Higher Hirota difference equation and its reductions

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Let A and B be two elements of an associative algebra over $\mathbb{C}$ with unity I. Let for some $a_1, a_2, a_3 \in \mathbb{C}$ ($a_1 \neq a_2 \neq a_3 \neq a_1$) there exist inverse $(A - a_i I)^{-1}$.

We introduce dependence of B on three discrete variables $m_1, m_2, m_3 \in \mathbb{Z}$ by means of

$$B(m) \equiv B(m_1, m_2, m_3) = \left(\prod_{n=1}^{3} (A - a_n)^{m_n}\right) B \left(\prod_{n=1}^{3} (A - a_n)^{m_n}\right)^{-1},$$

and denote

$$B^{(1)}(m) = B(m_1 + 1, m_2, m_3), \quad B^{(2)}(m) = B(m_1, m_2 + 1, m_3), \quad \ldots,$$

Then $B(m)$ obeys linear difference equation

$$a_{12}\{B^{(12)} + B^{(3)}\} + a_{23}\{B^{(23)} + B^{(1)}\} + a_{31}\{B^{(31)} + B^{(2)}\} = 0$$
$$(a_{ij} = a_i - a_j),$$

that follows from commutator identity

$$a_{12}\{(A - a_1)(A - a_2)B(A - a_1)^{-1}(A - a_2)^{-1} + (A - a_3)B(A - a_3)^{-1}\} +$$
$$+ \mathrm{cycle}(1, 2, 3) = 0.$$

Let us consider (infinite) matrices F, G, ..., let T denotes shift matrix $T_{m_1,m_1'} = \delta_{m_1,m_1'+1}$. Then any matrix can be written as $F = \sum_{n\in\mathbb{Z}} f_n T^n$, where all $f_n = \mathrm{diag}\{f_n(m_1)\}_{m_1\in\mathbb{Z}}$ are diagonal, i.e., mutually commuting matrices. We associate to every matrix its symbol $\widetilde{F}(m_1, z) = \sum_{n\in\mathbb{Z}} f_n(m_1) z^n$, where $m_1 \in \mathbb{Z}$, $z \in \mathbb{C}$. The standard product of matrices F and G in terms of their symbols takes the form

$$\widetilde{FG}(n_1, z) = \oint_{|\zeta|=1} \frac{d\zeta}{2\pi i\zeta} \widetilde{F}(n_1, z\zeta) \sum_{m_1\in\mathbb{Z}} \zeta^{n_1-m_1} \widetilde{G}(m_1, z).$$

Symbol of the unity operator is $\widetilde{I}(n, z) = 1$, and symbol of the shift operator T: $\widetilde{T}(n, z) = z$. For any F:

$$\widetilde{TFT^{-1}}(n_1, z) = \widetilde{F}(n_1 + 1, z)$$

In what follows we consider set of "pseudo-matrix" operators given by their symbols with the above composition law. We impose condition that symbols are tempered distributions with respect to their variables. On the set of such operators one can define $\bar{\partial}$-differentiation: $F \to \bar{\partial}F$:

$$(\widetilde{\bar{\partial}F})(n, z) = \frac{\partial \widetilde{F}(n, z)}{\partial \overline{z}}$$

In particular, $\bar{\partial}T = 0$.

We consider operators A and B as operators of this kind with symbols $\widetilde{A}$ and $\widetilde{B}$. Dependence of the symbol of B on m_1, $B^{(1)} = (A - a_1)B(A - a_1)^{-1}$ is exactly as the one given by the similarity transformation with operator T. Thus we can put $A = T + a_1$, i.e., $\widetilde{A}(n, z) = (z + a_1)$. Then dependence on m_2, m_3:

$$B^{(2)} = (A - a_2)B(A - a_2)^{-1}, \qquad B^{(3)} = (A - a_3)B(A - a_3)^{-1}$$

is defined as

$$B^{(2)} = (T + a_{12})B(T + a_{12})^{-1}, \qquad B^{(3)} = (T + a_{13})B(T + a_{13})^{-1}.$$

