : \mathbb{D} \times [0, \infty)$ *is called a weak holomorphic vector field of order* d ($d \in [1, +\infty]$) *on the unit disk* $\mathbb{D}$, *if*

1. *The function* $[0, \infty) \ni t \mapsto G(z, t)$ *is measurable for all* $z \in \mathbb{D}$,
2. *The function* $z \mapsto G(z, t)$ *is holomorphic for all* $t \in [0, \infty)$, ,
3. *For any compact set* $K \subset \mathbb{D}$ *and for every* $T > 0$ *there exist a non negative function* $k_{z,T} \in L^d([0, T], \mathbb{R})$ *such that*

$$|G(z, t)| \leq k_{z,T}(t)$$

for all $z \in K$ *and for almost every* $t \in [0, T]$.

Herglötz vector fields in $\mathbb{D}$ can be decomposed by means of Herglötz function.

Definition 4. *(Herglötz function) Let $p : \mathbb{D} \times [0, \infty) \to \mathbb{C}$. p is called a Herglötz function if it satisfies the following conditions:*

1. $p(0, \cdot) \equiv 1$,
2. $p(\cdot, t)$ *is holomorphic for all* $t \geq 0$,
3. $p(z, \cdot)$ *is measurable for all* $z \in \mathbb{D}$,
4. $\Re(p(z, t)) \geq 0$ *for all* $z \in \mathbb{D}$ *and* $t \geq 0$.

Later F. Bracci, M.D. Contreras, S. Díaz Madrigal [2] established that the evolution families in unit disk $\mathbb{D}$ can be put one to one correspondence with the Herglötz vector fields.

Theorem 5 ([2], Theorem 5.2 and 6.2). *For any evolution family of order $d \geq 1$ in the unit disc there exists a (essentially) unique Herglötz vector field $G(z, t)$ of order d such that for all $z \in \mathbb{D}$*

$$\frac{\partial \varphi_{s,t}(z)}{\partial t} = G(\varphi_{s,t}(z), t) \ \text{a.e.}\ t \in [0, \infty). \tag{6}$$

Conversely for any Herglötz vector field of order $d \geq 1$ in $\mathbb{D}$ there exists a unique evolution family $(\varphi_{s,t})$ of order d such that (6) is satisfied.

Theorem 6. *Let $G(z, t)$ be a Herglötz field of order $d \geq 1$ in $\mathbb{D}$. Then there exists a (essentially) unique measurable function $\tau : [0, \infty) \to \mathbb{D}$ and a Herglötz function $p(z, t)$ of order d such that for all $z \in \mathbb{D}$*

$$G(z, t) = (z - \tau(t))(\bar{\tau}(t) z - 1) p(z, t) \ \text{a.e.}\ \ t \in [0, \infty). \tag{7}$$

Conversely given a measurable function $\tau : [0, \infty) \to \mathbb{D}$ and a Herglötz function $p(z, t)$ of order $d \geq 1$, the equation (7) defines a Herglötz vector field of order d.

3. Stochastic Case of Loewner Evolution

We consider the generalized Loewner evolution by a Brownian particle on the unit circle and study the following initial value problem

$$\begin{cases} \frac{d\phi_t(z,\omega)}{dt} = \frac{(\tau(t,\omega) - \phi_t(z,\omega))^2}{\tau(t,\omega)} p(\phi_t(z, \omega), t, \omega) \\ \phi_0(z, \omega) = z \end{cases} \tag{8}$$

since $t \geq 0$, $z \in \mathbb{D}$ and $\omega \in \Omega$. $\tau(t,\omega) = exp(ikB_t(\omega))$ where $k \in \mathbb{R}$, $B_t(\omega)$ is the 1-dimensional Brownian motion with respect to the standard Brownian filtration $\{\mathcal{F}_t\}_{t\geq 0}$ in the probability space $(\Omega, \mathcal{F}, P)$ and $p(z,t,\omega)$ is a Herglotz function for each fixed $\omega \in \Omega$. The equation of (8) is called random differential equation.

Next part of work, we connect with the randomized Loewner equation (8) with an Itô diffusion. Firstly we start with notions of Itô diffusion and infinitesimal generator together with higher dimension.

