SCHUMPETERIAN DYNAMICS:

A SURVEY OF DIFFERENT APPROACHES

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I. Introduction: Innovation and Imitation

- **Josef Schumpeter (1939) divided the mechanism of technological changes into two components: creation of new technologies by a firm (innovation process) and adoption of technologies created by other firms (imitation process).**
- **The process of productivity growth of production units due to both technology innovations and imitation of technologies from more advanced agents is called Schumpeterian dynamics.**

Imitation: two technologies

Transition process between two technologies-

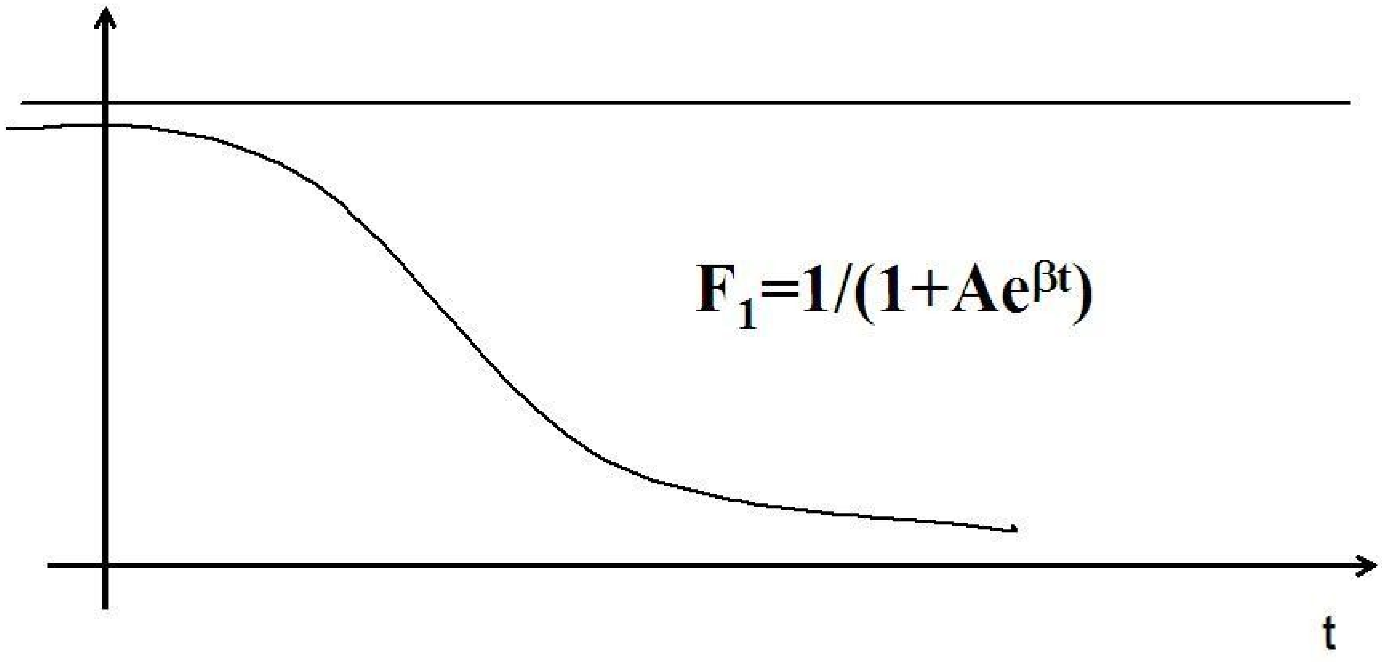
logistic curve - Griliches(1957), Davies (1979)

$$dF_1/dt = -\beta(1-F_1)F_1, \quad F_1(-\infty) = 1, \ \beta>0.$$

F_1 - the fraction of firms (or capacities) that use an old technology; the speed of the transition is proportional to F_1 and the proportionality

coefficient increases with expansion of the fraction of the firms that have adopted the new technology.

