The two dimensional inverse conductivity problem

Dedicated to Gennadi

Vincent MICHEL

IMJ

September 12, 2016

- (M, σ) is a **2-dimensionnal conductivity structure** when M is an abstract 2 dimensional manifold with boundary ($M \cap bM = \emptyset$) and $\sigma : T^*M \to T^*M$ is a tensor such that

$$\forall a, b \in T^*\overline{M},\ \sigma(a) \wedge b = \sigma(b) \wedge a,$$
$$\forall p \in \overline{M},\ \exists \lambda_p \in \mathbb{R}_+^*,\ \forall a \in T_p^*\overline{M},\ \sigma_p(a) \wedge a \geqslant \lambda_p \|a\|_p \mu_p.$$

where $\|.\|_p$ is a norm on $T_p^*\overline{M}$ and μ is a volume form for $\overline{M}$.

- (M, σ) is a **2-dimensionnal conductivity structure** when M is an abstract 2 dimensional manifold with boundary ($M \cap bM = \emptyset$) and $\sigma : T^*M \longrightarrow T^*M$ is a tensor such that

$$\forall a, b \in T^*\overline{M},\ \sigma(a) \wedge b = \sigma(b) \wedge a,$$
$$\forall p \in \overline{M},\ \exists \lambda_p \in \mathbb{R}_+^*,\ \forall a \in T_p^*\overline{M},\ \sigma_p(a) \wedge a \geqslant \lambda_p \|a\|_p \mu_p.$$

where $\|.\|_p$ is a norm on $T_p^*\overline{M}$ and μ is a volume form for $\overline{M}$.

- **Dirichlet operator** D_σ. For $u \in C^0(bM, \mathbb{R})$, $D_\sigma u \in C^0\left(\overline{M}\right)$ is defined by

$$d\sigma(dD_\sigma u) = 0 \ \ \& \ \ (D_\sigma u)|_{bM} = u$$

- (M, σ) is a **2-dimensionnal conductivity structure** when M is an abstract 2 dimensional manifold with boundary ($M \cap bM = \emptyset$) and $\sigma : T^*M \to T^*M$ is a tensor such that

$$\forall a, b \in T^*\overline{M},\ \sigma(a) \wedge b = \sigma(b) \wedge a,$$
$$\forall p \in \overline{M},\ \exists \lambda_p \in \mathbb{R}_+^*,\ \forall a \in T_p^*\overline{M},\ \sigma_p(a) \wedge a \geqslant \lambda_p \|a\|_p \mu_p.$$

where $\|.\|_p$ is a norm on $T_p^*\overline{M}$ and μ is a volume form for $\overline{M}$.

- **Dirichlet operator** D_σ. For $u \in C^0(bM, \mathbb{R})$, $D_\sigma u \in C^0(\overline{M})$ is defined by

$$d\sigma(dD_\sigma u) = 0 \ \ \& \ \ (D_\sigma u)|_{bM} = u$$

- **Neumann-Dirichlet operator** N_σ. For $u : bM \to \mathbb{R}$ sufficiently smooth, $N_\sigma u$ is defined by

$$N_\sigma u = \frac{\partial}{\partial \nu} D_\sigma u : bM \to \mathbb{R}$$

where $\nu \in T_{bM}\overline{M}$ is the outer unit normal vector field of bM.

- Using *isothermal coordinates*, one find out that

$$\sigma = (\det \sigma) \cdot *_\sigma$$

where $*_\sigma$ is the *Hodge star* operator associated to a complex structure $\mathcal{C}_\sigma$ on M.

- Using *isothermal coordinates*, one find out that

$$\sigma = (\det \sigma) \cdot *_\sigma$$

where $*_\sigma$ is the *Hodge star* operator associated to a complex structure $\mathcal{C}_\sigma$ on M.

- If M is a submanifold of $\mathbb{R}^3$, σ is isotropic when $\mathcal{C}_\sigma$ is induced by the standard euclidean metric of $\mathbb{R}^3$. Likewise, σ is said isotropic relatively to a complex structure $\mathcal{C}$ on M if $*_\sigma$ is the Hodge operator of $(M, \mathcal{C})$.

- Using *isothermal coordinates*, one find out that

$$\sigma = (\det \sigma) \cdot *_\sigma$$

where $*_\sigma$ is the *Hodge star* operator associated to a complex structure $\mathcal{C}_\sigma$ on M.

