How to find a copy of the Hamiltonian from the scattering map

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1 Statement of the problem

Let (M, g) be a n-dimensional smooth compact Riemannian manifold with boundary ∂M and $g^t(x, \xi) = (\gamma_{(x,\xi)}(t), \dot{\gamma}_{(x,\xi)}(t))$ is the geodesic flow on $TM_0 = \{(x, \xi) \in TM|\ \xi \in T_x,\ \xi \neq 0\}$, where $\gamma_{(x,\xi)}(t)$ is the geodesic, defined by initial data $\gamma_{(x,\xi)}(0) = x,\ \dot{\gamma}_{(x,\xi)}(0) = \xi$, and $\dot{\gamma}$ is the velocity vector. We assume that (M, g) is non-trapping, that is each maximal geodesic is finite. Formulate the following problem:

$$To\ find\ an\ isometric\ copy\ \varphi^* g, where\ \varphi : M \to \\ M\ is\ a\ diffeomorphism\ such\ that\ \varphi|_{\partial M} = id.$$

More simple problem we address here is formulated as follows. Let (M, H) be a smooth n-dimensional Hamiltonian manifold with hamiltonian $H(x, \xi) = g^{ij}(x)\xi_i\xi_j/2$, and $\varphi^t(x, \xi) = (\gamma_{(x,\xi)}(t), \dot{\gamma}_{(x,\xi)}(t))$ is the hamiltonian flow on $T^*M_0 = \{(x, \xi) \in T^*M|\ \xi \in T^*_x \neq 0\}$, where $(\dot{\gamma}_{(x,\xi)}(t))_i = g_{ij}(\gamma_{(x,\xi)}(t))\dot{\gamma}^j_{(x,\xi)}(t)$. In this talk we consider the following problem:

$$To\ find\ a\ hamiltonian\ H\ up\ to\ a\ symlectomorphism \\ f\ such\ that\ f|_{\partial T^* M_0} = id.$$

2 Hamiltonian flow

Hamiltonian vector field $\mathcal{H}$ is defined by the equality

$$\mathcal{H} = \frac{d\varphi^t}{dt}|_{t=0}.$$

It is identified with Poisson's bracket $[., H]$:

$$\mathcal{H}u = \frac{d}{dt}(u \circ \varphi^t)|_{t=0} = [u, H] = \sum_{i=1}^{n}(u_{x^i}H_{\xi_i} - u_{\xi_i}H_{x^i}). \tag{1}$$

Hamiltonian flow satisfies nonlinear Hamilton equation

$$\frac{d\varphi^t}{dt} = \mathcal{H}(\varphi^t),$$

or in coordinates

$$\begin{aligned}
\frac{d\gamma^i_{(x,\xi)}(t)}{dt} &= H_{\xi_i}(\gamma_{(x,\xi)}(t), \dot\gamma_{(x,\xi)}(t)) \\
\frac{d(\gamma_{(x,\xi)})_i(t)}{dt} &= -H_{x^i}(\gamma_{(x,\xi)}(t), \dot\gamma_{(x,\xi)}(t))
\end{aligned}$$

and Cauchy data

$$\varphi^0(x,\xi) = (x,\xi).$$

The following simple statement is very important.

Lemma 1 *Hamiltonian flow satisfies the linear kinetic equation*

$$L\varphi^t := (\partial_t - \mathcal{H})\varphi^t = 0.$$

Proof. The hamiltonian flow is one-parameter group, $\varphi^{t+s} = \varphi^t \circ \varphi^s = \varphi^s \circ \varphi^t$. Then due to (1) one has

$$\frac{d}{ds}\varphi^s = \frac{d}{dt}(\varphi^{t+s})|_{t=0} = \frac{d}{dt}(\varphi^s \circ \varphi^t)|_{t=0} = [\varphi^s, H] = \mathcal{H}\varphi^s.$$

So, the hamiltonian flow satisfies two equations: nonlinear ODE $d\varphi^t/dt = \mathcal{H}(\varphi^t)$ and linear PDE $L\varphi^t = 0$, and the Cauchy data $\varphi^0(x,\xi) = (x,\xi)$. ■

Remark 2 *If*

$$L\psi = 0, \quad \psi(x,\xi,0) = (x,\xi), \tag{2}$$

then by the argument of uniqueness of the solution to the Cauchy problem (2) $\psi(.,t) = \varphi^t$.

