On stability of special solutions for quasi-linear equations of traffic flow

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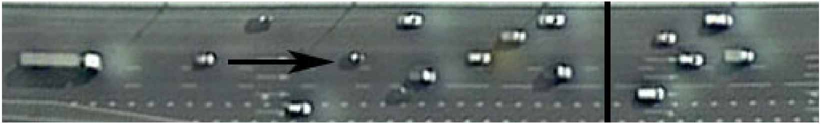
Main parameters

$$\rho(x,t) \text{ — flow density,}$$

$$v(x,t) \text{ — flow velocity,}$$

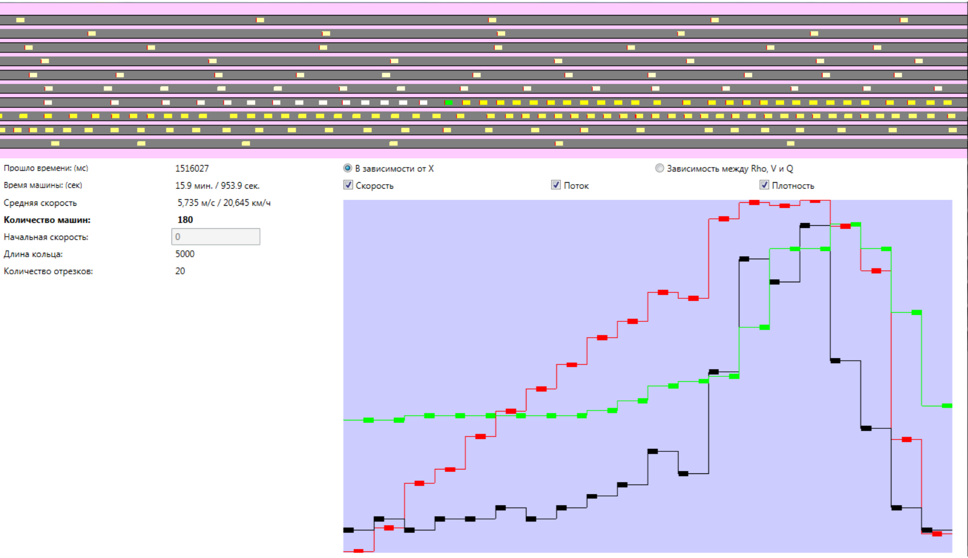
$$q(x,t) \text{ — flow rate.}$$



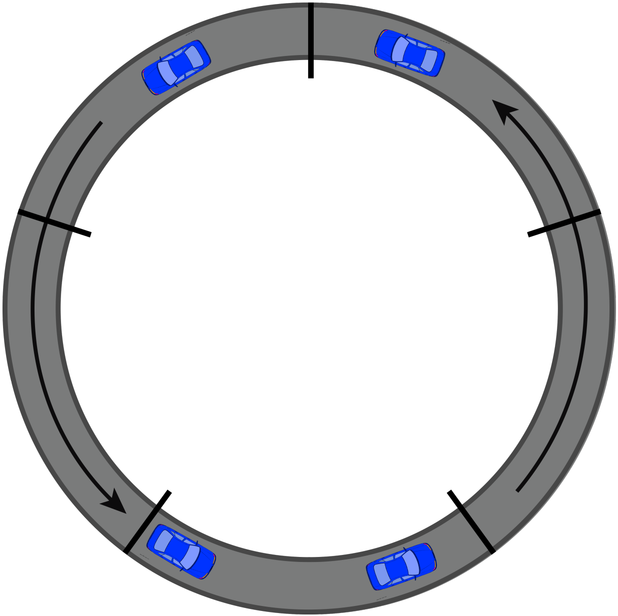
x

Moscow, MRR, north.

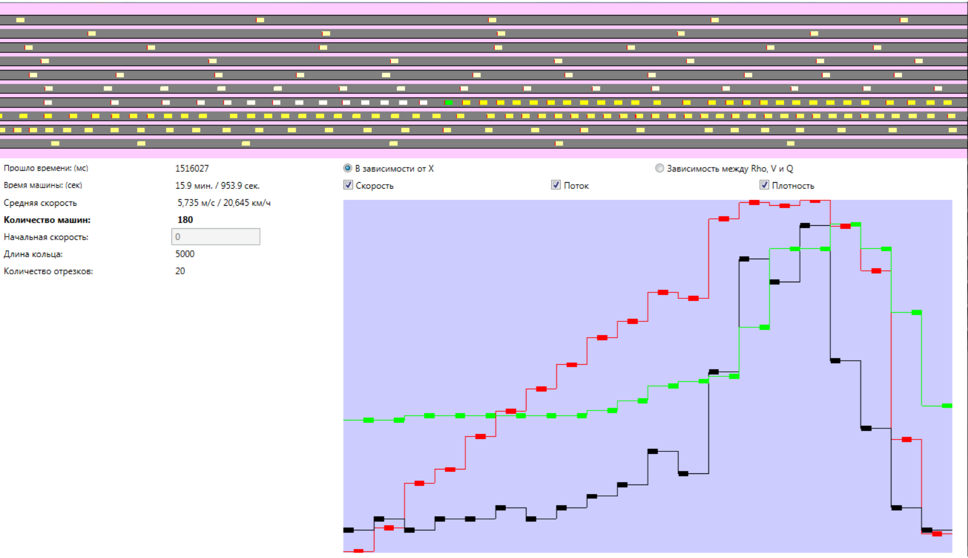
Interface of the program “Cars”