- If M is a submanifold of $\mathbb{R}^3$, σ is isotropic when $\mathcal{C}_\sigma$ is induced by the standard euclidean metric of $\mathbb{R}^3$. Likewise, σ is said isotropic relatively to a complex structure $\mathcal{C}$ on M if $*_\sigma$ is the Hodge operator of $(M, \mathcal{C})$.
- Dirichlet problem for $\left(\overline{M}, \sigma\right)$. For $u \in C^0(bM)$, seek U such that

$$dsd^\sigma U = 0 \;\; \& \;\; U\,|_{bM} = u$$

where $s = \det \sigma$, $d^\sigma = i\left(\overline{\partial}^\sigma - \partial^\sigma\right)$, $\overline{\partial}^\sigma$ is the standard Cauchy-Riemann operator associated to the Riemann surface $(M, \mathcal{C}_\sigma)$ and $\partial^\sigma = d - \overline{\partial}^\sigma$.

Inverse conductivity problem

Data : bM, $\nu \in T_{bM}\overline{M}$, $\sigma\,|_{bM}$ and N_σ

Problem : reconstruct M as a Riemann surface equipped with the conductivity tensor σ.

Remark : Let $\varphi : \overline{M} \to \overline{M}$ be a C^1-diffeomorphism such that $\varphi\,|_{bM} = Id_{bM}$ and $\widetilde{\sigma} = \varphi_* \sigma$. Then $N_{\widetilde{\sigma}} = N_\sigma$ and $\widetilde{\sigma} \neq \sigma$ but $(M, \mathcal{C}_{\widetilde{\sigma}})$ and $(M, \mathcal{C}_\sigma)$ represent the same (abstract) Riemann surface.

Consequence : non uniqueness up to a diffeomorphism gives different representations of the same Riemann surface.

- non exhaustive list for uniqueness results,and for a domain in $\mathbb{R}^2$, reconstructions process by Kohn-Vogelius (1984), Novikov (1988), Lee-Uhlman (1989), Sylvester (1990), Lassas-Uhlman (2001), Belishev (2003), Nachman (1996),
- Henkin-Michel (2007) : About reconstruction when C_σ is known and $\det \sigma = 1$
- Henkin-Santecesaria (2010) : Construction of $(\mathcal{M}, \widetilde{\sigma})$ where $\mathcal{M}$ is a bordered nodal Riemann surface of $\mathbb{CP}_2$ which represents C_σ except perhaps at a finite set of points and such that the pushforward $\widetilde{\sigma}$ of σ to $\mathcal{M}$ is isotropic.
- Henkin-Novikov (2011) : Reconstruction, isotropic case (C_σ is known)

- **Plan for solving the reconstruction problem** :

- non exhaustive list for uniqueness results,and for a domain in $\mathbb{R}^2$, reconstructions process by Kohn-Vogelius (1984), Novikov (1988), Lee-Uhlman (1989), Sylvester (1990), Lassas-Uhlman (2001), Belishev (2003), Nachman (1996),
- Henkin-Michel (2007) : About reconstruction when C_σ is known and $\det \sigma = 1$
- Henkin-Santecesaria (2010) : Construction of $(\mathcal{M}, \widetilde{\sigma})$ where $\mathcal{M}$ is a bordered nodal Riemann surface of $\mathbb{CP}_2$ which represents C_σ except perhaps at a finite set of points and such that the pushforward $\widetilde{\sigma}$ of σ to $\mathcal{M}$ is isotropic.
- Henkin-Novikov (2011) : Reconstruction, isotropic case (C_σ is known)

- **Plan for solving the reconstruction problem** :
  1. use of improved results of H-M to complete H-S and produce a Riemann surface S representing (M, C_σ) and where the conductivity is isotropic.