Dressing operator K with symbol $\widetilde{K}(n, z)$ is introduced as solution of the $\overline{\partial}$-problem:

$$\overline{\partial} K = KB, \qquad \lim_{z \to \infty} \widetilde{K}(n, z) = 1$$

under assumption of its unique solvability. Evolutions of this operator are defined by means of $\overline{\partial} K^{(m_j)} = K^{(m_j)} B^{(m_j)}$ that are compatible by construction. In particular, $K^{(1)} = TKT^{-1}$. Next, $\overline{\partial}(K^{(2)}(T + a_{12})) = (K^{(1)}(T + a_{12}))B$. Writing

$$K = 1 + uT^{-1} + \dots, \quad z \to \infty,$$

where u is a multiplication operator (its symbol does not depend on z) we get

$$K^{(2)}(T + a_{12}) = (T + a_{12} + u^{(2)} - u^{(1)})K.$$

Analogousely: $K^{(3)}(T + a_{13}) = (T + a_{13} + u^{(3)} - u^{(1)})K.$

Let us introduce:

$$\varphi(n,z)=\widetilde{K}(n,z)z^{n_1}(z+a_{12})^{n_2}(z+a_{13})^{n_3}.$$

Then the above equations can be written in the form:

$$\begin{aligned}
\varphi^{(2)} &= \varphi^{(1)}+\big(u^{(2)}-u^{(1)}+a_{12}\big)\varphi,\\
\varphi^{(3)} &= \varphi^{(2)}+\big(u^{(3)}-u^{(2)}+a_{23}\big)\varphi,\\
\varphi^{(1)} &= \varphi^{(3)}+\big(u^{(1)}-u^{(3)}+a_{31}\big)\varphi,
\end{aligned}$$

so that the Lax pair is given by any two of them. Condition of compatibility reads as

$$u^{(12)}(u^{(2)}-u^{(1)}+a_{12})+a_{12}u^{(3)}+\text{cycle}=0,$$

that is the Hirota difference equation. It gives “non-linearization” of the original linear equation

$$B^{(12)}a_{12}+a_{12}B^{(3)}+\text{cycle}=0.$$

In order to write the Hirota difference equation in a more familiar form we introduce

$$w(m) = u(m) - m_1a_1 - m_2a_2 - m_3a_3,$$

that remove all a_i from the Lax pair

$$\begin{aligned}
\varphi^{(2)} &= \varphi^{(1)} + \left(v^{(2)} - v^{(1)}\right)\varphi, \\
\varphi^{(3)} &= \varphi^{(2)} + \left(v^{(3)} - v^{(2)}\right)\varphi, \\
\varphi^{(1)} &= \varphi^{(3)} + \left(v^{(1)} - v^{(3)}\right)\varphi,
\end{aligned}$$

and

$$v^{(12)}(v^{(2)} - v^{(1)}) + \text{cycle}(1, 2, 3) = 0,$$

while asymptotic condition is essential: $u(m)$ is decaying.

As we have seen evolutions of operator B are given by means of

$$B^{(1)} = TBT^{-1}, \quad B^{(2)} = (T + a_{12})B(T + a_{12})^{-1}, \quad B^{(3)} = (T + a_{13})B(T + a_{13})^{-1}.$$

This means that symbol of B has representation

$$\widetilde{B}(m, z) = \oint\limits_{|\zeta|=1} \frac{d\zeta}{2\pi i\zeta}\zeta^{m_1}\left(\frac{z\zeta + a_{12}}{z + a_{12}}\right)^{m_2}\left(\frac{z\zeta + a_{13}}{z + a_{13}}\right)^{m_3} b(\zeta, z),$$

where $b(\zeta, z)$ is some function. It is natural to impose conditions

$$\left|\frac{z\zeta + a_{12}}{z + a_{12}}\right| = \left|\frac{z\zeta + a_{13}}{z + a_{13}}\right| = 1,$$

that are equivalent to $\frac{\overline{z}}{z} = \frac{\overline{a}_{12}}{a_{12}}\zeta = \frac{\overline{a}_{13}}{a_{13}}\zeta$. Then $\overline{a}_{12}/a_{12} = \overline{a}_{13}/a_{13}$, so we can choose all a_j to be real. This means that function $b(\zeta, z)$ has support on the surface $\zeta = \overline{z}/z$, i.e., representation for the symbol of B has the form

$$\widetilde{B}(m, z) = \left(\frac{\overline{z}}{z}\right)^{m_1}\left(\frac{\overline{z} + a_{12}}{z + a_{12}}\right)^{m_2}\left(\frac{\overline{z} + a_{13}}{z + a_{13}}\right)^{m_3} f(z),$$

where $f(z)$ is an arbitrary function of $z \in \mathbb{C}$.

