Inverse scattering in multidimensions

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We give a short review of old and recent results on inverse scattering in multidimensions related with works of G.M. Henkin. This talk is based, in particular, on the following references:

• G.M. Henkin, R.G. Novikov, The $\bar{\partial}$-equation in the multidimensional inverse scattering problem, Russ. Math. Surv. 42(3), 109-180, 1987;

• G.M. Henkin, N.N. Novikova, The reconstruction of the attracting potential in the Sturm-Liouville equation through characteristics of negative discrete spectrum, Stud. Appl. Math. 97, 17-52, 1996;

• R.G. Novikov, The $\bar{\partial}$-approach to monochromatic inverse scattering in three dimensions, J. Geom. Anal. 18(2), 612-631, 2008;

• R.G. Novikov, Formulas for phase recovering from phaseless scattering data at fixed frequency, Bull. Sci. Math. 139(8), 923-936, 2015.

We consider the Schrödinger equation

$$-\Delta\psi + v(x)\psi = E\psi, \quad x \in \mathbb{R}^d,\ E > 0,\ d \geq 2, \tag{1}$$

where

$$|v(x)| \leq \frac{q}{(1+|x|)^\sigma},\ q \geq 0,\ \sigma > d. \tag{2}$$

For equation (1) we consider the classical scattering eigenfunctions ψ^+ specified by the following asymptotics as $|x| \to \infty$:

$$\psi^+(x,k) = e^{ikx} + c(d,|k|)\frac{e^{i|k||x|}}{|x|^{(d-1)/2}} f\big(k, |k|\frac{x}{|x|}\big) + o\big(\frac{1}{|x|^{(d-1)/2}}\big), \tag{3}$$

$x \in \mathbb{R}^d,\ k \in \mathbb{R}^d,\ k^2 = E,\ c(d,|k|) = -\pi i(-2\pi i)^{(d-1)/2}|k|^{(d-3)/2}$,

where a priori unknown function f on

$$\mathcal{M}_E = \{k, l \in \mathbb{R}^d : \quad k^2 = l^2 = E\} \tag{4}$$

arising in (3) is the classical scattering amplitude for equation (1).

Direct scattering: $v \to \psi^+ \to f$
Inverse scattering: $f \to v$

Works of G.M. Henkin make important contributions into inverse scattering in multidimensions. To present some of these contributions and their impacts, we need to recall some preliminary results and some historical points.

Direct scattering:

$$\psi^+(x,k) = e^{ikx} + \int\limits_{\mathbb{R}^d} G^+(x-y,k)v(y)\psi^+(y,k)dy \qquad (5)$$

(Lippmann-Schwinger integral equation for ψ^+),

$$G^+(x,k) = -(2\pi)^{-d} \int\limits_{\mathbb{R}^d} \frac{e^{i\xi x}d\xi}{\xi^2 - k^2 - i0} \qquad (6)$$

(Green function with the Sommerfeld radiation condition for $\Delta + k^2$),

$$f(k,l) = (2\pi)^{-d} \int\limits_{\mathbb{R}^d} e^{-ily}v(y)\psi^+(y,k)dy \qquad (7)$$

(integral formula for f).

Born approximation for $q \to 0$:

$$\psi^+ \approx e^{ikx}, \quad f(k,l) \approx \hat{v}(k-l), \tag{8}$$

$$\hat{v}(p) = (2\pi)^{-d} \int\limits_{\mathbb{R}^d} e^{ipx} v(x) dx, \quad p \in \mathbb{R}^d. \tag{9}$$

Note that

$$(k,l) \in \mathcal{M}_E \Rightarrow k - l \in \mathcal{B}_{2\sqrt{E}},$$

$$p \in \mathcal{B}_{2\sqrt{E}} \Rightarrow \quad \exists \ \ (k,l) \in \mathcal{M}_E \ \text{ such that } \ p = k - l \ \ (\text{for } \ d \geq 2),$$

$$\mathcal{B}_r = \{p \in \mathbb{R}^d : \ \ |p| \leq r\}.$$

In the Born approximation f on $\mathcal{M}_E$ is reduced to $\hat{v}$ on $\mathcal{B}_{2\sqrt{E}}$.
In addition, in this approximation f on $U_{\zeta \in [E_0, E]} \mathcal{M}_\zeta$, $0 < E_0 \leq E$, is also reduced to $\hat{v}$ on $\mathcal{B}_{2\sqrt{E}}$.

The most natural way for inverse scattering

$$f \ \text{ on } \ \mathcal{M}_E \to v \ \text{ on } \ \mathbb{R}^d$$

(as well as with $U_{\zeta\in[E_0,E]}\mathcal{M}_\zeta,\ 0 < E_0 \le E$, in place of $\mathcal{M}_E$) in the Born approximation:

$$v(x) = v^{lin}_{appr}(x,E) + v^{lin}_{err}(x,E),$$

$$v^{lin}_{appr}(x,E) = \int\limits_{|p|\le 2\sqrt{E}} e^{-ipx}\hat{v}(p)dp \quad \text{(stable)}, \tag{10}$$

$$v^{lin}_{err}(x,E) = \int\limits_{|p|\ge 2\sqrt{E}} e^{-ipx}\hat{v}(p)dp =$$

$$O(E^{-(n-d)/2}) \ \text{ in } \ L^\infty \ \text{ as } \ E \to +\infty$$

$$\text{if } \ v \in W^{n,1}(\mathbb{R}^d), \quad n > d.$$

De facto, the main goal of ISP in multidimensions consisted in appropriate extending formulas (10) to the general global case.

The first global result

Born approximation for $E \to \infty$:

$$f(k,l) = \hat{v}(k-l) + O(E^{-1/2}), \quad E \to +\infty, \tag{11}$$

$$(k,l) \in \mathcal{M}_E$$

(e.g. [L.D.Faddeev 1956]).