Definition 7. *Let $X_t(\omega)$ be a stochastic process and $X_t(\omega) : [0,\infty) \times \Omega \to \mathbb{R}^n$. If this process is called a (time homogeneous) Itô diffusion, then it satisfies a stochastic differential equation of the form*

$$dX_t(\omega) = b(X_t)dt + \sigma(X_t)dB_t, \quad t \geq s; \quad X_s = x$$

where B_t is m dimensional Brownian motion and $b : \mathbb{R}^n \to \mathbb{R}^n$, $\sigma : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ satisfy the condition satisfy the existence and uniqueness theorem [10] for the stochastic differential equations which in this case

$$|b(x) - b(y)| + |\sigma(x) - \sigma(y)| \leq D|x - y|$$

$x, y \in \mathbb{R}^n$ i.e. $b(.)$ and $\sigma(.)$ are Lipschitz continuous.

A second order partial differential operator A can be associated to an Itô diffusion X_t. The basic connection between A and X_t is that A is the generator of the process X_t.

Definition 8. *Let $\{X_t\}$ be a (time homogeneous) Itô diffusion in $\mathbb{R}^n$ and $X_0 = x$. The infinitesimal generator A of X_t is defined by*

$$Af(x) = \lim_{t\downarrow 0} \frac{E^x[f(X_t)] - f(x)}{t}$$

$x \in \mathbb{R}^n$. $\mathcal{D}_A(x)$ denotes the set of functions $f : \mathbb{R}^n \to \mathbb{R}$ such that the limit exist at x, while the set of functions for which the limit exists for all $x \in \mathbb{R}^n$ is denoted by $\mathcal{D}_A$.

Theorem 9. *Let $\{X_t\}$ be an Itô diffusion*

$$dX_t(\omega) = \mathbf{b}(X_t)dt + \sigma(X_t)d\mathbf{B}_t.$$

If $f \in C_0^2(\mathbb{R}^n)$ (compact support in $\mathbb{R}^n$ with continuous derivatives up to order 2) then $f \in \mathcal{D}_A$ and

$$Af(x) = \sum_i b_i(x)\frac{\partial f}{\partial x_i} + \frac{1}{2}\sum_{i,j}(\sigma\sigma^T)_{i,j}(x)\frac{\partial^2 f}{\partial x_i \partial x_j}.$$

Recently G. Ivanov and A. Vaislev [5] considered (8) for (2). They found a substitution which transforms the randomized Loewner equation with $p(w,t) = \tilde{p}\left(\frac{w}{\tau(t)}\right)$ to an Itô diffusion and obtained the infinitesimal generator of the Itô diffusion in the form (3).

We obtain an inverse statement. Namely we prove that under rather general suppositions on $\tau(t) = \tau(t, B_t)$ it is possible to find that a substitution which transforms the classical Loewner equation in $\mathbb{D}$ to an Itô diffusion if and only if τ is given by (2).

Theorem 10. *Let us consider Loewner random differential equation*

$$\begin{cases} \frac{d\phi_t(z,w)}{dt} = \frac{(\tau_1(t,w)-\phi_t(z,w))^2}{\tau_1(t,w)}\tilde{p}(\frac{\phi_t(z,w)}{\tau_1(t,w)}) \\ \phi_0(z,w) = z \end{cases} \tag{9}$$

where $|\tau_1(t,\omega)| = 1$ and $\tilde{p}$ is an arbitrary Herglötz function.Let $\breve{C}$ is the set of functions $m(x,y) \in \mathbb{C}^2(\mathbb{D}\times\mathbb{R})$ such that $\frac{\partial m}{\partial y}$ doesn't vanish. Suppose $\psi_t = m(\phi_t, B_t)$, $m \in \breve{C}$ and $\tau_1(t,\omega) = \tau(B_t)$ then ψ_t is an Itô diffusion for an arbitrary Herglötz function $\tilde{p}$ if and only if $\tau(B_t) = e^{-ikB_t}$.

In the next theorem, we generalize necessary and sufficient condition of (10) for higher dimensions when τ depends on some independent Brownian motions

$$\tau(t) = \tau(\mathbf{B}_t)$$

where $\mathbf{B}_t = (B_t^1, B_t^2, \ldots, B_t^n)$.The theorem contains new random version of Loewner differential equation which can be called Loewner equation driven by a Brownian vector particle on the unit circle.