Logistic curve



Innovation and Imitation: many technologies with different efficiencies

- **Even in industries producing a homogeneous good, technologies of different efficiencies coexist, so that one may observe a distribution of firms on efficiency levels.**
- **Efficiency may be defined as profit or added value per unit of capacity, or *total factor productivity*.**
- **Cobb-Douglas production function:**

$Y = AK^{\alpha} L^{1-\alpha}$, Y –output, K-capital, L-labor, A –TFP

- **Considering an industry with many firms, one can describe its development as evolution of efficiency distribution. This fact is emphasized in the production function theory of Houthakker (1956) and Johansen (1972).**

What is this presentation about

- **Different mechanisms of innovation and imitation generate various patterns of Schumpeterian dynamics described by a wide range of non-linear equations, including**
- **Burgers - type equations,**
- **Kolmogorov-Petrovskii-Piskunov-type equations,**
- **Boltzmann equation, etc.**

An explosion of researches, Lucas, Acemoglu.

- **I discuss the economic essence of these mechanisms in the context of economic growth theory and recent results of their investigations.**
- **Some related unsolved problems will be also formulated.**

II. Distribution of firms by TFP : stylized facts-1 König et al. (2015b)

- **A large data set containing information about the productivity of western European firms in the period between 1995 and 2003. Main empirical findings:**

1. The distribution of high-productivity firms is well described by a power law.

2. The distribution of low-productivity firms is also well approximated by a power law, although this approximation is less accurate, arguably due to noisy data at low productivity levels for small firms.

3. The distribution is characterized by a constant growth rate over time, where both the right and the left power law are fairly stable (see Table).

- **This implies that the evolution over time of the empirical productivity distribution can be described as a 'traveling wave‘ (see also Sato (1975)).**

Distribution of firms by TFP : stylized facts-2
König et al. (2015b)

- **Estimated power law exponents for the right (λ) and left (ρ) tail of the probability density function for the total factor productivity (TFP) distribution of (17,404) French firms, 1995 - 2003**

year	λ	> mean(A)	$R^2(\lambda)$	ρ	< geomean(A)	$R^2(\rho)$
1995	3.80	35.2%	0.99	2.13	51.7%	0.97
1996	3.85	35.0%	0.99	2.50	51.8%	0.99
1997	3.77	34.6%	1.00	2.52	52.4%	0.98
1998	3.79	35.0%	0.99	2.54	52.3%	0.98
1999	3.77	34.7%	0.99	2.55	52.4%	0.99
2000	3.72	34.0%	0.99	2.31	52.9%	0.97
2001	3.71	34.2%	1.00	2.43	52.4%	0.98
2002	3.67	33.5%	0.99	2.26	52.3%	0.97
2003	3.53	33.0%	0.99	1.99	52.1%	0.96
average	3.73			2.36		

- **The percentage of firms on which the regression is computed is shown as well as the corresponding coefficient of determination R2.**

Efficiency distribution: stylized facts-2

- **While entry, exit and reallocation are important determinants of firm dynamics, they altogether account for only 25% of total productivity growth**

  So, we must explain the determinants of the accumulation of technical knowledge among incumbent firms (Konig et al, 2015a)
- **Established firms are the main source of innovations that improve existing products, while new firms invest in more radical and "original" innovations (Acemoglu, Cao, 2014).**

Size distribution of firms: stylized facts-3

- **«As many have noted, the size distribution of firms exhibits a striking pattern." Using 1997 data from the U.S. Census, Axtell [2001] finds that the right tail probabilities of this distribution, with firm size measured by employment S, is well approximated**

by a Pareto distribution:

$1/S^{\zeta}$, with a tail index ζ around 1.06.

(Luttmer, 2006, p. 2). This is close to Zipf's law.

"… firms closer to the technology frontier engage in more research and development investments (Griffith et al. 2003), and that large firms spend more on research and development than smaller ones. For example, Mandel (2011) finds that US firms with 5,000 or more employees spend more than twice as much per worker on research and development as those with 100-500 employees." (Lorentz et al, 2015)

II. Modeling Schumpeterian dynamics: movement mechanisms

- Speeds of innovation and imitation depend on labor and capital expenditures.
- Imitation speed may arise from observation

of more advanced firms or from meetings with them to get technologies.