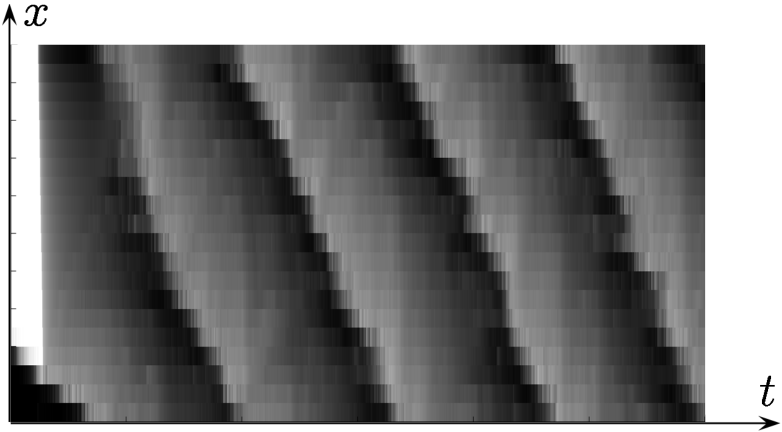
Ring road



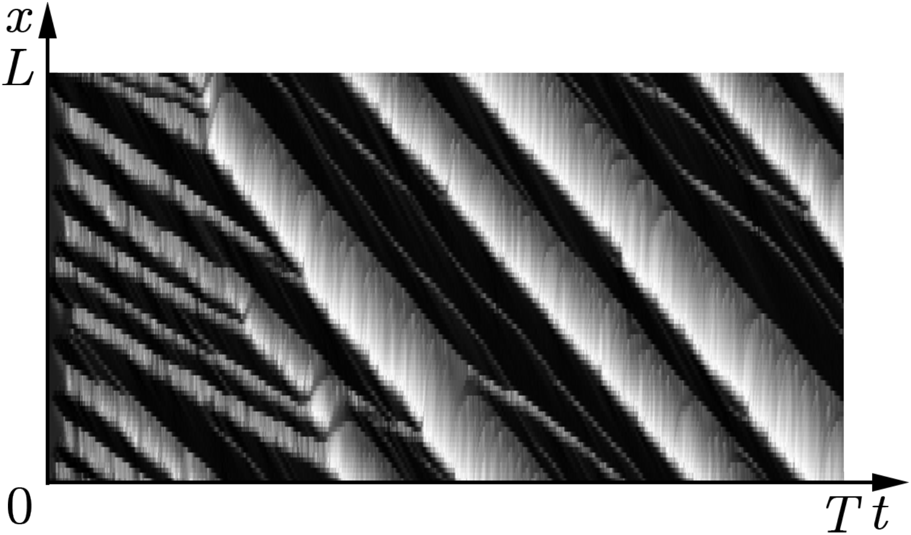
Interface of the program “Cars”



Jam movement on ring road ($N = 140$ cars, $L = 5$ km, $T = 3$ hours)

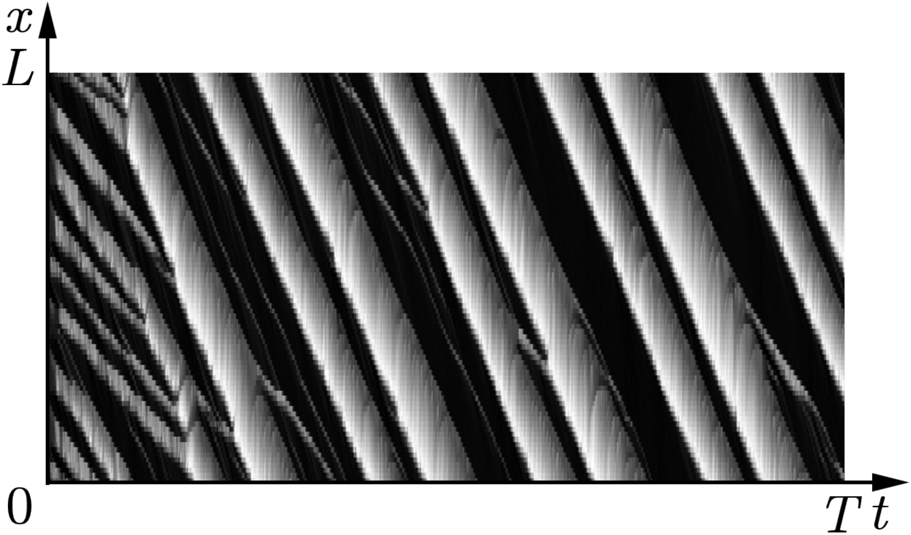


Tendency to stable condition



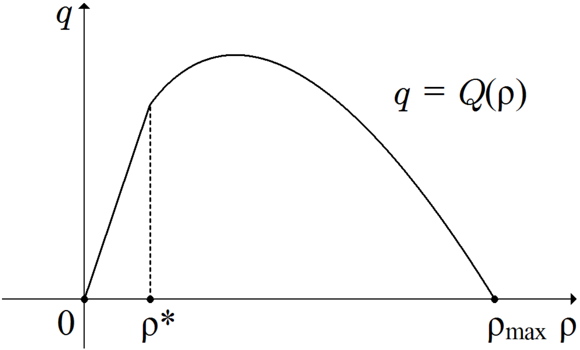
$L = 20$ km, $\quad T = 6$ h.

Tendency to stable condition



$$L = 20 \text{ km}, \quad T = 12 \text{ h}.$$

Fundamental diagram



$$q = Q(\rho), \quad Q \text{ is concave function},$$
$$Q(\rho) > 0, \quad 0 < \rho < \rho_{\max},$$
$$Q(0) = 0, \quad Q(\rho_{\max}) = 0.$$

Quasi-linear differential equation

Main equation of road traffic:

$$\frac{\partial \rho}{\partial t} + \frac{\partial Q(\rho)}{\partial x} = 0, \qquad \rho = \rho(x,t).$$

Integral identity:

$$\frac{d}{dt}\int\limits_{\alpha}^{\beta} \rho(x,t)dx = Q(\rho(\alpha,t)) - Q(\rho(\alpha,t)),$$

for a.e. $\alpha, \beta \in \mathbb{R}, \quad t \geqslant 0.$

Cauchy problem

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \dfrac{\partial Q(\rho)}{\partial x} = 0, \quad x \in \mathbb{R}, \quad t > 0, \\ \rho(x,0) = \mu(x), \end{cases}$$

with initial function

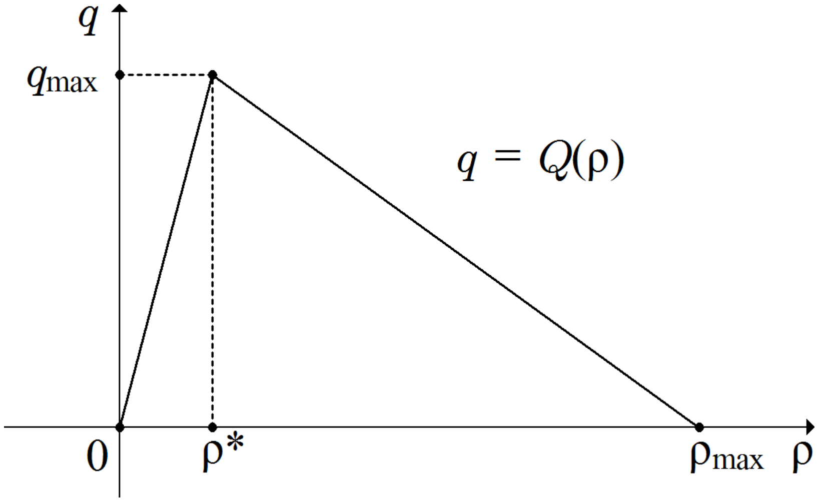
$$0 \leqslant \mu(x) \leqslant \rho_{\max}, \qquad x \in \mathbb{R}.$$

Technical solution of this problem is

$$\rho = \mu(x - k(\rho)t),$$

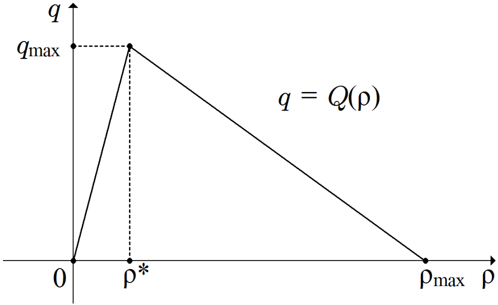
where $k = k(\rho) = Q'(\rho)$ is a slope of the characteristic.

Nagel–Schreckenberg diagram



K. Nagel, M. Schreckenberg. *A cellular automaton model for freeway traffic*. Journal de Physique I France. 1992. Vol. 2. No 12. P. 2221–2229.