Theorem 11. *Consider Loewner random differential equation*

$$\begin{cases} \frac{d\phi_t(z,w)}{dt} = \frac{(\tau_1(t,w)-\phi_t(z,w))^2}{\tau_1(t,w)}\tilde{p}(\frac{\phi_t(z,w)}{\tau_1(t,w)}) \\ \phi_0(z,w) = z \end{cases} \tag{10}$$

where $|\tau_1(t,\omega)| = 1$ for each fixed $w \in \Omega$ (Ω is a sample space) and $\tilde{p}$ is an arbitrary Herglötz function. Suppose $\psi_t = m(\phi_t, B_t^{(1)}, B_t^{(2)}, \ldots, B_t^{(n)})$ where $B_t^{(i)}$ are independent Brownian motions, $m \in \breve{C}$ and $\tau_1(t,\omega) = \tau(\mathbf{B}_t)$ then, ψ_t is an $n \times 1$ dimensional Itô diffusion with coefficients from $H(\mathbb{D})$ for an arbitrary Herglötz function $\tilde{p}$ if and only if $\tau(\mathbf{B}_t) = e^{\mathbf{k}\cdot\mathbf{B}_t}$ where $\mathbf{k} = (k_1, \ldots, k_n)$ and $\mathbf{k} \in \mathbb{C}^n$.

Furthermore the infinitesimal generator of ψ_t (when it is an Itô diffusion) is given by this form

$$A = \left(-\frac{z}{2}|\mathbf{k}|^2 + (1-z)^2\tilde{p}(z)\right)\frac{d}{dz} - \frac{1}{2}|\mathbf{k}|^2 z^2 \frac{d^2}{dz^2}. \tag{11}$$

Proof. For $n = 1$ sufficiency part was proved by G. Ivanov and A. Vasilev. We use similar argument to prove sufficiency for arbitrary n. By the complex Itô formula, the process

$$\frac{1}{\tau(\mathbf{B}_t)} = e^{-ik_1 B_t^{(1)} - ik_2 B_t^{(2)} - \ldots - ik_n B_t^{(n)}}$$

satisfies the stochastic differential equation (SDE)

$$\begin{aligned} d(e^{-ik_1 B_t^{(1)} - \ldots - ik_n B_t^{(n)}}) = -\sum_{j=1}^{n} ik_j e^{-ik_1 B_t^{(1)} - \ldots - ik_n B_t^{(n)}} dB_t^{(j)} \\ -\tfrac{1}{2}\sum_{j=1}^{n} k_j^2 e^{-ik_1 B_t^{(1)} - \ldots - ik_n B_t^{(n)}} dt. \end{aligned} \tag{12}$$

Let us denote $\psi_t(z,w) = \frac{\phi_t(z,w)}{\tau(\mathbf{B}_t)}$. Applying the integration by parts formula for ψ_t, we obtain

$$\begin{aligned} d(\psi_t) &= \phi_t d(e^{-ik_1 B_t^{(1)} - \ldots - ik_n B_t^{(n)}}) + (e^{-ik_1 B_t^{(1)} - \ldots - ik_n B_t^{(n)}}) d\phi_t \\ &= e^{-ik_1 B_t^{(1)} - \ldots - ik_n B_t^{(n)}} \frac{\left(e^{ik_1 B_t^{(1)} + \ldots + ik_n B_t^{(n)}} - \phi_t(z,w)\right)^2}{e^{ik_1 B_t^{(1)} + \ldots + ik_n B_t^{(n)}}} \tilde{p}(\psi_t) dt \\ &\quad -i\psi_t \sum_{j=1}^{n} k_j dB_t^{(j)} - \tfrac{\psi_t}{2}\sum_{j=1}^{n} k_j^2 dt \\ &= -i\psi_t \mathbf{k}\cdot d\mathbf{B}_t + (-\tfrac{|\mathbf{k}|^2}{2}\psi_t + (\psi_t - 1)^2 \tilde{p}(\psi_t)) dt \end{aligned} \tag{13}$$

So ψ_t is an Itô diffusion in $\mathbb{R}^n$.

Now for the necessity part, from our supposition $\psi_t = m(\phi_t, B_t^{(1)}, \ldots, B_t^{(n)})$. Apply Itô formula;

$$d(\psi_t) = \frac{\partial m}{\partial x} d\phi_t + \sum_{i=1}^{n} \frac{\partial m}{\partial y_i} dB_t^{(i)} + \frac{1}{2} \sum_{i=1}^{n} \frac{\partial^2 m}{\partial y_i^2} dt. \tag{14}$$

and if (14) is an $n \times 1$ dimensional Itô diffusion with analytic coefficients then there are functions $f_i \in H(\mathbb{D})$ such that

$$\frac{\partial m}{\partial y_i} = f_i(m(x, \mathbf{y})). \tag{15}$$

Taking derivative of (15) with respect to y_j we obtain

$$\frac{f_i'(m(x, \mathbf{y}))}{f_i(m(x, \mathbf{y}))} = \frac{f_j'(m(x, \mathbf{y}))}{f_j(m(x, \mathbf{y}))}.$$