- If the most advanced firm exists then “the distance to frontier” might be important.
- New firms may imitate incumbents stochastically or choose the best technology.

Modeling Schumpeterian dynamics: notations

- $F_n(t)$ - a fraction of firms that have efficiency level n or less at the moment $t \in [0, \infty)$; n-integer.

$\forall$ $\mathfrak{I} = \{F_n(t)\}$ - distribution function.

- $\{f_n(t)\}$ – density function
- Standard initial conditions:

$$F_n(0) = 0,\ n \le 0; \qquad 0 \le F_n(0) \le 1$$

$$\sum_{n=1}^{\infty} (1-F_n(0)) < \infty.$$

- $F(x,t)$, $f(x,t)$- continuous case, x- efficiency level

Modeling Schumpeterian dynamics: straightforward assumptions-Burgers type eq

$$df_n /dt= \sum_{k=1}^{n-1} \varphi(F_k, f_n, t)f_k - \sum_{k=n+1}^{\infty} \varphi(F_n, f_k, t)f_n, \quad n=1,2...$$

$\varphi(F_k, f_n, t)$- fraction of firms f_k at a level k jumping on the level n in the moment t per unite time. This equation includes the most important particular cases.

Assume that each firm from f_k can observe 1- F_k but can jump on the next level only:

$$\varphi(F_k, f_n, t)=0, \; k\neq n-1$$

$$\varphi(F_{n-1}, f_n, t)= \varphi(F_{n-1}),$$

Then $df_n /dt= \varphi(F_{n-1}) f_{n-1} - \varphi(F_n)f_n, \; f_0=0.$

$$dF_n/dt = \varphi(F_n) (F_{n-1} - F_n).$$

This is a difference-differential analogue of the Burgers eq.

Modeling Schumpeterian dynamics: straightforward assumptions-KPP type eq.

$$\frac{df_n}{dt} = \sum_{k=1}^{n-1} \varphi(F_k, f_n, t) f_k - \sum_{k=n+1}^{\infty} \varphi(F_n, f_k, t) f_n, \quad n=1,2...$$

Assume that per unit time, the fraction βf_n of f_k, $k < n$ jumps on the level n due to meetings with f_n and imitation; besides the fraction α of f_{n-1} jumps on the level n due to innovation.

$\varphi(F_k, f_n, t) = \beta f_n$, $k < n-1$, $\varphi(F_k, f_n, t) = 0$, $k \geq n$,

$\varphi(F_{n-1}, f_n, t) = \alpha + \beta f_n$,

Then

$df_n/dt = -\alpha f_n + \alpha f_{n-1} - \beta(1-F_n) f_n + \beta F_{n-1} f_n$,

$$dF_n/dt = -\alpha (F_n - F_{n-1}) - \beta(1 - F_n) F_n.$$

This is a difference- differential analogue of the Kolmogorov – Petrovsky – Piskunov's Equation.

Modeling Schumpeterian dynamics: straightforward assumptions-Boltzmann type eq.

$$\frac{df_n}{dt} = \sum_{k=1}^{n-1} \varphi(F_k, f_n, t) f_k - \sum_{k=n+1}^{\infty} \varphi(F_n, f_k, t) f_n, \quad n=1,2\ldots$$

Assume that per unit time, the fraction $\psi_k(t)f_n$ of f_k jumps on the level n due to meetings with f_n and imitation-innovation:

$$\varphi(F_k, f_n, t) = \psi_k(t) f_n, \; k<n, \quad \varphi(F_k, f_n, t)=0, \; k \geq n,$$

$$\frac{df_n}{dt} = f_n \sum_{k=1}^{n-1} \psi_k(t) f_k - \psi_n(t) f_n \sum_{k=n+1}^{\infty} f_k,$$

$$\frac{dF_n}{dt} = -(1-F_n) \sum_{k=1}^{n} \psi_k(t) f_k.$$

This is a difference- differential analogue of the Boltzmann's Equation.