Analytical representation



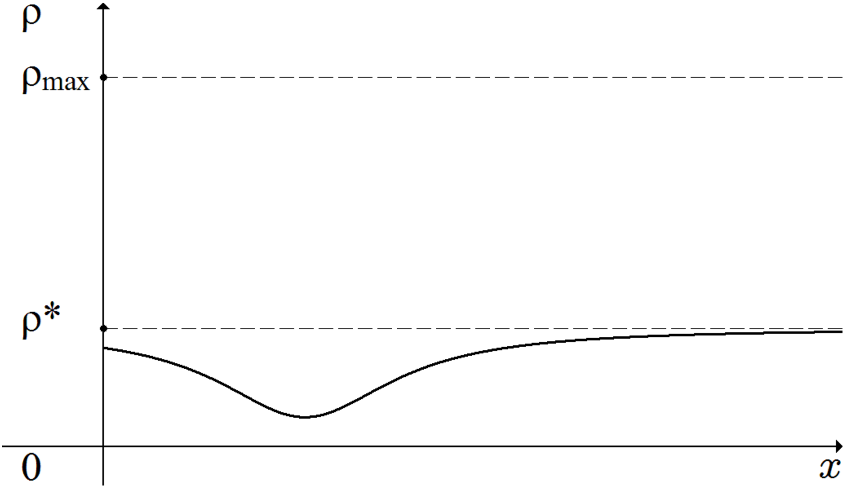
$$Q(\rho) = \begin{cases} k_1\,\rho, & 0 \leqslant \rho \leqslant \rho^*, \\ k_2\,(\rho_{max} - \rho), & \rho^* \leqslant \rho \leqslant \rho_{max}. \end{cases}$$

$$k_1 = \frac{q_{max}}{\rho^*}, \qquad k_2 = \frac{q_{max}}{\rho_{max} - \rho^*}.$$

Typical solutions

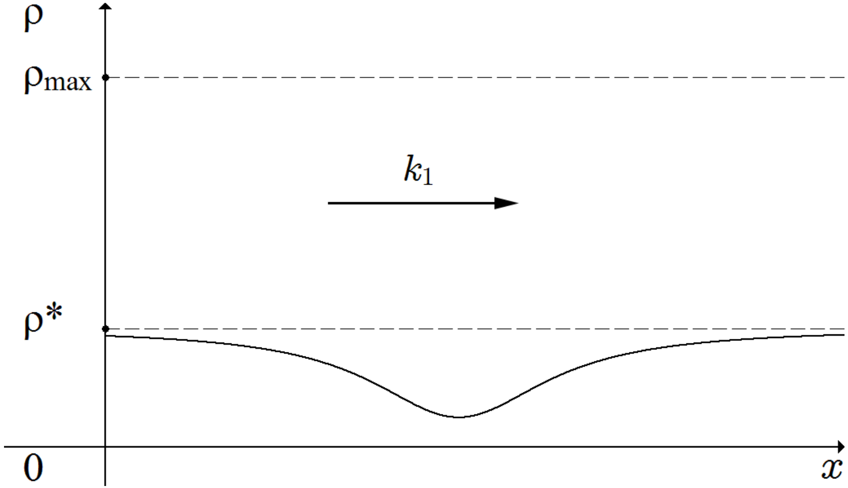
1. If $\mu(x) \leqslant \rho^*$ on $\mathbb{R}$, then
$$\rho(x,t) = \mu(x - k_1 t)$$
(forward wave).
2. If $\mu(x) \geqslant \rho^*$ on $\mathbb{R}$, then
$$\rho(x,t) = \mu(x + k_2 t)$$
(backward wave).