- non exhaustive list for uniqueness results,and for a domain in $\mathbb{R}^2$, reconstructions process by Kohn-Vogelius (1984), Novikov (1988), Lee-Uhlman (1989), Sylvester (1990), Lassas-Uhlman (2001), Belishev (2003), Nachman (1996),
- Henkin-Michel (2007) : About reconstruction when C_σ is known and $\det \sigma = 1$
- Henkin-Santecesaria (2010) : Construction of $(\mathcal{M}, \widetilde{\sigma})$ where $\mathcal{M}$ is a bordered nodal Riemann surface of $\mathbb{CP}_2$ which represents C_σ except perhaps at a finite set of points and such that the pushforward $\widetilde{\sigma}$ of σ to $\mathcal{M}$ is isotropic.
- Henkin-Novikov (2011) : Reconstruction, isotropic case (C_σ is known)

- **Plan for solving the reconstruction problem** :
  1. use of improved results of H-M to complete H-S and produce a Riemann surface S representing (M, C_σ) and where the conductivity is isotropic.
  2. use of H-N to produce the function $s : S \to \mathbb{R}_+^*$ such that $s \cdot *$ is the pushforward of σ.

- non exhaustive list for uniqueness results,and for a domain in $\mathbb{R}^2$, reconstructions process by Kohn-Vogelius (1984), Novikov (1988), Lee-Uhlman (1989), Sylvester (1990), Lassas-Uhlman (2001), Belishev (2003), Nachman (1996),
- Henkin-Michel (2007) : About reconstruction when C_σ is known and $\det \sigma = 1$
- Henkin-Santecesaria (2010) : Construction of $(\mathcal{M}, \widetilde{\sigma})$ where $\mathcal{M}$ is a bordered nodal Riemann surface of $\mathbb{CP}_2$ which represents C_σ except perhaps at a finite set of points and such that the pushforward $\widetilde{\sigma}$ of σ to $\mathcal{M}$ is isotropic.
- Henkin-Novikov (2011) : Reconstruction, isotropic case (C_σ is known)

- **Plan for solving the reconstruction problem** :
  1. use of improved results of H-M to complete H-S and produce a Riemann surface S representing (M, C_σ) and where the conductivity is isotropic.
  2. use of H-N to produce the function $s : S \to \mathbb{R}_+^*$ such that $s \cdot *$ is the pushforward of σ.
  3. $(S, s \cdot *)$ is a solution to our inverse conductivity problem.

Tools for step 1

1. Based on Henkin-Michel (2014) : explicit formulas for a Green function of the bordered nodal Riemann surface $\mathcal{M}$. This enable to compute for a given $u : bM \to \mathbb{R}$ the C_σ-harmonic extension $\widetilde{u}$ $(dd^\sigma \widetilde{u} = 0)$ of u from N_σ
2. Based on Henkin-Michel (2012) : embedding S of M in $\mathbb{CP}_4$ by a generic canonical map $(\partial\widetilde{u_0} : \partial\widetilde{u_1} : \partial\widetilde{u_2} : \partial\widetilde{u_3})$; S is given as the solution of boundary problem. Then we seek an atlas for S. For generic data, S is covered by preimages of regular parts of the images Q and Q' of $S\setminus\{(0:0:0:1)\}$ and $S\setminus\{(0:0:1:0)\}$ under the projections $\mathbb{CP}_4 \to \mathbb{CP}_3$, $(w_0 : w_1 : w_2 : w_3) \mapsto (w_0 : w_1 : w_2)$ and $(w_0 : w_1 : w_2 : w_3) \mapsto (w_0 : w_1 : w_3)$. This reduces the problem by one dimension.

Let Q be the (possibly singular) nodal complex curve which is the image of S by the projection $\mathbb{CP}_4 \to \mathbb{CP}_3$, $(w_0 : w_1 : w_2 : w_3) \mapsto (w_0 : w_1 : w_2)$. Some generic assumptions are made on $\overline{Q}$ such like $(0:1:0) \notin \overline{Q}$ and $bQ \subset \{w_0 w_1 w_2 \neq 0\}$. Let for $z = (x, y) \in \mathbb{C}^2$, L_z be the line of equation $\Lambda_z(w) \overset{def}{=} xw_0 + yw_1 + w_2 = 0$. Then from Dolbeault-Henkin (1997), for z near a generic z_*,

$$G_{\partial Q,k}(z) \overset{def}{=} \frac{1}{2\pi i} \int_{\partial Q} \left(\frac{w_1}{w_0}\right)^k \frac{d\frac{\Lambda_z(w)}{w_0}}{\frac{\Lambda_z(w)}{w_0}} = \sum_{1 \leqslant j \leqslant p} h_j(z)^k + P_k(z) \quad \text{(SWD)}$$