3 Scattering map

Let M be a compact manifold with smooth boundary ∂M. The boundary of tangent space ∂T^*M_0 may be decomposed into the sets of outer (+) and inner (-) covectors:

$$\partial T^*M_0 = \partial_+ T^*M_0 \cup \partial_- T^*M_0,$$

where

$$\begin{aligned}
\partial_\pm T^*M_0 &= \{(x,\xi) \in T^*M_0 |\ x \in \partial M,\ \pm(\nu(x),\xi) \geq 0\}, \\
\partial_0 T^*M_0 &= \{(x,\xi) \in T^*M_0 |\ x \in \partial M,\ (\nu(x),\xi) = 0\},
\end{aligned}$$

ν is the outer unit normal to the boundary ∂M. Denote by $\tau(x,\xi)$ the length of geodesic ray $\gamma_{(x,\xi)}(t), t \geq 0$. Note, that τ satisfies equation

$$\mathcal{H}\tau = \frac{d}{dt}(\tau \circ \varphi^t)|_{t=0} = \frac{d}{dt}(\tau - t)|_{t=0} = -1.$$

We put $\tau(x,\xi) = 0$ for $(x,\xi) \in \partial_+ T^*M_0$. Denote by X the extended phase space

$$X = \{(x,\xi,t)|\ (x,\xi) \in T^*M_0,\ -\tau(x,-\xi) \le t \le \tau(x,\xi)\}.$$

Introduce notations:

$$\begin{array}{rcl} \partial_- X &=& \{(x,\xi,t)|\ (x,\xi) \in \partial_- T^*M_0,\ 0 \le t \le \tau(x,\xi)\} \\ \partial_+ X &=& \{(x,\xi,t)|\ (x,\xi) \in \partial_+ T^*M_0,\ -\tau(x,-\xi) \le t \le 0\}, \\ \partial_0 X &=& \partial_0 T^*M_0. \end{array}$$

Introduce the scattering map $\alpha_H : \partial_\mp X \to \partial_\pm X$. It is defined by the formulas

$$\begin{array}{rcl} \alpha_H(x,\xi,t) &=& (\gamma_{(x,\xi)}(\tau(x,\xi)), \dot\gamma_{(x,\xi)}(\tau(x,\xi)), t - \tau(x,\xi)), \quad (x,\xi,t) \in \partial_- X \\ \alpha_H(x,\xi,t) &=& (\gamma_{(x,-\xi)}(\tau(x,-\xi)), \dot\gamma_{(x,-\xi)}(-\tau(x,-\xi)), \tau(x,-\xi) + t),\ (x,\xi,t) \in \partial_+ X. \end{array}$$

Clearly, $\alpha_H|_{\partial_0 X} = id$, and α_H is involution, that is $\alpha_H^2 = id$.

The solution to the Cauchy problem

$$\begin{array}{rcl} Lu &=& 0, \quad in\ X, \\ u|_{t=0} &=& u_0, \quad in\ T^*M_0 \end{array}$$

is given by the formula

$$u(x,\xi,t) = u_0(\varphi^t(x,\xi)),\ -\tau(x,-\xi) \le t \le \tau(x,\xi)$$

and its trace $u' = u|_Y$ is even function w.r.t. involution α_H :

$$u' = u' \circ \alpha_H,$$

where $Y = \partial_- X \cup \partial_+ X$. And other way around, any even function (w.r.t. α_H) u' on Y may be extended onto X as a solution u of the kinetic equation:

$$Lu = 0, \quad u|_Y = u'.$$

For such extension we use notation $u = \tilde{u}'$. We use also notation

$$u_0 = u|_{t=0}.$$

4 Copy

The map $\phi : T^*M_0 \to X,\ \phi(x,\xi,t) = \varphi^t(x,\xi)$ satisfies kinetic equation

$$L\phi = 0, \quad \phi(x,\xi,0) = (x,\xi).$$

For any t the map $\varphi^t = (\gamma_{(.)}(t), \dot\gamma_{(.)}(t))$ is symplectomorphism, that is, for any local coordinates on T^*M_0

$$[\gamma^i,\gamma^j] = 0,\ [\gamma^i,\dot\gamma_j] = \delta^i_j,\ [\dot\gamma_i,\dot\gamma_j] = 0, \quad i,j = 1,...,n.$$

Further, it easy to check that the equality

$$L[u,v] = [Lu,v] + [u,Lv] \tag{3}$$

holds.