III. Modeling Schumpeterian dynamics: a stochastic differential equation and KPP eq.

- **Suppose that the log productivity x_t of a particular producer evolves according to**

$$dx_t = \alpha dt + \sigma dW_t + \Delta_t dN_t \ ,$$

where α represents deterministic innovation by this producer, W_t is a standard Brownian motion (stochastic innovation), N_t is a Poisson process with arrival rate β that counts opportunities to imitate. When an imitation opportunity arrives, the producer randomly selects another producer from the population and copy his technology if it is more productive. The resulting increase in productivity is represented by $\Delta_t \geq 0$. In a large population, any initial discreteness in the initial productivity distribution is smoothed out instantaneously, and we get Kolmogorov – Petrovsky – Piskunov's Equation:

$$\forall \ \partial F/\partial t = - \alpha \partial F/\partial x + 0.5\sigma^2(\partial^2 F/\partial x^2) - \beta F(1- F)$$

where F is the distribution of log productivity x at time t ($\partial F/\partial x$ can be excluded by a substitution). (Luttmer (2012), Konig (2015))

Modeling Schumpeterian dynamics: a stochastic differential equation and Burgers eq.

- **A collection of N groups of interacting agents A_k**

with productivity $X_k(t)$, k = 1, 2,… K, $X_{k+1}(0) > X_k(0)$. The speed of $X_k(t)$ is a sum of three components:

1. **deterministic innovation (α);**
2. **stochastic innovation (Brownian motion, with parameter σ);**
3. **a term proportional (γ) to the fraction of more productive firms.**

Then for K $\to \infty$ we get Burgers equation

$$\partial F/\partial t = -(\alpha + \gamma(1-F))(\partial F/\partial x) + 0.5\sigma^2(\partial^2 F/\partial x^2),$$

F(x, t) – distribution of firms by productivity x.

(Hongler et al, 2016).

Modeling Schumpeterian dynamics: Boltzmann eq.

- ***f (x, t)*** **–density of agents distributions by productivity *x*. The *f (x, t)* agents devotes a fraction s(x,t) of his time to meet random persons and to imitate higher productivity . The rate of meetings is *μ(s(x,t))f (x, t)* where μ is a given function. The outflow from the position *x* is the first term of the right side of the equation**

$$\partial f/\partial t = -\mu(s(x,t))\, f(x,t) \int_x^{\infty} f(y,t)\,dy +$$

$$+ f(x,t) \int_0^x \mu(s(y,t)) f(y,t)\,dy\ ,$$

The second term is the inflow to the position x. Integrating this equation one gets Boltzmann equation for distribution function *F (x, t)*:

$$\partial F/\partial t = -(1 - F(x,t)) \int_0^x \mu(s(y,t)) f(y,t)\,dy\ .$$

(Lucas Jr., Moll, 2014).

Schumpeterian dynamics and economic growth-1

Dynamic optimal planning problem:

- $$\max_{s(x,t)} \int_0^\infty e^{-rt} \left(\int_0^\infty [1- s(x,t)]x\, f(x, t)dx \right) dt$$

$$\partial f/\partial t = - \mu(s(x,t))\, f(x, t) \int_x^\infty f(y,t)dy +$$

$$+ f(x, t) \int_0^x \mu(s(y,t)) f(y,t)dy\,,$$

f (x, 0) is given.

(Lucas Jr., Moll, 2014).

Schumpeterian dynamics and economic growth-2

There are no stability results. However , authors (Lucas Jr., Moll, 2014) prove that there exist a balanced growth path (BGP) where

1)production grows at a constant rate γ

$$Y(t) = e^{\gamma t} \int_0^{\infty} [1- s(x,0)]xf(x,0)dx ,$$

2) cumulative distribution of *lnx* and efforts as a function of *lnx* behave as wave trains with speed γ,

3) if we start with BGP distribution then BGP turns out to be an optimal trajectory.

The authors also consider independent optimal behavior of each agents and compare results.