Forward wave



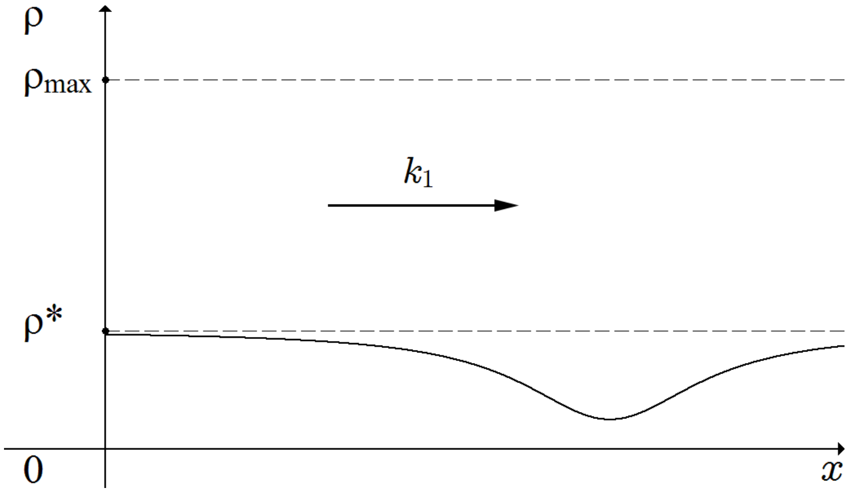
for $\quad t = 0$

Forward wave



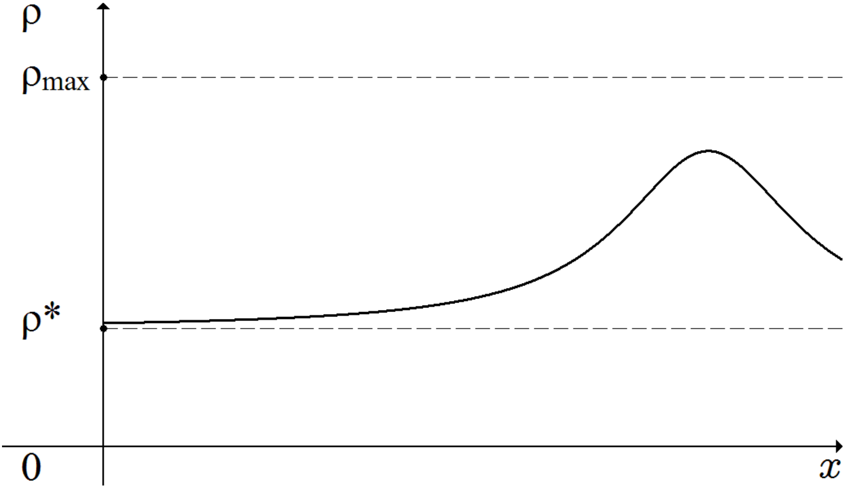
for $\quad t = 0.3$

Forward wave



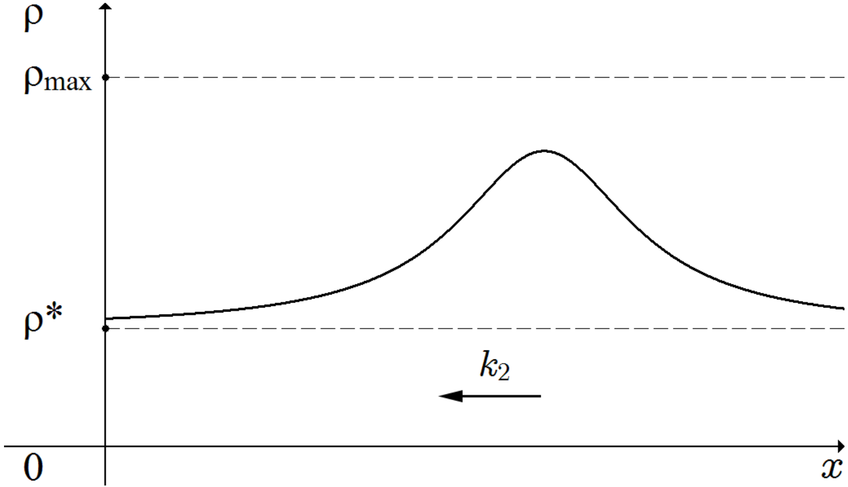
for $\quad t = 0.6$

Backward wave



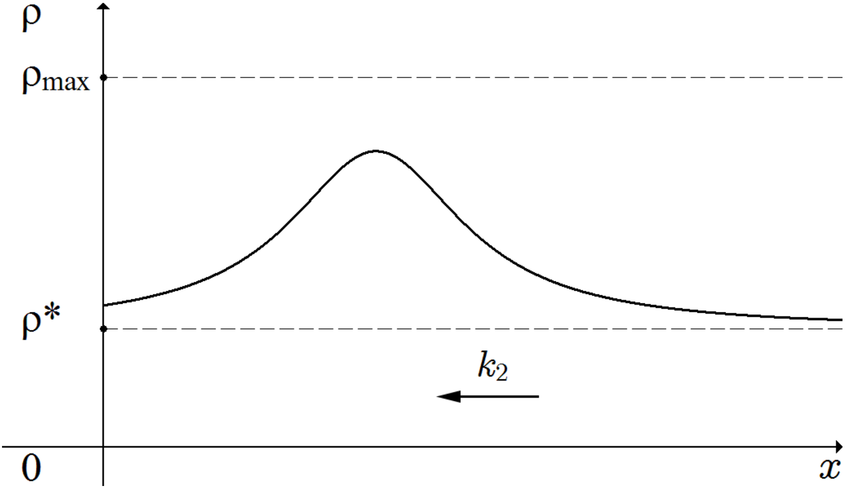
for $t = 0$

Backward wave



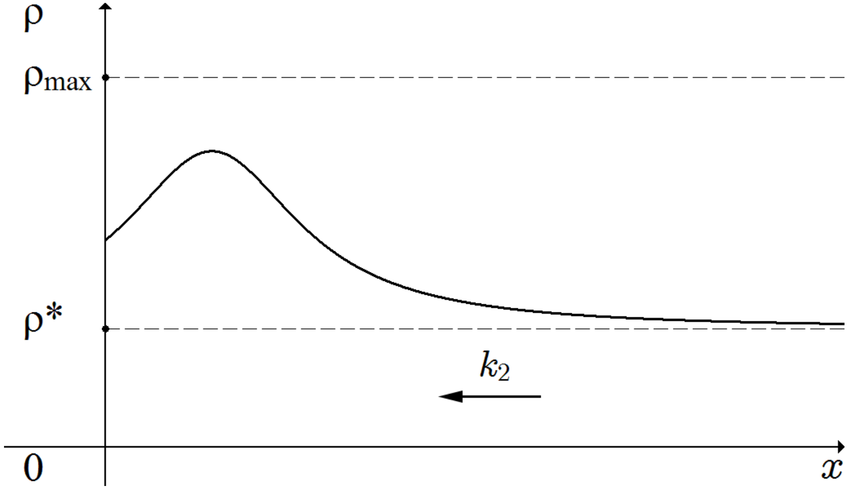
for $t = 0.7$

Backward wave



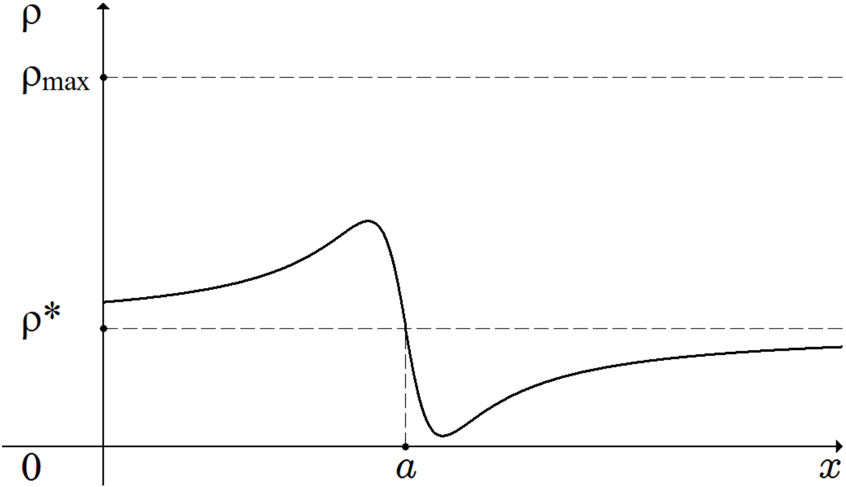
for $\quad t = 1.4$

Backward wave



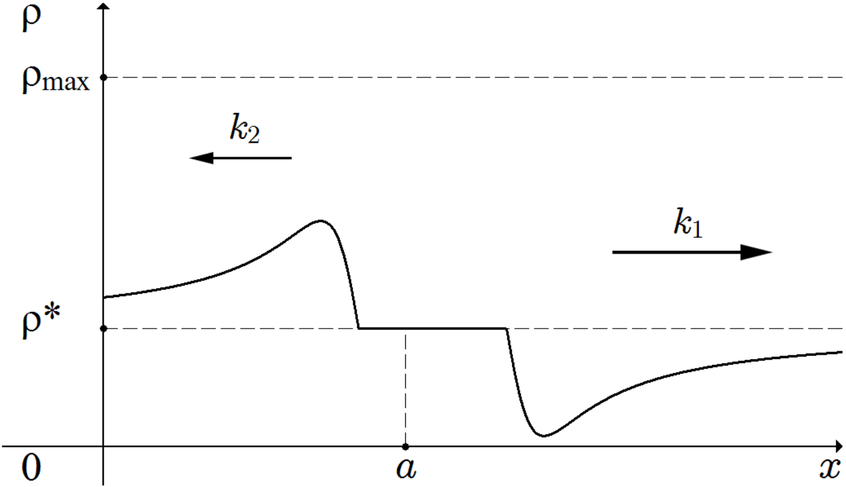
for $\quad t = 2.1$

Combined solution: depression wave



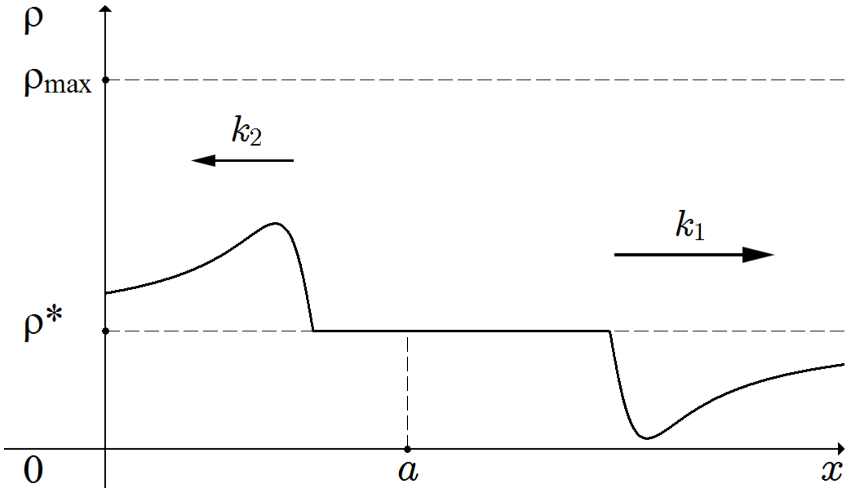
for $t = 0$

Combined solution: depression wave



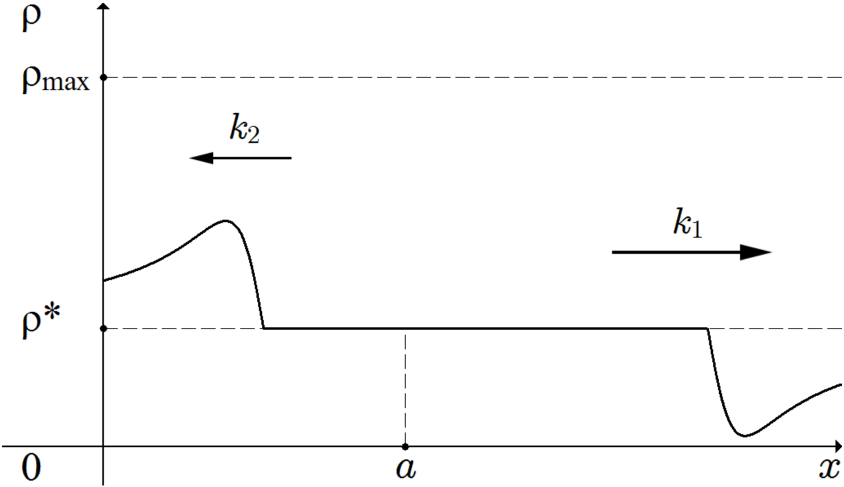
for $t = 0.2$

Combined solution: depression wave



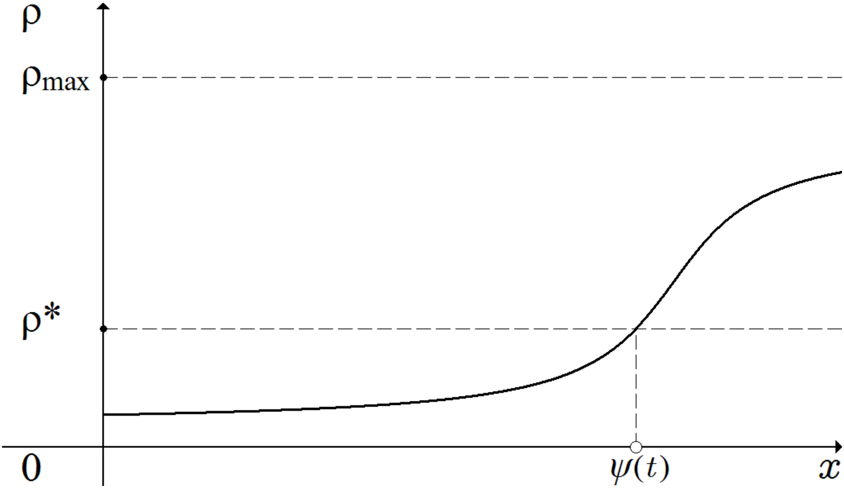
for $t = 0.4$

Combined solution: depression wave



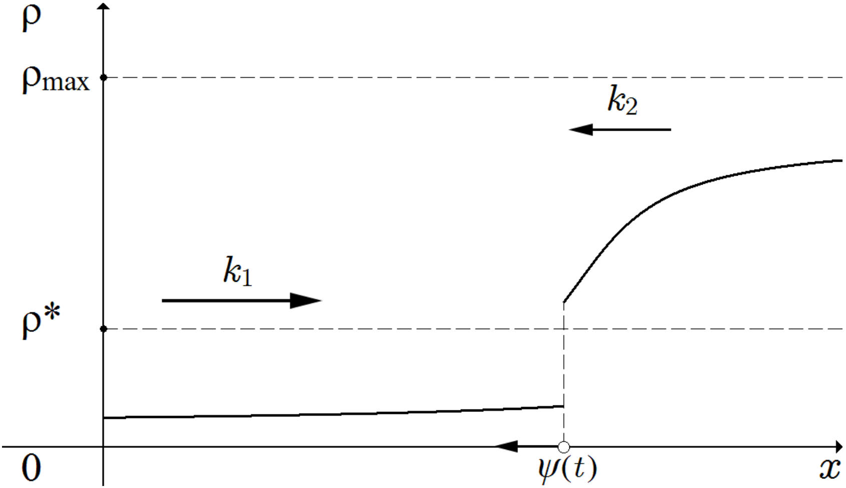
for $t = 0.6$

Combined solution: shock wave



