Tomography and integral geometry

Roman Novikov*

* Centre de Mathématiques Appliquées, Ecole Polytechnique

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1. Introduction

Tomography is known first of all as a research domain related with the problem of determining the structure of an object from X-ray photographs. This tomography uses X-ray photons as a probing tool.

At present, in addition to this X-ray tomography, several other types of tomography are also known, where instead of incident X-rays some other types of radiation are used. For example: electron tomography uses electrons; neutron tomography uses neutrons; acoustic tomography uses sonic or ultrasonic waves.

These problems arise in medicine, biology, different domains of physics, industry, etc .

On the mathematical level, these problems are often reduced to studies of classical Radon transforms and their different generalizations (or, by other words, to problems of integral geometry).

The objective of this lecture is to give an introduction to this research domain.

The word tomography is derived from Ancient Greek "tomos"= "slice, section" and "grapho"= "to write" .
In tomography the reconstruction of the object structure is realized, usually, slice by slice.
In addition, on the mathematical level in the X-ray tomography one deals with the reconstruction of the attenuation coefficient $a = a(x)$, $x \in \mathbb{R}^3$, of X-rays photons in the medium.
One of the main formulas of the X-ray tomography:

$$\frac{I_1}{I_0} = \exp\Big(-Pa(\gamma)\Big), \quad Pa(\gamma) = \int\limits_{\gamma} a(x)dx, \tag{1}$$

where γ is an arbitrary ray (oriented straight line) of propagation of X-ray photons, I_0 is the intensity of radiation before passing through the body, I_1 is the intensity of radiation after passing through the body.
The transform P arising in (1) is known as ray transform or Radon transform along straight lines.

Note that the set of all rays (oriented straight lines) in $\mathbb{R}^d$ can be identified with

$$TS^{d-1} = \{(x,\theta) \in \mathbb{R}^d \times S^{d-1} : \quad x\theta = 0\}. \qquad (2)$$

In addition, $\gamma = (x,\theta) \in TS^{d-1}$ is considered as the straight line

$$\gamma = (x,\theta) = \{y \in \mathbb{R}^d : \; y = x + s\theta, \; s \in \mathbb{R}\},$$

where θ gives the orientation.
Note also that

$$dim\, TS^{d-1} = 2d - 2.$$

The transform P on the plane (i.e., for $d = 2$) was considered for the first time in [Radon, 1917]. A similar transform on the sphere S^2 was considered earlier in [Minkowski, 1904], [Funk, 1916].
The mathematics of the X-ray tomography were developed, in particular, in [Radon, 1917], [John, 1937], [Cormack, 1963], by Gel'fand, Gindikin, Graev (in 1960ths and beyond), [Helgason, 1965]. These mathematics are strongly related with studies and inversion of the transform P.
In 1979, Cormack and Hounsfield won the Nobel Prize in Physiology and Medicine for the synthesis of ideas, led to the creation of the first X-ray tomograph.

2. The ray transform P and the Fourier transform

The transform P can be defined by the formula

$$Pf(x,\theta) = \int\limits_{\mathbb{R}} f(x+s\theta)ds, \quad (x,\theta) \in TS^{d-1}, \tag{3}$$

where f is a test function on $\mathbb{R}^d$. The Fourier transform of f is defined by the formula

$$\hat{f}(\xi) = (2\pi)^{-d/2} \int\limits_{\mathbb{R}^d} e^{i\xi x} f(x)dx, \quad \xi \in \mathbb{R}^d. \tag{4}$$

We consider also $P_\theta f$ and $\widehat{P_\theta f}$, where

$$P_\theta f(x) \stackrel{\mathrm{def}}{=} Pf(x,\theta),\ \theta \in S^{d-1},\ x \in X_\theta,$$

$$\widehat{P_\theta f}(\xi) = (2\pi)^{-\frac{d-1}{2}} \int\limits_{X_\theta} e^{i\xi x} P_\theta f(x)dx, \quad \xi \in X_\theta, \theta \in S^{d-1},$$

$$X_\theta = \{x \in \mathbb{R}^d : \ x\theta = 0\}.$$

Projection Theorem. The following formula holds

$$(2\pi)^{1/2}\hat{f}(\xi) = \widehat{P_\theta f}(\xi), \quad \xi \in X_\theta, \quad \theta \in S^{d-1}.$$

Proof:

$$\int\limits_{X_\theta} e^{i\xi x} P_\theta f(x)dx = \int\limits_{X_\theta} e^{i\xi x} \int\limits_{\mathbb{R}} f(x+s\theta)dsdx \stackrel{\xi\theta=0}{=} \int\limits_{\mathbb{R}^d} e^{i\xi y} f(y)dy.$$

Projection theorem permits to reconstruct f from Pf according to the following scheme

$$Pf \mapsto \hat{f} \mapsto f.$$

In addition, each method for finding f from Pf for $d = 2$ gives also a method for finding f from Pf for $d \geq 3$. Indeed, for $d \geq 3$, for finding $f(x)$ for any fixed $x \in \mathbb{R}^d$ one can spend through x a two-dimensional plane $Y \approx \mathbb{R}^2$ and reconstruct f on Y from Pf on $TS^1(Y)$, where $TS^1(Y)$ denotes the set of all rays in Y. Therefore, the case of $d = 2$ is of particular interest.