Space reduction of the HDE. Condition $\widetilde{B}(m,z)$ is independent on some variable leads to $z_{\mathrm{Im}}=0$, that cancels dependence on all other variables, because

$$\widetilde{B}(m,z)=\left(\frac{\overline{z}}{z}\right)^{m_1}\left(\frac{\overline{z}+a_{12}}{z+a_{12}}\right)^{m_2}\left(\frac{\overline{z}+a_{13}}{z+a_{13}}\right)^{m_3}f(z).$$

Nontrivial reduction is, say, $B^{(2)}=B^{(-1)}$, that gives two conditions: either $z_{\mathrm{Im}}=0$, or $z_{\mathrm{Re}}=-a_1$ $(a_2=-a_1)$. Thus

$$\widetilde{B}(m,z)=a(z_{\mathrm{Re}})\delta(z_{\mathrm{Im}})+\left(\frac{a_1+iz_{\mathrm{Im}}}{a_1-iz_{\mathrm{Im}}}\right)^{m_1-m_2}\left(\frac{a_3+iz_{\mathrm{Im}}}{a_3-iz_{\mathrm{Im}}}\right)^{m_3}b(z_{\mathrm{Im}})\delta(z_{\mathrm{Re}}+a_1)$$

Introducing $\lambda=z+a_1$ and $w(m_1,m_3)=u(m_1,m_3)-m_1a_1-m_3a_3$ we can write Lax pair

$$\begin{aligned}&(\lambda-a_1)\,\varphi^{(-1)}-(\lambda+a_1)\,\varphi^{(1)}=\left(w^{(-1)}-w^{(1)}\right)\varphi,\\&\varphi^{(3)}=(\lambda+a_1)\,\varphi^{(1)}+\left(w^{(3)}-w^{(1)}\right)\varphi,\end{aligned}$$

and equation $(w^{(1,3)}-w)(w^{(3)}-w^{(1)})=(w^{(-1,3)}-w)(w^{(3)}-w^{(-1)})$, or

$$\frac{(w^{(3)}-w^{(-1)})^{(1)}}{w^{(3)}-w^{(-1)}}=\frac{(w^{(3)}-w^{(1)})^{(-1)}}{w^{(3)}-w^{(1)}}$$

Higher Hirota difference equation.

Let us introduce evolutions of operator B by means of

$$B^{(1)}=(A-a_1)B(A-a_1)^{-1}, \qquad B^{(2)}=(A-a_2)B(A-a_2)^{-1},$$
$$B^{(3)}=(A^2-a_3^2)B(A^2-a_3^2)^{-1}$$

Let $\Delta_i B=B^{(i)}-B$. Then we have identity

$$\left[(\Delta_1 a_1-\Delta_2 a_2)^2-a_3^2(\Delta_1-\Delta_2)^2\right]\Delta_3 B=a_{12}\Delta_1\Delta_2\big(a_{12}\Delta_1\Delta_2+2\Delta_1 a_1-\Delta_2 a_2\big)B$$

