Asymptotics of solutions of the Cauchy problem for a quasilinear first order equation in several space variables

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We investigate the solution in $\Pi^{+}=\left\{\left(t,x\right):t>0,\ x\in\mathbb{R}^{n}\right\}$ of the Cauchy problem for a scalar quasilinear first-order conservation law in several space variables:

$$\frac{\partial u}{\partial t}+\sum_{i=1}^{n}\frac{\partial\varphi_{i}\left(u\right)}{\partial x_{i}}=0,\ t>0,\ x=\left(x_{1},x_{2},\ldots,x_{n}\right)\in\mathbb{R}^{n}, \quad (1)$$

$$u\big|_{t=0}=u_{0}\left(x\right),\ x\in\mathbb{R}^{n}. \quad (2)$$

The initial function $u_{0}\left(x\right)$ is measurable and bounded, $\varphi_{i}\left(u\right)\in C^{1}\left(\mathbb{R}\right),\ i=1,\ldots,n$.

Definition 1(S.N. Kruzhkov//Mat. Sb., 1970, v. 81, No.2 =Math. USSR-Sb. 10(1970)). A bounded measurable function $u(t,x)$ is called a generalized solution(g.s.) of the problem (1), (2) in Π^+, if:

1) the Kruzhkov inequality holds:

$$\forall k = \text{const and } g(t,x) \in C_0^\infty(\Pi^+), g(t,x) \geq 0$$

$$\iint_{\Pi^+} \left\{ \left|u(t,x) - k\right| g_t + \sum_{i=1}^{n} \operatorname{sign}\left(u(t,x) - k\right)\left[\varphi_i\left(u(t,x)\right) - \varphi_i(k)\right] g_{x_i} \right\} dxdt \geq 0$$

2) there is a set $\mathcal{E} \subset (0,+\infty)$ of zero measure such that $u(t,x)$ is defined a.e. in $\mathbb{R}^n$ for $t \in (0,+\infty) \setminus \mathcal{E}$ and for any ball $K_r = \left\{x : |x| = \sqrt{x_1{}^2 + \ldots + x_n{}^2} \leq r\right\}$ we have

$$\int_{K_r} \left|u(t,x) - u_0(x)\right| dx \to 0,\ t \to 0,\ t \in (0,+\infty) \setminus \mathcal{E}.$$

Theorem 1(S.N. Kruzhkov//Mat. Sb., 1970, v. 81, No.2 =Math. USSR-Sb. 10(1970)). There exists a unique generalized solution of the problem (1), (2) in the sense of Definition 1.

We study the large-time behavior of the solution $u(t,x)$ of the problem (1), (2) when the initial data $u_0(x)$ is a perturbation of a function

$$\tilde{u}_0(x) = \begin{cases} u^-, & x_\mu < 0, \\ u^+, & x_\mu \geq 0, \end{cases} \qquad (3)$$

where u^- and u^+ are constants, μ is a given unit vector:

$\mu = (\mu_1, \ldots, \mu_n), |\mu| = 1, x_\mu = (x, \mu) = x_1\mu_1 + \ldots + x_n\mu_n.$

Let $n=1$, $f(u)=\varphi_1(u)$. Consider the Riemann problem(the problem of the decay of a discontinuity) for the equation

$$\frac{\partial u}{\partial t}+\frac{\partial f(u)}{\partial x}=0 \tag{4}$$

with the initial function

$$u_0(x)=\begin{cases} u^-, x<x_0, \\ u^+, x \geq x_0, \end{cases} \quad u^- \neq u^+, \tag{5}$$

If $u^- < u^+$, $f(u) \in C^2(\mathbb{R})$, $f''(u) > 0, u \in [u^-, u^+]$, then the rarefaction wave

$$H\left(\frac{x-x_0}{t}\right) = \begin{cases} u^-, x - x_0 < f'(u^-)t, \\ f'^{-1}\left(\frac{x-x_0}{t}\right), f'(u^-) \le \frac{x-x_0}{t} < f'(u^+), \\ u^+, x - x_0 \ge f'(u^+)t. \end{cases} \tag{6}$$

is a solution of the Riemann problem (4), (5).

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Theorem 2 (N.S. Petrosyan//Dinamika sploshnoy Sredy. – 1978, No. 36, p. 86-96). Let $u_0(x) \to u^{\pm}$ as $x \to \pm\infty$, $u^- < u^+$. Let $f(u) \in C^2(\mathbb{R})$, $f''(u) > 0$ for $u \in \left[u^-, u^+\right]$ and $f''(u)$ has only isolated zeros. Then $u(t,x) - H(x/t) \to 0$ as $t \to +\infty$ uniformly $\forall x \in \mathbb{R}$, where $H(x/t)$ is the rarefaction wave (5).

This assertion may be extended to the multidimensional case.

Put

$$x=(x,\mu)\mu+\hat{x}=x_\mu\mu+\hat{x},\ \hat{x}\in\mathbb{R}^{n-1},$$

$$\varphi(u)=(\varphi_1(u),\dots,\varphi_n(u)),$$

$$f(u)=\varphi_\mu(u)=(\varphi(u),\mu)=\varphi_1(u)\mu_1+\ldots+\varphi_n(u)\mu_n.$$

Theorem 3. Assume that $u_0(x)\to u^{\pm}$ as $x_\mu\to\pm\infty$ uniformly for $\hat{x}\in\mathbb{R}^{n-1}$, $u^-<u^+$. Assume that $f''(u)=\varphi''_\mu(u)\geq 0, u\in[u^-,u^+]$, and $f''(u)=\varphi''_\mu(u)$ has only isolated zeros. Then $u(t,x)-H(x_\mu/t)\to 0$ as $t\to+\infty$ uniformly for $x\in\mathbb{R}^n$.