In addition, f on $\mathcal{M}_E$ proceeded via (11) and the inverse Fourier transformation yields $u_1(x,E)$,

$$u_1(x,E) = v(x) + O(E^{-(n-d)/(2n)}) \ \text{ in } \ L^\infty \ \text{ as } \ E \to +\infty \tag{12}$$

if $v \in W^{n,1}(\mathbb{R}^d)$, $n > d$ (in addition to the initial assumptions (2)).

The convergence of $u_1(x,E)$ to $v(x)$, $E \to +\infty$, is slow and not optimal, in particular,

$$\alpha_1 = \frac{n-d}{2n} \le \frac{1}{2} \ \text{ even if } \ n \to +\infty. \tag{13}$$

Multidimensional analog of the Gel'fand-Levitan-Marchenko theory constructed by Faddeev in the period 1965-1974

Achievements:

• Jost type solutions of the Schrodinger equation (1) in multidimensions (Faddeev solutions, CGO-solutions). Lippmann-Schwinger-Faddeev integral equations. Related results on inverse scattering including multidimensional analogs of the Gel'fand-Levitan integral equation.

• Formal characterization of the full scattering amplitude f (at least, for small potentials in dimension d=3) in terms of some its extension into complex domain.

Problems:

• Exceptional point conjecture about unique solvability of the Lippmann-Schwinger-Faddeev integral equations for real (nonzero)momenta (at least, for "good" potentials).

• De facto, no progress with respect to uniqueness and reconstruction. The Born formula at high energies (11) is more efficient, in practice.

Some results of [Henkin-Novikov, 1987]

• A complete synthesis of the $\bar{\partial}$- method of the soliton theory, the aforementioned ideas of Faddeev and methods of modern complex analysis.
Note that the $\bar{\partial}$- method in the framework of the soliton theory had appeared in the period 1981-1985; see [Beals, Coifman, 1985]. However, in the framework of the Radon-Penrose transform such a method had appeared previously in [Henkin, 1980].
As about methods of complex analysis we mean the methods presented in [Henkin, 1985].

• Solving the Faddeev exceptional point problem by the result that real (nonzero) exceptional points necessarily exist for the potentials with negative eigenvalues. (This disproves the Faddeev conjecture.)

- The fixed energy analog of the Born formula at high energies:

$$\hat{v}(p) = \lim_{\substack{k,l\in\mathbb{C}^d, k^2=l^2=E,\ k-l=p \\ |Im\, k|=|Im\, l|\to\infty}} h(k,l), \quad p\in\mathbb{R}^d, d\geq 3, \qquad (14)$$

where h is the Faddeev extension of the scattering amplitude f into complex domain. At present, formula (14) is one of keystone results in monochromatic quantum and acoustic inverse scattering and in electrical impedance tomography in dimension $d \geq 3$.

• The global uniqueness result that in dimension $d \geq 2$ the scattering amplitude f given in a neighborhood of a fixed positive energy E for the Schrodinger equation with short range potential v (i.e. under condition (2)) uniquely determines the Fourier transform $\hat{v}$ (of v) in the ball $\mathcal{B}_{2\sqrt{E}}$. This uniqueness result is a globalisation of its Born approximation prototype. The proof of this global result is constructive, but the related reconstruction is, unfortunately, unstable.
In view of this instability the next results of [Henkin-Novikova, 1996] and [Novikov, 1999, 2005] are of particular interest.

Results of [Henkin-Novikova, 1996]
This work develops an efficient numerical inverse scattering algorithm for the horizontal homogeneous liquid half space from vibrosounding surface data at fixed frequency.
De facto, the main results consist in obtaining global analogs of formulas (10) for the aforementioned case of acoustic inverse scattering in the horizontal homogeneous half space. In addition, the approximate but stable reconstruction at fixed frequency is given by a multisoliton type formula.
This work very much stimulated modern mathematical studies on approximate but efficient monochromatic inverse scattering algorithms in multidimensions and, in particular, studies of [Novikov, 1999, 2005].

Results of [Novikov 1999, 2005]:

$$f \ \text{ on } \ \mathcal{M}_E \to v_{appr}(\cdot, E) \ \text{ on } \ \mathbb{R}^d \tag{15}$$

- stable reconstruction via integal equations such that -

$$\|v - v_{appr}(\cdot, E)\|_{L^\infty(\mathbb{R}^d)} = O(E^{-(n-d)/2}), \text{ as } E \to +\infty \ \text{ for } \ d = 2,$$

$$\|v - v_{appr}(\cdot, E)\|_{L^\infty(\mathbb{R}^d)} = O(E^{-(n-d)/2} \ln E), \text{ as } E \to +\infty \text{ for } d = 3, \tag{16}$$

under the assumptions that $v \in W_s^{n,1}(\mathbb{R}^d)$, $n > d$, $s > 0$, where

$$W_s^{n,1}(\mathbb{R}^d) = \{v : \ (1 + |x|)^s \partial^J v(x) \in L^1(\mathbb{R}^d), \ |J| \le n\}.$$

These results are nonlinear analogs of (10).
For more information and the exact references to [Novikov 1999, 2005], see the review given in [R.Novikov, 2008].

Finally, note that in quantum mechanical scattering experiments (in the framework of model described by equation (1)) only phaseless scattering data, like $|f|^2$, can be measured directly, whereas scattering data with phase information, like f, are not accessible for direct measurements. However, recent work [Novikov, 2015] gives explicit formulas for finding f from appropriate phaseless scattering data.
Therefore, the multidimensional inverse scattering theory (with phase information) discussed in this talk can be applied to phaseless inverse scattering as well.

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