where $P_k \in \mathbb{C}(Y)_k[X]$ and $h_1, ..., h_k$ are holomorphic solutions of the **shock wave** equation

$$\frac{\partial h}{\partial y} = h \frac{\partial h}{\partial x}$$

such that

$$Q \cap L_z = \{(1 : h_j(z) : -x - yh_j(z)) ;\ 1 \leqslant j \leqslant p\}$$

- **Difficulty :** If K is algebraic and
$$K \cap L_z = \left\{ \left(1 : \varphi_j(z) : -x - y\varphi_j(z) \right) ; \ 1 \leqslant j \leqslant q \right\},$$
$\sum\limits_{1 \leqslant j \leqslant q} \varphi_j(z)^k \in \mathbb{C}(Y)_k[X]$. How distinguish Q from $K \cup Q$?

- **Difficulty :** If K is algebraic and

$$K \cap L_z = \left\{ \left(1 : \varphi_j(z) : -x - y\varphi_j(z) \right) ;\ 1 \leqslant j \leqslant q \right\},$$

$\sum\limits_{1 \leqslant j \leqslant q} \varphi_j(z)^k \in \mathbb{C}(Y)_k[X]$. How distinguish Q from $K \cup Q$?

- For $\deg P_1 \leqslant 2$, Agaltsov-Henkin (2015) gives an algorithm to get P_1 and (h_j). For $\deg P_1 \geqslant 3$, we propose a different method based on the plan :

- **Difficulty :** If K is algebraic and

$$K \cap L_z = \left\{ \left(1 : \varphi_j(z) : -x - y\varphi_j(z) \right) ;\ 1 \leqslant j \leqslant q \right\},$$

$\sum_{1 \leqslant j \leqslant q} \varphi_j(z)^k \in \mathbb{C}(Y)_k[X]$. How distinguish Q from $K \cup Q$?

- For $\deg P_1 \leqslant 2$, Agaltsov-Henkin (2015) gives an algorithm to get P_1 and (h_j). For $\deg P_1 \geqslant 3$, we propose a different method based on the plan :

  1. Find a decomposition $G_{\partial Q,1} = \sum_{1 \leqslant j \leqslant d} g_j + P$ of type (SWD).

- **Difficulty :** If K is algebraic and

$$K \cap L_z = \left\{ \left(1 : \varphi_j(z) : -x - y\varphi_j(z) \right) ;\ 1 \leqslant j \leqslant q \right\},$$

$\sum_{1 \leqslant j \leqslant q} \varphi_j(z)^k \in \mathbb{C}(Y)_k[X]$. How distinguish Q from $K \cup Q$?

- For $\deg P_1 \leqslant 2$, Agaltsov-Henkin (2015) gives an algorithm to get P_1 and (h_j). For $\deg P_1 \geqslant 3$, we propose a different method based on the plan :
  1. Find a decomposition $G_{\partial Q,1} = \sum_{1 \leqslant j \leqslant d} g_j + P$ of type (SWD).
  2. If $\sum_{1 \leqslant j \leqslant d} g_j \in \mathbb{C}(Y)_1[X]$, then from e.g Wood (1984), Q is a domain in an algebraic curve and requires a special process.

- **Difficulty :** If K is algebraic and

$$K \cap L_z = \left\{ \left(1 : \varphi_j(z) : -x - y\varphi_j(z)\right) ;\ 1 \leqslant j \leqslant q \right\},$$

$\sum\limits_{1 \leqslant j \leqslant q} \varphi_j(z)^k \in \mathbb{C}(Y)_k[X]$. How distinguish Q from $K \cup Q$?

- For $\deg P_1 \leqslant 2$, Agaltsov-Henkin (2015) gives an algorithm to get P_1 and (h_j). For $\deg P_1 \geqslant 3$, we propose a different method based on the plan :

1. Find a decomposition $G_{\partial Q,1} = \sum\limits_{1 \leqslant j \leqslant d} g_j + P$ of type (SWD).
2. If $\sum\limits_{1 \leqslant j \leqslant d} g_j \in \mathbb{C}(Y)_1[X]$, then from e.g Wood (1984), Q is a domain in an algebraic curve and requires a special process.
3. If $\sum\limits_{1 \leqslant j \leqslant d} g_j \notin \mathbb{C}(Y)_1[X]$, let J be the maximal subset of $\{1, ..., d\}$ such that $\sum\limits_{j \in J} g_j \in \mathbb{C}(Y)_1[X]$ and write $G_{\partial Q,1} = \sum\limits_{j \notin J} g_j + \widetilde{P}$ where $\widetilde{P} \in \mathbb{C}(Y)_1[X]$.