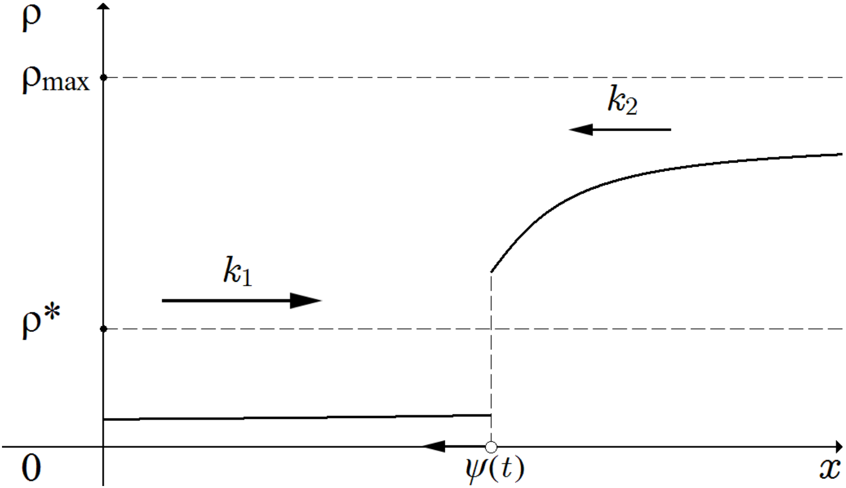
for $t = 0$

Combined solution: shock wave



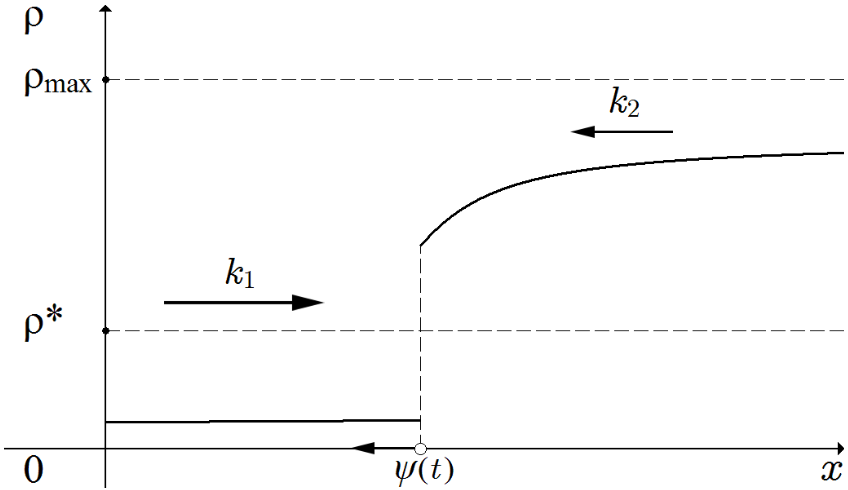
for $\quad t = 0.4$

Combined solution: shock wave



for $\quad t = 0.8$

Combined solution: shock wave



for $\quad t = 1.2$

Hugoniot condition

Speed of the shock wave is

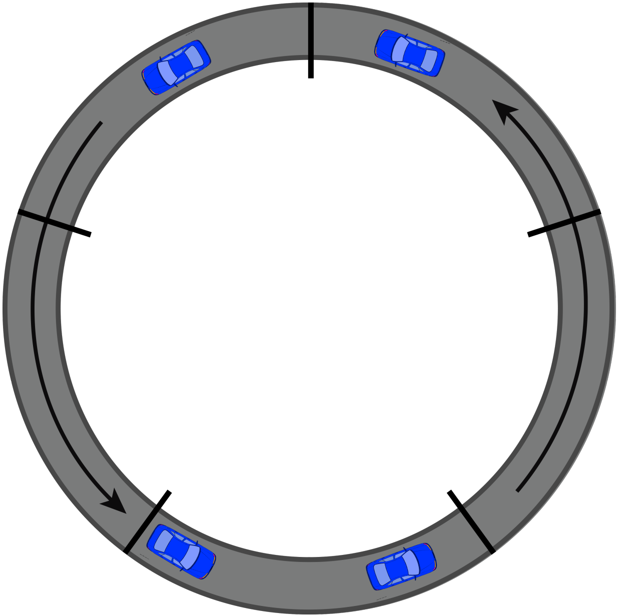
$$\psi'(t) = \frac{Q(\rho_{\text{right}}) - Q(\rho_{\text{left}})}{\rho_{\text{right}} - \rho_{\text{left}}},$$

where $\psi(t)$ is a boundary of the shock wave.

Integral identity:

$$\frac{d}{dt}\int\limits_{\alpha}^{\beta} \rho(x,t)dx = Q(\rho(\alpha,t)) - Q(\rho(\alpha,t)).$$

Ring road idea



Specifics of the ring road

L is length of the ring road.
All functions are L-periodic for x.
The amount of cars on the road is main value

$$M \equiv \int_0^L \mu(x)dx.$$

Two cases:

1. $M < \rho^* L$ (almost free movement);
2. $M > \rho^* L$ (crowded traffic).

Stability of solutions on $t \to \infty$