Schumpeterian dynamics and economic growth-3

A number of authors (Acemoglu, Cao (2015), Konig et. Al. (2015), Luttmer (2012), etc.) construct general equilibrium models where productivity follows Shumpeterian dynamics mechanisms and prove that productivity or firm size distributions generated by their models converge to wave trains with Pareto tails.

For future investigations

1. **General theory of Schumpeterian growth (conservation law). Different sizes of observations and jumps** ***(see Tashlitskaya, Shananin, 2000; Hongler et al., 2016)*****.**
2. **How to choose among different models.**
3. **Modeling economic growth with Burgers type dynamics (see.,** ***Polterovich, Henkin 1989, in Russian)*****.**
4. **Multidimensional Schumpeterian dynamics: innovation and imitation of technologies (physical capital) and skills (human capital) (*****see Henkin, Polterovich, 1991)*****.**
5. **Depreciation: firm size (capital) decreases, the distribution moves back** ***(see Gelman, Levin, Polterovich, Spivak, 1993)*****.**
6. **Empirics for developing countries.**
7. **Multiwave behavior (for developing countries): slow exit due to support of the weak firms by the state, imitation of more advanced firms from abroad, more local imitation at the tail.**
8. **Schumpeterian dynamics for countries: growth modeling** ***(see Polterovich, Tonis, 2004)*****.**

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Thank you for your attention!

Apendix:

Some earlier results

Iwai model (1984)

- Iwai undertook the first attempt to show that the ``logistic'' character of diffusion curves and stability of the form of the efficiency distribution both are consequences of a ``dynamic equilibrium'' between innovation and imitation processes.
- The Iwai model is based on two main assumptions.
  1. The probability of transition to an efficiency level is the same for all less efficient firms. Therefore the rate of change of the cumulative distribution function at every point is defined by its value at that point.
- 2. The exponential speed of the emergence of new, the most effective technologies is postulated directly, and thus the speed of the efficiency distribution is established a priori. (It is not a result of interactions.)
- Both assumptions seem to be artificial.

The simplest model -1

Polterovich, Henkin (1988, 1989)

- F_n- a fraction of firms that have efficiency level n or less.

  $\mathfrak{I} = \{F_n\}$ - a distribution function.

- To describe the evolution of the distribution curve $\{F_n\}$ in time, we introduce four hypothesis.

The simplest model -1a
Four hypothesis

1. Firms can not jump over levels: if a firm has a level n then it may transit to the level n+1 only.
2. The speed of the transition is the sum of two components: an innovation component and an imitation component.
3. The speed of the transition from a level n to the next level per unit of time as a result of the imitation is proportional to the fraction of more efficient firms.
4. The speed of the transition as a result of the innovation is constant.

Innovation processes are spontaneous whereas propensity to imitation depends on the position of the firm among other firms.

The simplest model -2

$$dF_n/dt = \alpha\,(F_{n-1} - F_n) + \beta(1 - F_n)\,(F_{n-1} - F_n),$$

n –integer. Or

$$dF_n/dt = (\alpha + \beta(1 - F_n))\,(F_{n-1} - F_n)\,. \qquad (1)$$

$$F_n(0) = 0,\ n<0; \qquad 0 \le F_n(0) \le 1; \qquad (2)$$

$$\sum_{n=1}^{\infty} (1 - F_n(0)) < \infty.$$

- **$\alpha > 0$ – speed of innovation process,**
- **$\beta(1 - F_n)$- fraction of firms moving from the level n to the level n+1 per unit of time due to imitation.**

The simplest model -3

$$\varphi(F_n) = \alpha + \beta(1 - F_n) \qquad (3)$$

- speed of transition from the level n to the level n+1 = a sum of innovation and imitation components.

- Eq. (1) may be linearized by substitution
  $F_n = (1/\beta)(\mu - z_{n-1}/z_n),\ 1 \leq n < \infty,\ (4)$
  $z_0 = \exp(\mu t),\ \mu = \alpha + \beta$ (and solved in an explicit form.)

Levi, Ragnisco, Brushi (1983) described a class of equations that admit linearizing substitutons, it includes (1).