- **Difficulty :** If K is algebraic and

$$K \cap L_z = \left\{ \Big(1 : \varphi_j(z) : -x - y\varphi_j(z)\Big) \,;\; 1 \leqslant j \leqslant q \right\},$$

$\sum\limits_{1 \leqslant j \leqslant q} \varphi_j(z)^k \in \mathbb{C}(Y)_k[X]$. How distinguish Q from $K \cup Q$?

- For $\deg P_1 \leqslant 2$, Agaltsov-Henkin (2015) gives an algorithm to get P_1 and (h_j). For $\deg P_1 \geqslant 3$, we propose a different method based on the plan :
    1. Find a decomposition $G_{\partial Q,1} = \sum\limits_{1 \leqslant j \leqslant d} g_j + P$ of type (SWD).
    2. If $\sum\limits_{1 \leqslant j \leqslant d} g_j \in \mathbb{C}(Y)_1[X]$, then from e.g Wood (1984), Q is a domain in an algebraic curve and requires a special process.
    3. If $\sum\limits_{1 \leqslant j \leqslant d} g_j \notin \mathbb{C}(Y)_1[X]$, let J be the maximal subset of $\{1, ..., d\}$ such that $\sum\limits_{j \in J} g_j \in \mathbb{C}(Y)_1[X]$ and write $G_{\partial Q,1} = \sum\limits_{j \notin J} g_j + \widetilde{P}$ where $\widetilde{P} \in \mathbb{C}(Y)_1[X]$.
    4. With Henkin (1995) and Collion (1996), one can prove that $(g_j) = (h_j)$ and $\widetilde{P} = P$.

Building a SWD $G_{\partial Q,1} = \sum_{1 \leqslant j \leqslant d} g_j + P$

1. A function N is the sum of d different shock wave functions iff there exists $s_1, \ldots, s_d$ such that $s_1 = -N$, and

$$-s_d \frac{\partial N}{\partial x} + \frac{\partial s_d}{\partial y} = 0, \quad -s_k \frac{\partial N}{\partial x} + \frac{\partial s_k}{\partial y} = \frac{\partial s_{k+1}}{\partial x}, \ 1 \leqslant k \leqslant d-1, \quad (1)$$

and the discriminant of $T^d + s_1 T^{d-1} + \cdots + s_d \in \mathcal{O}(D)[T]$ is not 0.

2. Let $N = G_1 - P$ where P is of the form $\frac{B'}{B} X + \frac{A}{B}$ with $A, B \in \mathbb{C}[Y]$, $B(0) = 1$ and $\deg A < \deg B$. $(s_1, \ldots, s_d)$ is a solution of (1) iff there exists one variable holomorphic functions $\mu_1, \ldots, \mu_d$ such that

$$s_k = \frac{e^H}{1 \otimes B} \left(\mathcal{E}^0 (\mu_k \otimes 1) + \cdots + \mathcal{E}^{d-k} (\mu_d \otimes 1) \right) \quad (2)$$

where H is such that $\frac{\partial H}{\partial y} = \frac{\partial G_1}{\partial x}$, $\mathcal{E}^j = \mathcal{E}^{j-1} \circ \mathcal{E}$ and $\mathcal{E} = \int^y e^{-H} \frac{\partial}{\partial x} e^H$.

- For a given $\mu = (\mu_1, ..., \mu_d)$, $\left((-1)^j s_j\right)_{1 \leqslant j \leqslant d}$ defined by (2) are the symmetric functions of a d-uple of shock wave functions iff $s_1 = -N$ which turns out to be equivalent to a linear differential system

$$\forall n \in \mathbb{Z}, \quad \sum_{0 \leqslant m < j \leqslant d} c_{j,m}^{0,n} \mu_j^{(m)} = K_n^0 (B, A, .) \tag{3}$$

where $\left(c_{j,m}^{0,n}\right)$ depends only on G_1 and $K_n^0 (B, A, .)$ vanishes for $n \geqslant d$, is linear with respect to (A, B) and has coefficients depending only on G_1 .