$$M \equiv \int_0^L \mu(x)dx.$$

Theorem

There is an alternative.

1. *If* $M < L\rho^*$, *then* $\exists\, t^* > 0$ *such that*

$$\rho(x,t) = f(x - k_1\, t), \quad \forall\, t \geqslant t^*,$$

where $0 \leqslant f(s) \leqslant \rho^*$ *for* $\forall s \in \mathbb{R}$.

2. *If* $M > L\rho^*$, *then* $\exists\, t^* > 0$ *such that*

$$\rho(x,t) = g(x + k_2\, t), \quad \forall\, t \geqslant t^*,$$

where $\rho^* \leqslant g(s) \leqslant \rho_{\max}$ *for* $\forall s \in \mathbb{R}$.

Combinatorial result

Lemma

In every set of real numbers

$$a_1, a_2, \ \ldots \ , a_n,$$

with

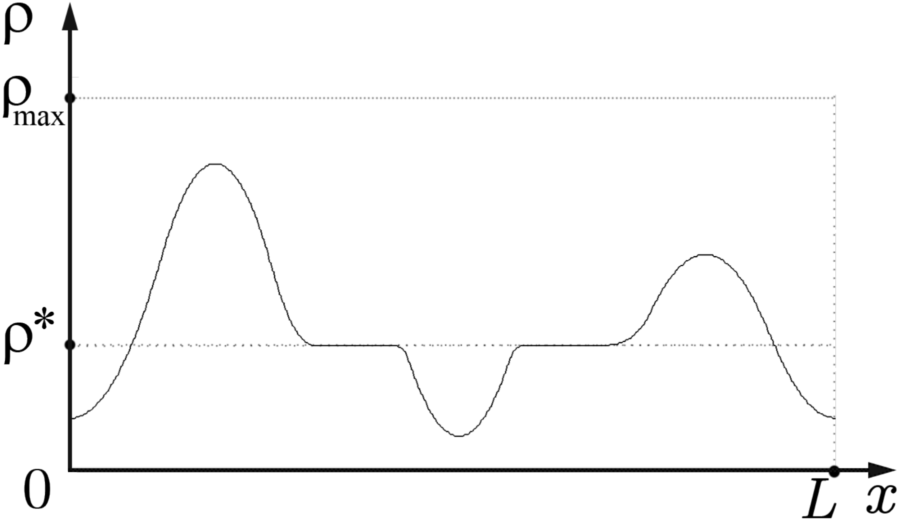
$$a_1 + a_2 + \ldots + a_n > 0,$$

there exists a dominant element a_k, *such as*

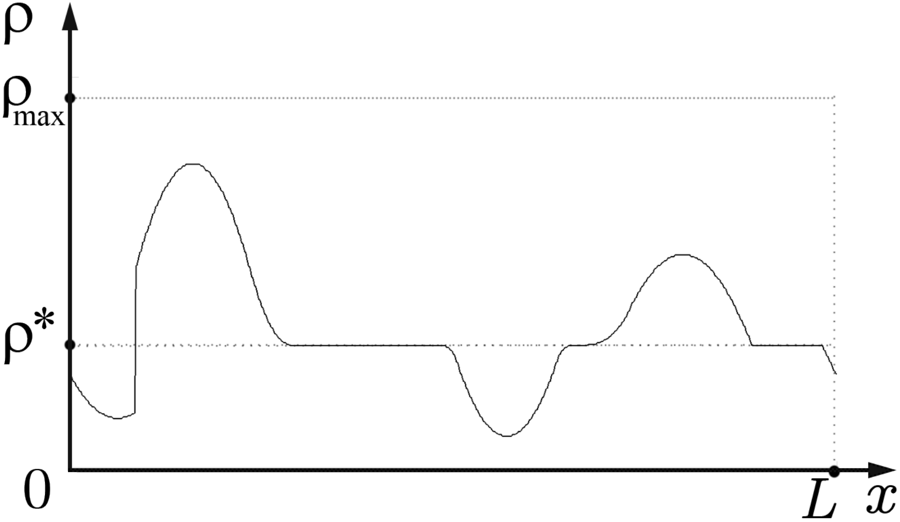
$$\begin{aligned} & a_k > 0, \\ & a_k + a_{k+1} > 0, \\ & a_k + a_{k+1} + a_{k+2} > 0, \\ & \ldots \ldots \ldots \\ & a_k + a_{k+1} + a_{k+2} + \ldots + a_{k+n-1} > 0. \end{aligned}$$

Index changes cyclically.