The dressing operator K is defined as above: $\overline{\partial}K=KB$, $\lim_{z\to\infty}\widetilde{K}(m,z)=1$, as well as shifts with respect to m_1 and m_2:

$$K^{(2)}(T+a_{12})=K^{(1)}T+(u^{(2)}-u^{(1)}+a_{12})K, \tag{1}$$

For the shift with respect to m_3 we have $\bar{\partial}\big(K^{(3)}[(T+a_1)^2-a_3^2]\big)= K^{(3)}[(T+a_1)^2-a_3^2]B$, so that $K^{(3)}[(T+a_1)^2-a_3^2]=[T^2+XT+Y]K$, where X and Y are multiplication operators. In order to derive them we need the second order term of expansion

$$K=1+uT^{-1}+vT^{-2}+\dots, \qquad z\to\infty,$$

and by (1)

$$v^{(2)}-v^{(1)}=(u^{(2)}-u^{(1)}+a_{12})u-a_{12}u^{(2)}$$

In this way we get Lax pair:

$$
\begin{aligned}
\varphi^{(2)} &= \varphi^{(1)} + (u^{(2)} - u^{(1)})\,\varphi \\
\varphi^{(3)} &= \varphi^{(1,2)} + (u^{(3)} - u^{(1,2)})\frac{\varphi^{(1)} + \varphi^{(2)}}{2} + \\
&\quad + \left[v^{(3)} - v^{(1,2)} - \frac{1}{2}(u^{(1)} + u^{(2)})(u^{(3)} - u^{(1,2)})\right]\varphi
\end{aligned}
$$

and equation:

$$
\begin{aligned}
&(v^{(1)} + v^{(2)} - 2v)^{(3)} - (v^{(1)} + v^{(2)} - 2v)^{(1,2)} + (u^{(1)} + u^{(2)})(u^{(3)} - u^{(1,2)}) - \\
&\qquad - u^{(1,2)}\left[(u^{(1)} + u^{(2)})^{(3)} - (u^{(1)} + u^{(2)})^{(1,2)}\right] = 0 \\
&v^{(2)} - v^{(1)} = (u^{(2)} - u^{(1)})u,
\end{aligned}
$$

where we performed substitution

$$
\begin{aligned}
u(n) &\to u(n) + a_1 n_1 + a_2 n_2 \\
v(n) &\to v(n) + \ldots \sim \frac{1}{2}(a_1 n_1 + a_2 n_2)^2 - n_3 a_3^2, \quad |n| \to \infty.
\end{aligned}
$$

This system also admits (1+1)-dimensional reduction $u^{(2)} = u^{(-1)}, \ldots$

$$(v_1 + v_{-1})_3 + (u_1 + u_{-1})u_3 - (u_1 + u_{-1})_3 u = 0$$
$$v_1 - v_{-1} = (u_1 - u_{-1})u,$$

where

$$v_1 = v^{(1)} - v, \qquad v_{-1} = v^{(-1)} - v, \qquad u_3 = u^{(3)} - u, \ldots$$

But it also admits reduction $u^{(3)} = u$:

$$(u^{(1)} + u^{(2)})(u^{(1,2)} - u) - u^{(1,2)}\left[(u^{(1)} + u^{(2)})^{(1,2)} - (u^{(1)} + u^{(2)})\right] +$$
$$+ (v^{(1)} + v^{(2)} - 2v)^{(1,2)} - (v^{(1)} + v^{(2)} - 2v) = 0$$
$$v^{(2)} - v^{(1)} = (u^{(2)} - u^{(1)})u,$$

Limiting cases. Evolutions

$$B^{(1)} = (A - a_1)B(A - a_1)^{-1}, \quad B^{(2)} = (A - a_2)B(A - a_2)^{-1},$$
$$B^{(3)} = (A^2 - a_3^2)B(A^2 - a_3^2)^{-1}$$

become ill-defined or singular only if $a_k \to \infty$ for some k, or if $a_1 = a_2$, or $a_3^2 = a_1^2$, $a_3^2 = a_2^2$. These limiting values must be considered specially. Say, in tthe limit $a_3 \to \infty$: $a_3^2(B^{(3)} - B) \to -[A^2, B]$. Let us introduce continuous variable t_3: $B_{t_3} = [A^2, B] \equiv [(T+a_1)^2, B]$. Then $\bar\partial$-problem for K gives $\bar\partial\big(K_{t_3}+K(T+a_1)^2\big) = \big(K_{t_3} + K(T + a_1)^2\big)B$. Like above we derive Lax pair:

$$\varphi^{(2)} = \varphi^{(1)} +(u^{(2)} - u^{(1)})\,\varphi$$
$$\varphi_{t_3} = \varphi^{(1,2)} +(u - u^{(1,2)})\frac{\varphi^{(1)} + \varphi^{(2)}}{2}+$$
$$+\left[v - v^{(1,2)} - \frac{1}{2}(u^{(1)} + u^{(2)})(u - u^{(1,2)})\right]\varphi$$

and nonlinear equation

$$\frac{\partial}{\partial t_3}\ln(u^{(2)} - u^{(1)})^2 = v^{(1)} + v^{(2)} - 2v - (v^{(1)} + v^{(2)} - 2v)^{(1,2)} + u(u^{(1)} + u^{(2)})-$$
$$- u^{(1,2)}\left[2(u^{(1)} + u^{(2)}) - (u^{(1)} + u^{(2)})^{(1,2)}\right]$$
$$v^{(2)} - v^{(1)} = (u^{(2)} - u^{(1)})u.$$

Limit $a_2 \to a_1$. Here $B^{(2)} = B^{(1)} + a_{12}[T^{-1}, B^{(1)}] + o(a_{12})$, so we can introduce another continuous variable t_2: $B_{t_2} = [T^{-1}, B]$. Then $\bar\partial(K_{t_2}T + K) = (K_{t_2}T + K)B^{(-1)}$ and in the same way as before we get Lax pair:

$$\varphi_{t_2} = (1 + u_{t_2})\,\varphi^{(-1)}$$
$$\varphi_{t_3} = \varphi^{(1,1)} + (u - u^{(1,1)})\,\varphi^{(1)} + \big(v - v^{(1,1)} + (u^{(1,1)} - u)u^{(1)}\big)\,\varphi,$$

where v is the second coefficient of decomposition $K = 1 + uT^{-1} + vT^{-2} + \ldots$.

The compatibility condition is the second equation in Toda hierarchy:

$$\partial_{t_3} \ln(1 + u_{t_2}) = v^{(-1)} - v - (u^{(1)} - u^{(-1)})u + (u^{(1,1)} - u)u^{(1)}$$
$$v_{t_2} = -u + (1 + u_{t_2})u^{(-1)},$$

or

$$\partial_{t_3} \ln(1 + u_{t_2}) = \big(v - (u^{(1,1)} - u)u^{(1)}\big)^{(-1)} - \big(v - (u^{(1,1)} - u)u^{(1)}\big),$$
$$v_{t_2} = -u + (1 + u_{t_2})u^{(-1)}.$$

We get another chain if instead of $a_2 \to a_1$ we consider **limit** $a_2 \to \infty$. In this case $B^{(2)} = B - a_2^{-1}[A, B] + \ldots$. Let now $B_{t_2} = [A, B]$, while as above

$$B^{(1)} = (A - a_1)B(A - a_1)^{-1}, \qquad B_{t_3} = [A^2, B]$$

Such operator obeys equation

$$(B^{(1)} - B)_{t_3} = B^{(1)}_{t_2 t_2} + B_{t_2 t_2}$$

Then we derive Lax pair:

$$\begin{aligned} \varphi_{t_2} &= \varphi^{(1)} - (u^{(1)} - u)\,\varphi, \\ \varphi_{t_3} &= \varphi^{(1,1)} - (u^{(1,1)} - u)\,\varphi^{(1)} + \{-u^{(1)}_{t_2} - u_{t_2} + (u^{(1)} - u)^2\}\,\varphi\,. \end{aligned}$$

and equation:

$$(u^{(1)} - u)_{t_3} = \{u^{(1)}_{t_2} + u_{t_2} - (u^{(1)} - u)^2\}_{t_2}$$

This chain also admits (1+1)-dimensional reduction $B_{t_3} = 0$. Then Lax pair reads as

$$\varphi^{(1,1)} - (u^{(1,1)} - u)\,\varphi^{(1)} + \{-u^{(1)}_{t_2} - u_{t_2} + (u^{(1)} - u)^2\}\,\varphi = z^2\,\varphi,$$
$$\varphi_{t_2} = \varphi^{(1)} - (u^{(1)} - u)\,\varphi,$$

and

$$(u^{(1)} + u)_{t_2} = (u^{(1)} - u)^2.$$