The simplest model -3a

A family of wave solutions:

$F_n^*(t, d) = F^*(n-ct, d) \quad = 1/[1+exp(\beta(n-ct+d))]$, **(3)**

where d – parameter of a shift,

$c= \beta/\ln(\mu/\alpha)$ **- speed of waves,**

$\mu = \alpha + \beta$.

Wave train

F

1

F_n^*

n

The simplest model -4 Stability

- **Theorem 1. (H-P, 1988). Let $\mathfrak{I} = \{F_n\}$ be a solution of (1), (2). Then**

a) There exists a shift d:

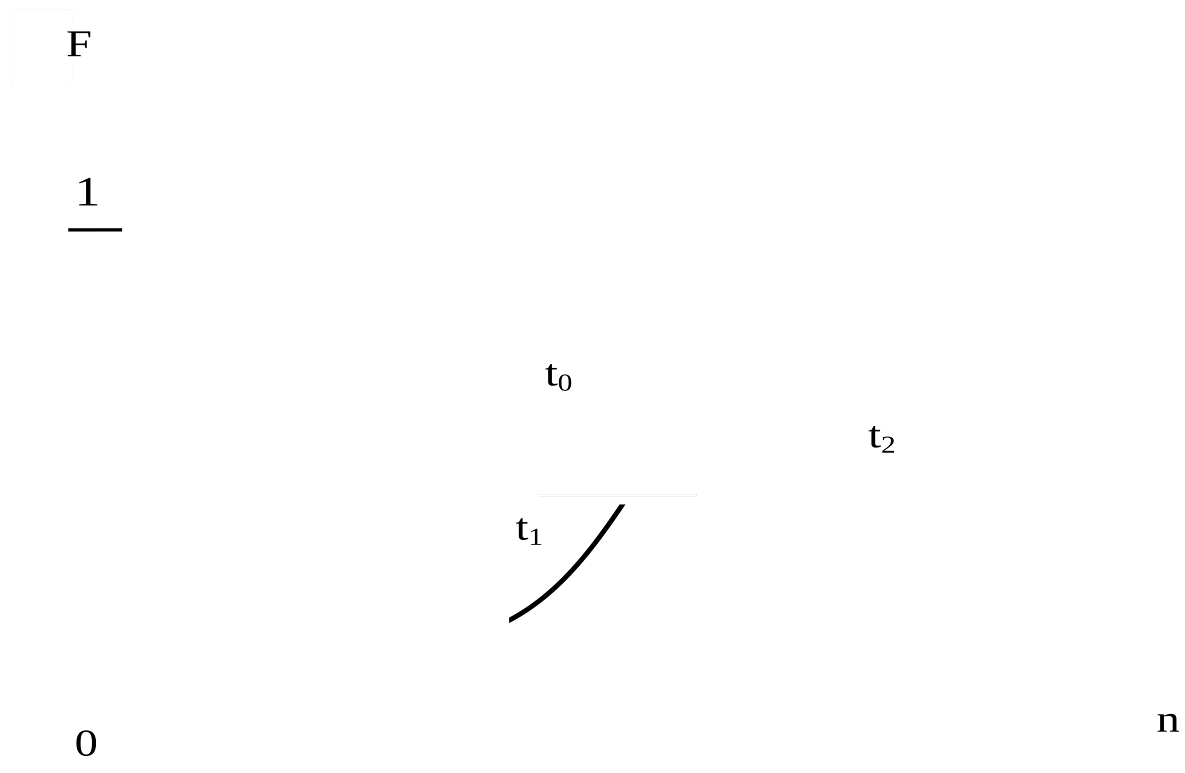
$$\sup_n |F_n(t) - F^*_n(t, d)| \to 0, \quad t \to \infty.$$

b) If $F_n(0) = 1$ for all $n \geq N$-positive integer, then

$$|F_n(t) - F^*_n(t, d)| \leq \lambda \exp(\gamma t), \; 0 \leq n < \infty, \; t \geq T_0,$$

where $\gamma = \gamma(\alpha,\beta)$; λ, T_0 depend on α,β, N and on initial conditions (the value of the first integral).