Example



Example



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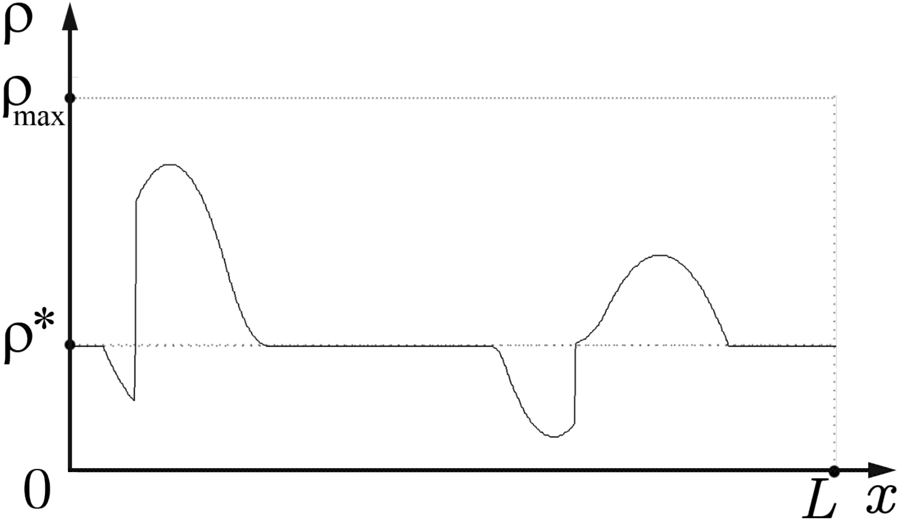

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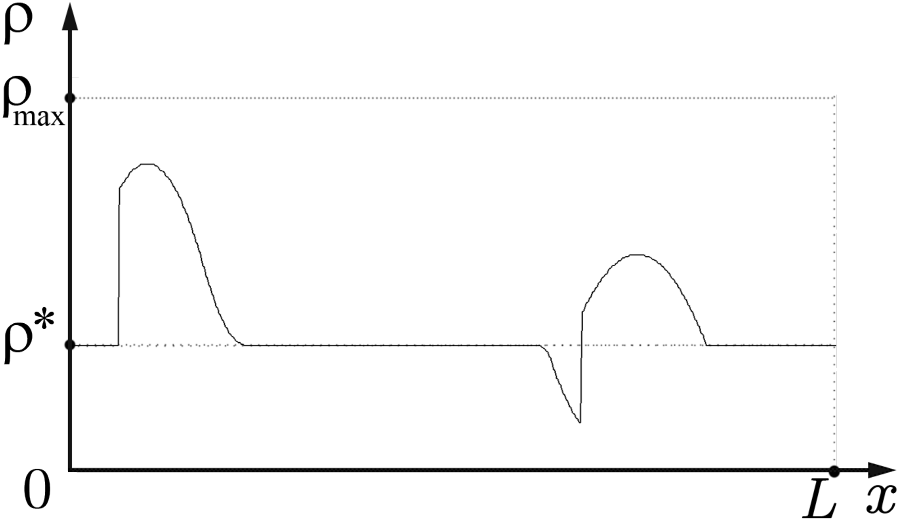

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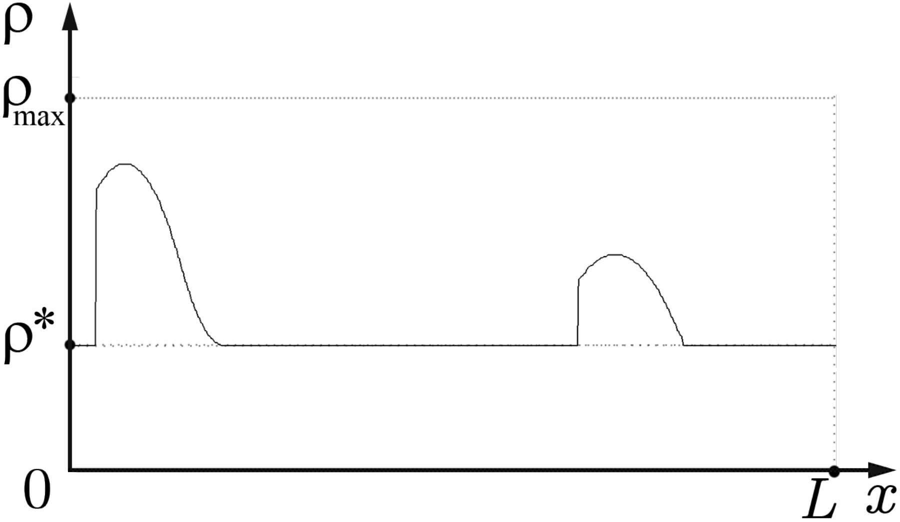

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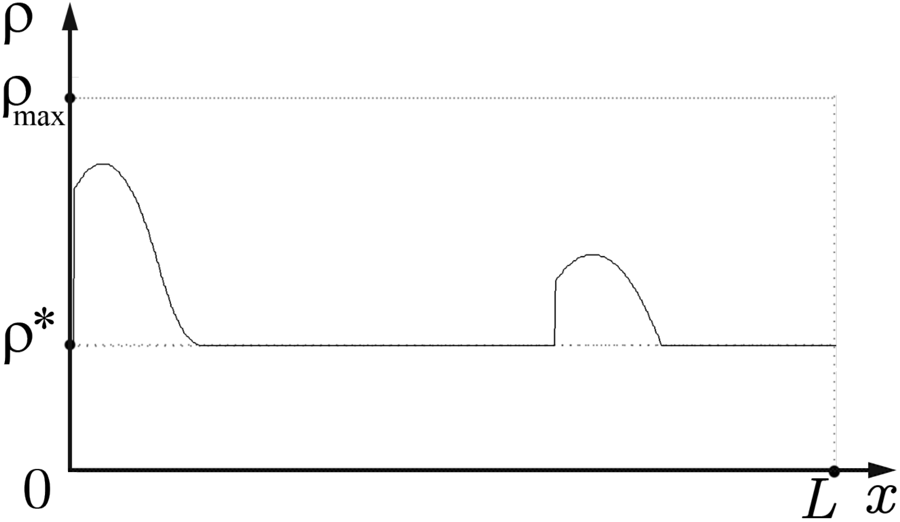