- For a given $\mu = (\mu_1, ..., \mu_d)$, $\left((-1)^j s_j\right)_{1 \leqslant j \leqslant d}$ defined by (2) are the symmetric functions of a d-uple of shock wave functions iff $s_1 = -N$ which turns out to be equivalent to a linear differential system

$$\forall n \in \mathbb{Z}, \quad \sum_{0 \leqslant m < j \leqslant d} c_{j,m}^{0,n} \mu_j^{(m)} = K_n^0 \, (B, A, .) \tag{3}$$

where $\left(c_{j,m}^{0,n}\right)$ depends only on G_1 and $K_n^0 \, (B, A, .)$ vanishes for $n \geqslant d$, is linear with respect to (A, B) and has coefficients depending only on G_1 .
- The compatibility system has at least a solution. This solution gives birth to a solution μ for (3) and thus to a SWD.

Process

- If G_1 is affine in x, solving the compatibility system on (A, B) and then (3) gives, after reduction, a parametrization of Q by its intersection with the lines L_z.

Process

- If G_1 is affine in x, solving the compatibility system on (A, B) and then (3) gives, after reduction, a parametrization of Q by its intersection with the lines L_z.
- If G_1 is affine in x, Q is a "special" domain in an algebraic curve K. This reverts to the 1st case by choosing coordinates such that at least one line L_z hits Q and $K \backslash Q$.

Green formula for singular complex curves

Smooth case symmetric dunction $g : \left(\overline{M} \times \overline{M}\right) \setminus \Delta_{\overline{M}} \to' \mathbb{R}$ such that for all $q \in M$,

- $g_q = g\left(q, .\right)$ is harmonic on $M \setminus \{q\}$ and continuous on $\overline{M} \setminus \{q\}$
- $g_q = -\frac{1}{2\pi} \ln |z|$ extends harmonically around q (z holomorphic coordinate z centered at q)

Singular case A Green function for a curve $\mathcal{Y}$ in $\mathbb{C}^2$ is a symmetric function $g : \left(\mathrm{Reg}\,\overline{\mathcal{Y}} \times \mathrm{Reg}\,\overline{\mathcal{Y}}\right) \setminus \Delta_{\mathrm{Reg}\,\overline{\mathcal{Y}}} \to \mathbb{R}$ s.t. for all $q \in \mathrm{Reg}\,\overline{\mathcal{Y}}$, $g_q = g\left(q, .\right)$ satisfies

$$i\partial\overline{\partial} g_q = \delta_q dV$$

in the sense of currents on $\mathcal{Y}$, δ_q being the Dirac measure supported by $\{q\}$ and $dV = i\partial\overline{\partial}\left|.\right|^2$ - this implies in particular that ∂g_q is a weakly holomorphic $(1,0)$-form on $\mathcal{Y} \setminus \{q\}$ in the sense of Rosenlicht

Nodal case

We demands that $\mathcal{Y}$ extends as a usual continuous function along any branch except for one branch passing trough q where it has an isolated logarithmic singularity.

Let us now detail a formula of Henkin-Michel (2014) establishing the existence of Green functions for a 1-paramter family of complex curves whose possible singularities are arbitrary. Let us consider a complex curve $\mathcal{Y}$ in an open subset of $\mathbb{C}^2$, Ω a Stein neighborhood of $\mathcal{Y}$ in $\mathbb{C}^2$, Φ a holomorphic function on Ω such that $\mathcal{Y} = \{\Phi = 0\}$ and $d\Phi|_{\mathcal{Y}} \neq 0$ then a strictly pseudoconvex domain Ω^* of $\mathbb{C}^2$ verifying

$$\mathcal{Y}_0 = \mathcal{Y} \cap \Omega^* \subset \Omega,$$

and lastly a symmetric function $\Psi \in \mathcal{O}\left(\Omega \times \Omega, \mathbb{C}^2\right)$ such that for all $(z, z') \in \mathbb{C}^2$,

$$\Phi\left(z'\right) - \Phi\left(z\right) = \left\langle \Psi\left(z', z\right), z' - z \right\rangle$$

where $\langle v, w \rangle = v_1 w_1 + v_2 w_2$ when $v, w \in \mathbb{C}^2$. We define on RegY a $(1,0)$-form ω by setting

$$\omega = \frac{-dz_1}{\partial\Phi/\partial z_2} \text{ on } \mathcal{Y}^1 = \mathcal{Y} \cap \{\partial\Phi/\partial z_2 \neq 0\}$$
$$\omega = \frac{+dz_2}{\partial\Phi/\partial z_1} \text{ on } \mathcal{Y}^2 = \mathcal{Y} \cap \{\partial\Phi/\partial z_1 \neq 0\}$$

and we consider

$$k\left(z',z\right)=\det\left[\frac{\overline{z'}-\overline{z}}{\left|z'-z\right|^{2}},\Psi\left(z',z\right)\right].$$