This motivates the following way to get a copy of hamiltonian H. Take arbitrary smooth map $\sigma : Y \to T^*M_0$ such that

$$\sigma \circ \alpha_H = \sigma, \quad \sigma|_{\partial_0 X} = id,$$

and locally σ is determined by functions $u^i, v_i,\ i = 1,...,n$ on Y which satisfy symplectic conditions

$$[u^i,u^j] = 0,\ [u^i,v_j] = \delta^i_j,\ [v_i,v_j] = 0$$

and initial data

$$u(x,\xi,0) = x,\ v(x,\xi,0) = \xi, \quad (x,\xi) \in \partial_0 X.$$

Then functions $\tilde u^i, \tilde v^i$ (extensions) due to 3 also satisfy (for any t)

$$[\tilde u^i,\tilde u^j] = 0,\ [\tilde u^i,\tilde v_j] = \delta^i_j,\ [\tilde v_i,\tilde v_j] = 0.$$

So, by construction, the map $\tilde\psi : X \to T^*M_0,\ \tilde\psi = (\tilde u, \tilde v)$ satisfies kinetic equation $L\tilde\psi = 0$. At $t=0$ according to our notation $\tilde\psi|_{t=0} = (\tilde u_0, \tilde v_0)$. The map

$$\begin{aligned} f \quad : \quad & T^*M_0 \to T^*M_0, \\ & (x,\xi) \xrightarrow{f} (\tilde u_0(x,\xi), \tilde v_0(x,\xi)) \end{aligned}$$

is symplectomorphism by construction. Then for $\psi : X \to T^*M_0$, defined by the equality

$$\tilde\psi(x,\xi,t) = \psi(f(x,\xi),t)$$

we have

$$\psi_t - [\psi,\tilde H] = 0, \quad \psi|_{t=0} = id,$$

where $\tilde H = H \circ f$ is a copy of H. By construction this copy has the same scattering map, $\alpha_H = \alpha_{\tilde H}$. One can find this copy by the following way. Due to

Remark after Lemma 1 we have $\psi(x,\xi,t)=\tilde{\varphi}^t(x,\xi)$, where $\tilde{\varphi}^t$ is the Hamiltonian flow on $(T^*M_0,\tilde{H})$, and therefore for any $(x,\xi,t)\in X$ ψ satifies the Hamiltonian system on the whole extended phase spase X :

$$\begin{aligned} \frac{d}{dt}\psi(x,\xi,t) &= \tilde{\mathcal{H}}(\psi(x,\xi,t)), \\ \psi(x,\xi,0) &= (x,\xi). \end{aligned}$$

In partiqular we have on the set Y equalities

$$\begin{aligned} \frac{d\psi}{dt}(x,\xi,t) &= \tilde{\mathcal{H}}(\psi(x,\xi,t)), \quad (x,\xi,t)\in Y, \\ \psi(x,\xi,0) &= (x,\xi)\in\partial_0 X. \end{aligned}$$

Since $\psi|_Y=(u,v)$ is given one can obtain the function $\tilde{H}(u,v)$ (up to a constant) from these equalities.

Thus, a copy $\tilde{H}$ may be obtained by the following way.

1). Take any even w.r.t. α_H functions $u^i, v_i,\ i=1,...,n$ on Y which satisfy a) symplectic conditions

$$[u^i,u^j]=0,\ [u^i,v_j]=\delta^i_j,\ [v_i,v_j]=0,$$

b) the map $\sigma:(x,\xi,t)\to(u(x,\xi,t),v(x,\xi,t))$ is the diffeomorhism from Y onto T^*M_0.

2) The copy $\tilde{H}$ of hamiltonian H is determined by choosing (u,v) and may be found out from the hamiltonian system

$$\begin{aligned} \frac{du^i}{dt}(x,\xi,t) &= \tilde{H}_{v_i}(u(x,\xi,t),v(x,\xi,t)), \\ \frac{dv^i}{dt}(x,\xi,t) &= -\tilde{H}_{u_i}(u(x,\xi,t),v(x,\xi,t)),\ (x,\xi,t)\in Y. \\ (u,v)|_{t=0} &= (x,\xi). \end{aligned}$$

Notice, that we nowhere used the spesial form of hamiltonian $H(x,\xi)=g^{ij}(x)\xi_i\xi_j/2$. And by construction all copies of unknown hamiltonian H generate the same scattering map α_H.