Hence

$$(\ln f_i(z))' = (\ln f_j(z))'$$

and

$$f_i(z) = c_{ij} f_j(z). \tag{16}$$

Let us denote

$$f_i(z) = c_i f(z) \tag{17}$$

and let $F(z)$ be an antiderivative of $\frac{1}{f(z)}$, hence

$$F(m(x, \mathbf{y})) = \mathbf{c} \cdot \mathbf{y} + q(x)$$

where $\mathbf{c} = (c_1, \ldots, c_n)$.

Since by supposition F' doesn't vanish, there exists an inverse function F^{-1} and

$$m(x, \mathbf{y}) = F^{-1}(\mathbf{c} \cdot \mathbf{y} + q(x)). \tag{18}$$

Let us denote

$$F^{-1}(z) = G(z). \tag{19}$$

Now for coefficients in dt, we have

$$\frac{\partial m}{\partial x} d\phi_t + \frac{1}{2} \sum_{i=1}^{n} \frac{\partial^2 m}{\partial y_i^2} dt = g(G(\mathbf{c} \cdot \mathbf{y} + q(x))dt.$$

where g is an analytic function in $\mathbb{D}$. If we substitute (15, 17, 19) then we get,

$$G'(\mathbf{c}\cdot\mathbf{y}+q(x))q'(x)\tfrac{d\phi_t}{dt}+f'(G(\mathbf{c}\cdot\mathbf{y}+q(x)))f(G(\mathbf{c}\cdot\mathbf{y}+q(x)))\tfrac{1}{2}\sum_{i=1}^{n}c_i^2$$

$$=g(G(\mathbf{c}\cdot\mathbf{y}+q(x))). \tag{20}$$

$$G'(\mathbf{c}\cdot\mathbf{y}+q(x))q'(x)\frac{(\tau(\mathbf{y})-x)^2}{\tau(\mathbf{y})}\tilde{p}(\frac{x}{\tau(\mathbf{y})})=g_1(\mathbf{c}\cdot\mathbf{y}+q(x)) \tag{21}$$

where we denote $g_1(z)=g(z)-f'(z)f(z)\frac{1}{2}\sum_{i=1}^{n}c_i^2.$

Definition of (18) shows that $G'(\mathbf{c}\cdot\mathbf{y}+q(x))$ doesn't vanish. Hence $g_1(\mathbf{c}\cdot\mathbf{y}+q(x))$ is not identically zero. So equation (21) can be written as

$$q'(x)\frac{(\tau(\mathbf{y})-x)^2}{\tau(\mathbf{y})}\tilde{p}(\frac{x}{\tau(\mathbf{y})})=H_1(\mathbf{c}\cdot\mathbf{y}+q(x)) \tag{22}$$

where $H_1(z)=\frac{g_1(z)}{G'(z)}$.

Let us differentiate (22) with respect to x and y_i. Then we obtain two equalities:

$$q''(x)\tfrac{(\tau(\mathbf{y})-x)^2}{\tau(\mathbf{y})}\tilde{p}(\tfrac{x}{\tau(\mathbf{y})})-2q'(x)\tfrac{(\tau(\mathbf{y})-x)}{\tau(\mathbf{y})}\tilde{p}(\tfrac{x}{\tau(\mathbf{y})})+q'(x)\tfrac{(\tau(\mathbf{y})-x)^2}{\tau^2(\mathbf{y})}\tilde{p}'(\tfrac{x}{\tau(\mathbf{y})})$$

$$=q'(x)H_1'(\mathbf{c}\cdot\mathbf{y}+q(x)) \tag{23}$$

and

$$q'(x)\frac{(\tau(\mathbf{y})-x)\frac{\partial\tau(\mathbf{y})}{\partial y_i}(\tau(\mathbf{y})+x)}{\tau^2(\mathbf{y})}\tilde{p}(\tfrac{x}{\tau(\mathbf{y})})-q'(x)\frac{(\tau(\mathbf{y})-x)^2\frac{\partial\tau(\mathbf{y})}{\partial y_i}x}{\tau^2(\mathbf{y})}\tilde{p}'(\tfrac{x}{\tau(\mathbf{y})})$$