When $q_{*}\in\operatorname{Reg}\mathcal{Y}_{0}$, H-M (2014) proves that the formula

$$g_{q_{*}}\left(q\right)=\frac{1}{4\pi^{2}}\int_{q'\in\mathcal{Y}_{0}}k\left(q',q\right)k\left(q_{*},q'\right)\ i\omega\left(q'\right)\wedge\overline{\omega}\left(q'\right). \qquad (4)$$

defines for $\mathcal{Y}_{0}$ a Green function in the above sense and that if $q_{*}\in\operatorname{Reg}\mathcal{Y}_{0}$

$$\partial g_{q_{*}}=\widetilde{k}_{q_{*}}\omega$$

where $\widetilde{k}_{q_{*}}=\frac{1}{2\pi}k\left(.,q_{*}\right)$.

Proposition

Suppose $\mathcal{Y}$ has only nodal singularities. In this case, when $q_* \in \operatorname{Reg} \mathcal{Y}_0$, g_{q_*} extends as usual harmonic function along the branches of $\mathcal{Y}_0 \setminus \{q_*\}$; in other words, ∂g_{q_*} extends as a standard holomorphic $(1,0)$-form along the branches of $\mathcal{Y}_0 \setminus \{q_*\}$.

Corollary

Suppose that $\mathcal{Y}$ is an open nodal Riemann surface of $\mathbb{C}^2$ and g is defined by (4). Then g is a simple Green function for $\mathcal{Y}$.

Corollary

Let (M, σ) *be a two dimensional conductivity structure. We select, which is always possible, a two dimensional conductivity structure* $(\widetilde{M}, \widetilde{\sigma})$ *extending plainly* (M, σ), *which means that* $M \subset\subset \widetilde{M}$, $\widetilde{\sigma}\,|_M = \sigma$ *and* $\widetilde{\sigma}\,|_p = Id_{T_p^*\widetilde{M}}$ *for all* $p \in b\widetilde{M}$. *On denote then by* $F : \widetilde{M} \to \mathbb{C}^2$ *the map obtained by applying H-S theorem to* $(\widetilde{M}, \widetilde{\sigma})$, *we set* $\mathcal{Y} = F\left(\widetilde{M}\right)$ *and fix a Stein neighborhood* Ω *of* $\mathcal{Y}$ *in* $\mathbb{C}^2$. *Lastly,* $\mathcal{M} = F(M)$ *being relatively compact in* $\mathcal{Y}$, *we can pick up in* $\mathbb{C}^2$ *a strictly pseudoconvex domain* Ω^* *s.t.* $\mathcal{M} \subset\subset \mathcal{Y}_0 = \mathcal{Y} \cap \Omega^* \subset \Omega$. *We note* g *the function defined by (4). Then,* $F^*g\,\Big|_{\overline{M}\times\overline{M}\setminus\Delta_M}$ *is a Green function for* (M, c_σ).

Corollary

$\mathcal{M}$ admits a principal Green function and if g is such a function, for all $u \in C^{\infty}(bM)$, $F^{*}\theta^{\mathcal{M}} f_{*}u$ is given by the formula

$$F^{*}\theta^{\mathcal{M}} f_{*}u = \left(F^{*}\partial\widehat{f_{*}u}\right)|_{bM} \ , \ \widehat{f_{*}u} : \mathrm{Reg}\,\overline{\mathcal{M}} \ni q \mapsto \left| \begin{array}{l} \frac{i}{2}\int_{\partial\mathcal{M}} (f_{*}u)\,\overline{\partial} g_{q} \ \ si \ q \in \mathcal{M} \\ f_{*}u\,(q) \ \ si \ q \in b\mathcal{M} \end{array} \right.$$