Evolution of an efficiency distribution



Two observations are explained

- **The curve of transition from a level n to n+1 is logistic.**
- **Distributions are stable.**

Logistic curve is not always observed in reality.

Generalization?

The simplest model -5
Similarity to Burgers Equation

- **The linearizing substitution (4) is similar to the well-known Florin--Cole--Hopf substitution for the Burgers equation,**

∀ **$\partial F/\partial t + \phi(F)(\partial F/\partial x) = \varepsilon\ (\partial^2 F/\partial x^2)$, $\varepsilon \geq 0$, $x \in \Re$, (with linear ϕ), and the Theorem 1 is quite similar to the corresponding Hopf theorem about Burgers equation (Hopf (1950)). Due to these facts we consider (1) as a difference-differential analogue of the Burgers equation.**

General equation-1: nonlinear speed of transition $\varphi(F_n)$

$$dF_n/dt = \varphi(F_n)\,(F_{n-1} - F_n), \quad (5)$$

n –integer, $-\infty < n < \infty$.

Initial conditions:

$$a \le F_n(0) \le b; \qquad (6)$$

$$\sum_{-\infty}^{0} (F_n(0)-a) < \infty, \quad \sum_{0}^{\infty}(b - F_n(0)) < \infty, \qquad (7)$$

a, b- constants, $a < b$, $\varphi: [a,b] \to R^1$.

A1. φ is positive, bounded on [a,b], and $1/\varphi$ is integrable.

General equation-2: nonlinear speed of transition $\varphi(F_n)$

Define:

$(b-a)\Phi(z) = \int_z^b dy/\phi(y),\ z \in [a,b],$ (8)

$\mathfrak{I} = \{F_n(t),\ -\infty < n < \infty\}$

$$B_{\mathfrak{I}}(t) = \sum_{n=1}^{\infty} \Phi(F_n(t)) - \sum_{n=-\infty}^{0} [\Phi(a) - \Phi(F_n(t))] - t, \quad (9)$$

a, b- constants, a < b, $\varphi: [a,b] \rightarrow R^1$.

General equation-3: existence, uniqueness, conservation law

Theorem 2. Under A1, there exists a unique solution

$\forall\, \mathfrak{I} = \{F_n(t),\, n \in (-\infty, \infty)\}$ of the problem (5)-(7).

- **For all $t \geq 0$:**
- **$F_n(t) \to a$, as $n \to -\infty$;**
- **$F_n(t) \to b$, as $n \to +\infty$;**
- **$B_{\mathfrak{I}}(t) \equiv B_{\mathfrak{I}}(0)$- conservation law;**
- **$F_n(t) \geq F_{n-1}(t)\ \forall n$, if $F_n(0) \geq F_{n-1}(0)\ \forall n$ – monotonicity preservation.**

Wave trains: definition

Wave trains are solutions of (5) such that

$F_n(t) = F(x), \quad x = n - ct,$

$a \le F(x) \le b,$

where c is a constant.

- **Wave train equation:**

$c(dF/dx) = \phi(F)(F(x)-F(x-1)). \quad (10)$

Wave trains: existence

- **A2. ϕ does not increase, $\phi(0) > \phi(1)$, ϕ satisfies the Lipshitz condition.**

Theorem 3. Let A1, A2. Then a wave train F*(x) exists iff

$$c = (b-a)/\Phi(a),\ \Phi(a) = \int_a^b dy/\phi(y).$$

- **Every wave train has the form F* (x- d), where d is a constant.**
- **There exist positive numbers λ_0, λ_1, λ_2, h >o such that**

$$\exp(\lambda_0 x) \geq F^*(x) - a \geq \exp(\lambda_1 x),\ \forall x \leq -h.$$

$$\exp(-\lambda_2 x) \geq b - F^*(x),\ \forall x \geq h.$$

Wave train density

Theorem 4. Let A1, A2; let ϕ be twice differentiable and $1/\phi$ be convex. Then the wave train density dF/dx has a unique local maximum point.