Note also that $TS^1 \approx \mathbb{R} \times S^1$:

$$(s, \theta) \in \mathbb{R} \times S^1 \mapsto (s\theta^\perp, \theta) \in TS^1,$$

$$(x, \theta) \in TS^1 \mapsto (x\theta^\perp, \theta) \in \mathbb{R} \times S^1,$$

where $\theta = (\theta_1, \theta_2) \in S^1$, $\theta^\perp = (-\theta_2, \theta_1)$.

3. Radon inversion formula

Theorem (Radon, 1917). The following formula holds:

$$f(x) = \frac{1}{4\pi} \int\limits_{S^1} \theta^{\perp} \nabla_x \tilde{q}_\theta(x\theta^{\perp}) d\theta, \tag{5}$$

$$\tilde{q}_\theta(s) = (Hq_\theta)(s) \stackrel{\text{def}}{=} \frac{1}{\pi} p.v. \int\limits_{\mathbb{R}} \frac{q_\theta(t)}{s-t} dt,$$

$$q_\theta(s) = Pf(s\theta^{\perp}, \theta),$$

where $x \in \mathbb{R}^2$, $\theta \in S^1$, $s \in \mathbb{R}$, $\theta^{\perp} = (-\theta_2, \theta_1)$ for $\theta = (\theta_1, \theta_2) \in S^1$.

Formula (5) is one of the main mathematical formulas of the X-ray tomography. The numerical algorithm realizing this formula is known as a filtered backprojection algorithm.

4. Attenuated ray transform and single-photon emission tomography

We consider the weighted ray transforms P_W defined by the formula

$$P_W f(x,\theta) = \int_{\mathbb{R}} W(x+t\theta,\theta) f(x+t\theta)dt,\ (x,\theta) \in TS^{d-1}, \quad (6)$$

where $W = W(x,\theta)$ is the weight, $f = f(x)$ is a test function on $\mathbb{R}^d$.

If $W = 1$, then $P = P_W$ is the classical ray (or Radon) transform.

If

$$W(x,\theta) = W_a(x,\theta) = \exp(-Da(x,\theta)), \;\; Da(x,\theta) = \int\limits_0^{+\infty} a(x+t\theta)dt, \tag{7}$$

where a is a complex-valued sufficiently regular function on $\mathbb{R}^d$ with sufficient decay at infinity, then P_W is known as the attenuated ray (or Radon) transform. This transform (at least, with $a \geq 0$) arises, in particular, in single-photon emission computed tomography (SPECT).

Transforms P_W with some other weight also arise in applications. For example, such transforms arise also in fluorescence tomography, optical tomography, positron emission tomography.

In single-photon emission computed tomography (SPECT) one considers a body containing radioactive isotopes emitting photons. The emission data p in SPECT consist in the radiation measured outside the body by a family of detectors during some fixed time. The basic problem of SPECT consists in finding the distribution f of these isotopes in the body from the emission data p and some a priori information concerning the body. Usually this a priori information consists in the photon attenuation coefficient a in the points of body, where this coefficient is found in advance by the methodes of the classical X-ray transmission tomography.
In SPECT the quantity $P_{W_a}f(\gamma)$, $\gamma = (x, \theta) \in TS^{d-1}$, describes the expected emission data along γ.

Exact and simultaneously explicit inversion formulas for the classical and attenuated Radon transforms for $d = 2$ were given for the first time in [Radon, 1917] and [R.Novikov, 2002], respectively. For some other weights W exact and simultaneously explicit inversion formulas are also known, see [Boman, Strömberg, 2004], [Gindikin, 2010], [R.Novikov, 2011].

Theorem (R.Novikov 2002). The following formula holds:

$$f = P_{w_a}^{-1} g, \quad \text{where} \quad g = P_{w_a} f, \tag{8}$$

$$P_{w_a}^{-1} g(x) = \frac{1}{4\pi} \int_{S^1} \theta^{\perp} \partial_x \big(\exp\left[-Da(x, -\theta)\right] \tilde{g}_\theta(\theta^{\perp} x) \big) \, d\theta,$$

$$\tilde{g}_\theta(s) = \exp\left(A_\theta(s)\right) \cos\left(B_\theta(s)\right) H(\exp\left(A_\theta\right) \cos\left(B_\theta\right) g_\theta)(s) +$$

$$\exp\left(A_\theta(s)\right) \sin\left(B_\theta(s)\right) H(\exp\left(A_\theta\right) \sin\left(B_\theta\right) g_\theta)(s),$$

$$A_\theta(s) = (1/2) Pa(s\theta^{\perp}, \theta), \quad B_\theta(s) = HA_\theta(s), \quad g_\theta(s) = g(s\theta^{\perp}, \theta),$$

$$Hu(s) = \frac{1}{\pi} p.v. \int_{\mathbb{R}} \frac{u(t)}{s - t} dt,$$

$$x \in \mathbb{R}^2, \quad \theta^{\perp} = (-\theta_2, \theta_1) \;\; \text{for} \;\; \theta = (\theta_1, \theta_2) \in S^1, \quad s \in \mathbb{R}.$$

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