Let $n=1$.Consider now the case of the shock wave. Denote by

$$\sigma(u,v)=\left(f(u)-f(v)\right)/(u-v),\ k=\sigma\left(u^-,u^+\right).$$

Assume that

$$u^- < u^+,\ \sigma\left(u^-,u\right)>k>\sigma\left(u,u^+\right)\ \forall u\in\left(u^-,u^+\right). \quad (7)$$

Then the shock wave

$$S\left(t,x-x_0\right)=u_0\left(x-kt\right)=\begin{cases} u^-,\ x-x_0<kt, \\ u^+,\ x-x_0>kt \end{cases}$$

is a solution of the Riemann problem (4), (5).

Theorem 4 (Kuo-Shung Cheng// J. Differ. Equat., 1981, v. 40, No. 3). Let $f(u) \in C^2(\mathbb{R})$ and $f''(u)$ has only isolated zeros. Let $u_0(x) \equiv u^{\pm}$, $\pm x > N = \text{const} > 0$. Assume that $u(t,x)$ is a piecewise solution of the problem (4), (2). Then
$\exists t_0 > 0 : u(t,x) = S(t, x - x_0), \quad t \ge t_0$,
where

$$x_0 = \frac{1}{u^+ - u^-} \int_{-N}^{N} \left(\frac{u^+ + u^-}{2} - u_0(x) \right) dx.$$

The multidimensional analog of the theorem 4 is established in the paper

P. Bauman and D. Phillips. Large–time behavior of solutions to a scalar conservation law in several space dimensions// Trans. Amer. Math. Soc. – 1986, v. 298, No. 1, p. 401-419.

Let

$$u^- < u^+,$$

$$k = \left(\varphi\left(u^+\right) - \varphi\left(u^-\right)\right) \Big/ \left(u^+ - u^-\right),$$

$$\tilde{\varphi}(u) \equiv \varphi(u) - \varphi\left(u^-\right) - \left(u - u^-\right)k.$$

The function $\tilde{u}_0(x-kt)$ is a solution of the problem (1), (2) if and only if $(\tilde{\varphi}(u),\mu)\le 0,\, u\in(u^-,u^+)$.

Assume, that $\tilde{u}_0$ is nondegenerate; that is $\exists\,\theta=\text{const}>0$:

$$\left(\tilde{\varphi}(u),\mu\right)\le-\theta\left(u^+-u\right)\left(u-u^-\right)0,\, u\in\left[u^-,u^+\right]. \quad (8)$$

Assume that $u_0(x)=\tilde{u}_0(x), |x|>R_1>0$. There are bounds $m_1<u^-$ and $m_2>u^+$(determined explicitly by the vector-function $\varphi(u)$) so that if $m_1\le u_0(x)\le m_2$ then $u^-\le u(t,x)\le u^+$ as $t\ge t_0$ for some $t_0>0$.

Theorem 5(P. Bauman and D. Phillips). Assume that $u_0(x) = \tilde{u}_0(x), |x| > R_1 > 0$ and $m_1 \le u_0(x) \le m_2$ There exists a constant $R_2 > 0$ and a set $M \subset \mathbb{R}^n$ such that if $v(x) = \begin{cases} u^+, x \in M, \\ u^-, x \in \mathbb{R}^n \setminus M, \end{cases}$, then:

1) $u(t,x) - v(x-kt) \to 0$ in $L_1(\mathbb{R}^n)$ as $t \to +\infty$;

2) $u(t,x) = v(x-kt)$ as $t > 0$ and $|x-kt| > R_2$;

3) $M \cap \{x \in \mathbb{R}^n : |x| > R_2\} = \{x \in \mathbb{R}^n : x_\mu < 0, |x| > R_2\}$.

The condition $m_1 \le u_0(x) \le m_2$ may be removed.

Theorem 6. Assume that $u_0(x) \equiv u^{\pm}$, $\pm x_\mu > N = \text{const} > 0$. Assume that $\varphi_\mu''(u)$ has only isolated zeros. Then $\exists T_0 > 0 : u^- \le u(t,x) \le u^+, t > T_0$.

Corollary. Let $u_0(x) = \tilde{u}_0(x)$ as $|x| > R_1 > 0$. Then $u_0(x) \equiv u^{\pm}, \pm x_\mu > R_1 = \text{const} > 0$. Hence, $\exists T_0 > 0 : u^- \le u(t,x) \le u^+, t > T_0$ and the assertions of the theorem 5 are true.

The proofs of the theorems 3 and 6 are based on the following comparison theorem.

Theorem(S.N. Kruzhkov). Let $u(t,x)$ and $v(t,x)$ be generalized solutions of the problem (1), (2) in Π^+ with initial functions $u_0(x)$ and $v_0(x)$ respectively, where $u_0(x) \le v_0(x)$ a.e. in $\mathbb{R}^n$. Then $u(t,x) \le v(t,x)$ a.e.in Π^+.

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