$$=c_iH_1'(\mathbf{c}\cdot\mathbf{y}+q(x));\quad i=1,2,\ldots,n. \tag{24}$$

Now (23) and (24) imply,

$$q''(x)\frac{(\tau(\mathbf{y})-x)^2}{\tau(\mathbf{y})}\tilde{p}\left(\frac{x}{\tau(\mathbf{y})}\right)-2q'(x)\frac{(\tau(\mathbf{y})-x)}{\tau(\mathbf{y})}\tilde{p}\left(\frac{x}{\tau(\mathbf{y})}\right)+q'(x)\frac{(\tau(\mathbf{y})-x)^2}{\tau^2(\mathbf{y})}\tilde{p}'\left(\frac{x}{\tau(\mathbf{y})}\right)$$

$$=\frac{q'(x)}{c_i}\left[q'(x)\frac{(\tau(\mathbf{y})-x)\frac{\partial\tau(\mathbf{y})}{\partial y_i}(\tau(\mathbf{y})+x)}{\tau^2(\mathbf{y})}\tilde{p}\left(\frac{x}{\tau(\mathbf{y})}\right)-q'(x)\frac{(\tau(\mathbf{y})-x)^2\frac{\partial\tau(\mathbf{y})}{\partial y_i}x}{\tau^2(\mathbf{y})}\tilde{p}'\left(\frac{x}{\tau(\mathbf{y})}\right)\right];$$
$$i=1,2,\ldots,n. \tag{25}$$

Take derivative of (25) with respect to y_i again ;

$$\left[\tilde{p}'\left(\frac{x}{\tau(\mathbf{y})}\right)\frac{x\frac{\partial\tau(\mathbf{y})}{\partial y_i}}{\tau^2(\mathbf{y})}(-q''(x)(\tau(\mathbf{y})-x)+3q'(x))-\frac{q'(x)x\frac{\partial\tau(\mathbf{y})}{\partial y_i}(\tau(\mathbf{y})-x)}{\tau^3(\mathbf{y})}\tilde{p}''\left(\frac{x}{\tau(\mathbf{y})}\right)\right.$$

$$\left.+q''(x)\frac{\partial\tau(\mathbf{y})}{\partial y_i}\tilde{p}\left(\frac{x}{\tau(\mathbf{y})}\right)\right]\cdot\left[q'(x)\frac{\frac{\partial\tau(\mathbf{y})}{\partial y_i}(\tau(\mathbf{y})+x)}{\tau(\mathbf{y})}\tilde{p}\left(\frac{x}{\tau(\mathbf{y})}\right)-q'(x)\frac{(\tau(\mathbf{y})-x)\frac{\partial\tau(\mathbf{y})}{\partial y_i}x}{\tau^2(\mathbf{y})}\tilde{p}'\left(\frac{x}{\tau(\mathbf{y})}\right)\right]$$

$$=\left[q'(x)\frac{\frac{\partial^2\tau(\mathbf{y})}{\partial y_{i^2}}\tau(\mathbf{y})(\tau(\mathbf{y})+x)-(\frac{\partial\tau(\mathbf{y})}{\partial y_i})^2x}{\tau^2(\mathbf{y})}\tilde{p}\left(\frac{x}{\tau(\mathbf{y})}\right)-\frac{q'(x)}{\tau^3(\mathbf{y})}(x\frac{\partial^2\tau(\mathbf{y})}{\partial y_{i^2}}\tau(\mathbf{y})(\tau(\mathbf{y})-x)\right.$$

$$\left.+3x^2(\frac{\partial\tau(\mathbf{y})}{\partial y_i})^2)\tilde{p}'\left(\frac{x}{\tau(\mathbf{y})}\right)+q'(x)\frac{(\tau(\mathbf{y})-x))x^2(\frac{\partial\tau(\mathbf{y})}{\partial y_i})^2}{\tau^4(\mathbf{y})}\tilde{p}''\left(\frac{x}{\tau(\mathbf{y})}\right)\right]\cdot\left[q''(x)(\tau(\mathbf{y})-x)\tilde{p}\left(\frac{x}{\tau(\mathbf{y})}\right)\right.$$

$$\left.-2q'(x)\tilde{p}\left(\frac{x}{\tau(\mathbf{y})}\right)+q'(x)(\tau(\mathbf{y})-x)\tilde{p}'\left(\frac{x}{\tau(\mathbf{y})}\right)\right];\quad i=1,2,\ldots,n. \tag{26}$$