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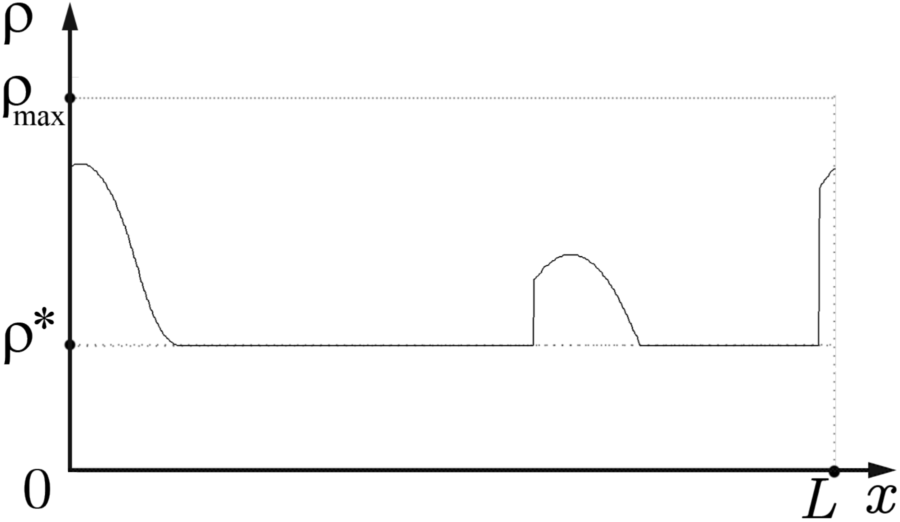

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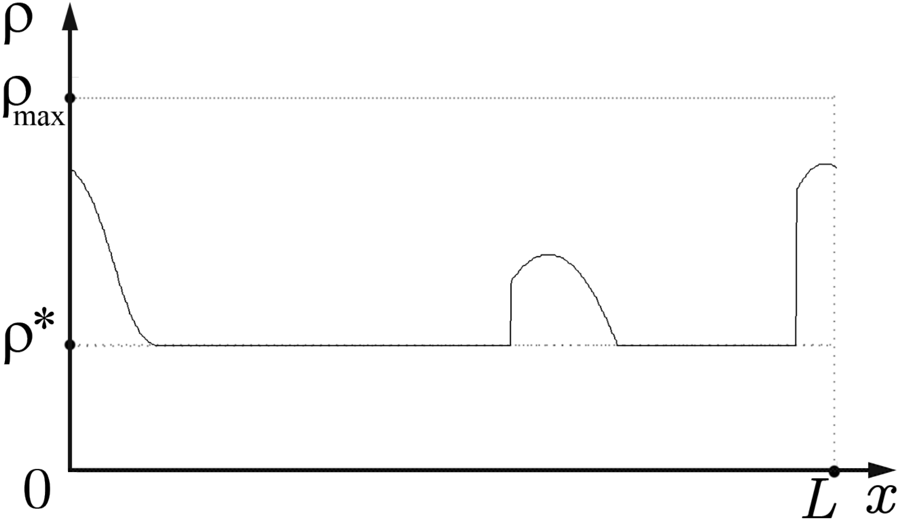

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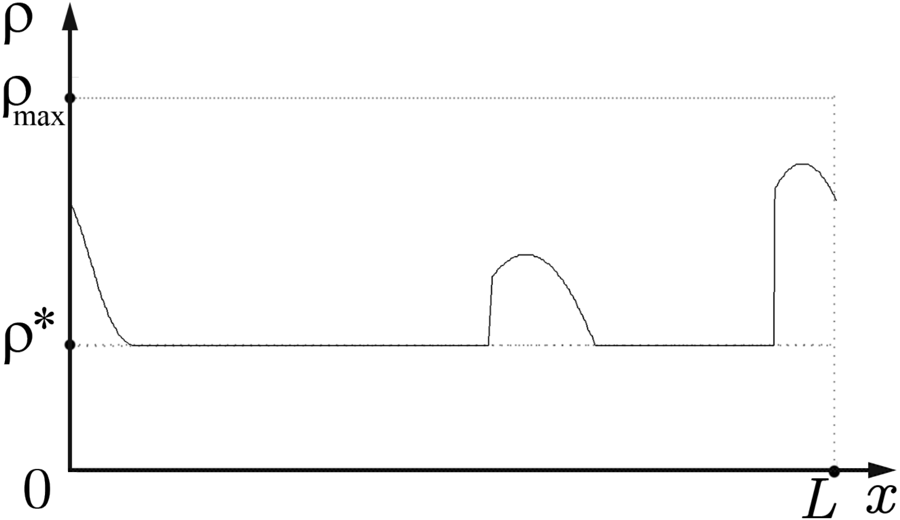

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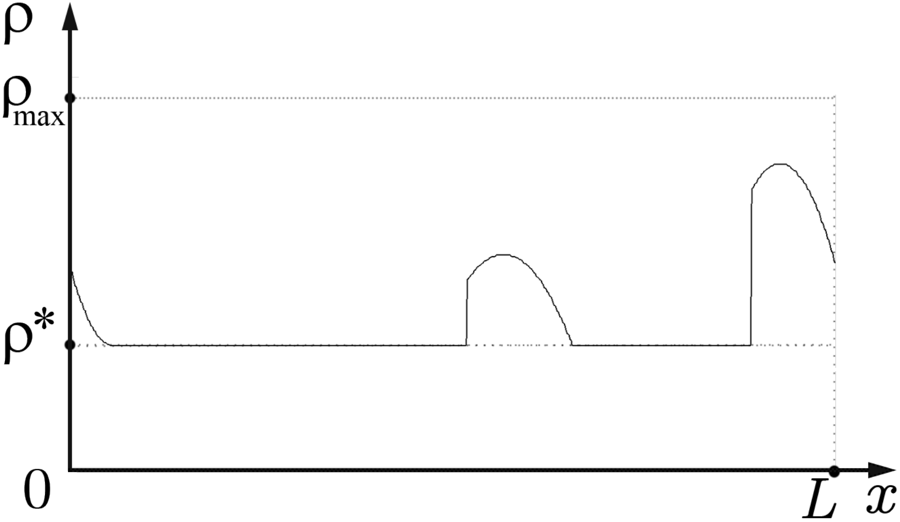

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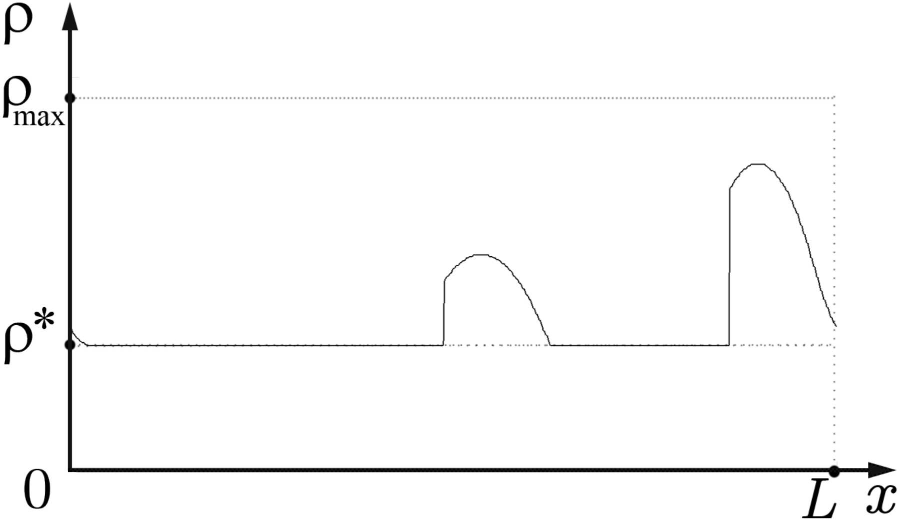


Thank you for your attention!