Observe that the functions $(\tilde{p}'(z))^2$, $\tilde{p}'(z)\,\tilde{p}(z)$, $(\tilde{p}(z))^2$, $\tilde{p}''(z)\,\tilde{p}(z)$, $\tilde{p}''(z)\,\tilde{p}'(z)$ and $(\tilde{p}''(z))^2$,where $\tilde{p}$ are arbitrary Herglötz functions, are independent. In fact they are independent even for $\tilde{p}(w)=\frac{1}{1-w}+a, a>0$, what can be checked by straightforward calculations. Hence coefficients in $(\tilde{p}'(z))^2$, $\tilde{p}'(z)\,\tilde{p}(z)$, $(\tilde{p}(z))^2$, $\tilde{p}''(z)\,\tilde{p}(z)$, $\tilde{p}''(z)\,\tilde{p}'(z)$and $(\tilde{p}''(z))^2$ in the left and right hand part of (26)coincide.

In particular,

$$x\left(\frac{\partial\tau(\mathbf{y})}{\partial y_i}\right)^2q''(x)=-\tau(\mathbf{y})\frac{\partial^2\tau(\mathbf{y})}{\partial y_i^2}q'(x) \tag{27}$$

and

$$q'(x)=-xq''(x). \tag{28}$$

Then

$$q(x) = \alpha \ln x, \tag{29}$$

where α is a constant. If we substitute this in (27) then we obtain

$$\left(\frac{\partial \tau(\mathbf{y})}{\partial y_i}\right)^2 = \tau(\mathbf{y}) \frac{\partial^2 \tau(\mathbf{y})}{\partial y_i^2}, \quad 1 \leq i \leq n. \tag{30}$$

It is easy to see that any solution of system (30) can be written as

$$\tau(\mathbf{y}) = h_1(y_2, \dots, y_n) e^{y_1 g_1(y_2, \dots, y_n)} = h_2(y_1, y_3, \dots, y_n) e^{y_2 g_2(y_1, y_3, \dots, y_n)} =$$

$$\dots = h_n(y_1, \dots, y_{n-1}) e^{y_n g_n(y_1, \dots, y_{n-1})},$$

where $h_1, \dots, h_n$ are sufficiently smooth functions.Then

$$\frac{\partial^n \ln \tau(\mathbf{y})}{\partial y_1 \dots \partial y_n} = \frac{\partial^{n-1} g_1(y_2, \dots, y_n)}{\partial y_2 \dots \partial y_n} = \dots = \frac{\partial^{n-1} \partial g_n(y_1, \dots, y_{n-1})}{\partial y_1 \dots \partial y_{n-1}} = c. \tag{31}$$

It gives the general solution of (30)

$$\ln \tau(\mathbf{y}) = c y_n \dots y_1 + \tilde{g}_1(y_2, \dots, y_n) + \dots + \tilde{g}_n(y_1, \dots, y_{n-1}), \tag{32}$$

where $\tilde{g}_1, \dots, \tilde{g}_n$ are arbitrary sufficiently smooth functions.If we put this $\tau(\mathbf{y})$ in (25) and take coefficient of $\tilde{p}'(z)$, then we obtain

$$\frac{c_i}{t_i} = \prod_{k=1}^{n} y_k + \sum_{k=1, k \neq i}^{n} \frac{\partial \tilde{g}_k(y_1, ..., \tilde{y}_i, ..., y_n)}{\partial y_i} \tag{33}$$

Taking derivative $\frac{\partial^{n-1}}{\partial y_2 \dots \partial y_n}$ of (33) with $i = 1$, we obtain $c = 0$.

Moreover we claim that (32) and (33) imply $\tau(\mathbf{y}) = \exp(\mathbf{K} \cdot \mathbf{y})$. Indeed for $n = 1$ it follows immediately from (30). For $n > 1$ we take derivative of (33) with respect to $y_2, \dots, y_{n-1}$ we obtain

$$\frac{\partial^{n-1} \tilde{g}_n(y_1, \dots, y_{n-1})}{\partial y_1 \dots \partial y_{n-1}} = 0.$$

Hence $\tilde{g}_n$ can be written as sum of functions of $n-2$ variables. By induction it implies $\tau(\mathbf{y}) = \exp(\sum\limits_{i=1}^{n} K_i y_i)$, and application of (30) finishes the proof of the necessity part.

Now Theorem (9) says that the generator A of the process ψ_t from (11) is given by (12). ■

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