Stability-1

- **Theorem 5. Let A1, A2, and F* be a wave train. Then for every solution $\mathfrak{I} = \{F_n\}$ of the problem (5)- (7) one can find a constant d such that**

$$\sup_n |F_n(t) - F^*(n-ct-d)| \to 0 \text{ as } t \to \infty.$$

(Simlar to Iljin, Oleinik (1960) for Burgers equation.)

Stability-1a

The constant d is the solution of the equation

$$\mathbf{B_{\mathfrak{I}}(0) = \sum_{n=-\infty}^{0} \Phi(F^*(n\text{-}d)) - \Phi(0)) + \sum_{n=1}^{\infty} \Phi(F^*(n\text{-}d))}$$

This means equality of the first integral expressions

$$\mathbf{B_{\mathfrak{I}(0)} = B_{F^*(n\text{-}d)}(0).}$$

A Model of Economic Growth-1

$dM_n/dt = (1- \phi_0(F_n))\lambda_n M_n + \phi_0(F_{n-1}))\lambda_{n-1} M_{n-1}$ (11)

- M_n – capacities of the level n;

$\forall \lambda_n$ - profit (in real term) per unit of capacities per unit of time.

- The fraction $\phi_0(F_n)$ of the profit $\lambda_n M_n$ creates new capacities of the level n+1, and the rest is spent on the expansion of the level n.

A Model of Economic Growth-2

Let $\lambda_k > 0, \lambda_k \uparrow \lambda, \lambda > 0$,

$$\sum_{k=1}^{\infty} k(\lambda - \lambda_k) < \infty;$$

$$F_n = \left(\sum_0^n M_k\right) / \left(\sum_0^\infty M_k\right),\ n = 0,1,\ldots \qquad (12)$$

- **Equation (11), (12) is equivalent to**

$$dF_n/dt = \phi(F_n)(F_{n-1} - F_n) + r_n,\quad \phi = \lambda\phi_0, \qquad (11a)$$

r_n is a residual term, unessential for asymptotic behavior.

The case of increasing ϕ: diffusion.

Theorem 7. Let ϕ be a positive function with a positive derivative ϕ'. Then

1) every solution $F_n(t)$ can be represented as

$$F_n(t) = \phi^{(-1)}(n/t) + o(1/t^{1/2}),$$

$\phi^{(-1)}$ is the inverse function to ϕ,

$o(1/t^{1/2})\, t^{1/2} \to 0$ as $t \to \infty$.

2) If $\phi'(y) \geq \xi > 0 \quad \forall y \in [0,1]$

then $F_n(t) - F_{n-1}(t) \leq 1/(\xi t + 1)$.

Nonmonotonic ϕ: An analogue of a I.M. Gelfand problem (1959)

- **Initial conditions**
- $$F(x, o) = \begin{cases} 0, \text{ if } x < x^- \\ 1, \text{ if } x > x^+ \\ g(x), \text{ otherwise}, \end{cases} \qquad (13)$$

where $a < b$, $g(x)$ is an L^∞ function,

- **What is the asymptotic behavior of the solutions $F(x,t)$, $t \to \infty$?**

Nonmonotonic ϕ: wave trains-1

- **Wave trains:**
- **$a \leq F^*(x) \leq b$, $F^*(x) \to a$, $x \to -\infty$,**
- **$F^*(x) \to b$, $x \to +\infty$, $F^*(x)$ – nondecreases.**
- **$c(dF^*(x)/dx) = \phi(F^*(x))\,(F^*(x)-F^*(x-1))$,**
- **$c = 1/\Gamma(b)$,**

$\forall\, \Gamma(z) = 1/(z-a)\ {}_a\!\int^z dy/\phi(y)$.

Nonmonotonic ϕ: wave trains-2

- **Theorem 3' (H-P (1990), Belenky (1990)). Let ϕ be positive and integable. If $\Gamma(z) < \Gamma(b) \ \ \forall z \in [a,b]$, then there exists a wave train F*(x) and every wave train can be represented as F*(x-d) for some d. If a wave train exists then $\Gamma(z) \leq \Gamma(b) \ \ \forall z \in [a,b]$.**

Non-monotonic ϕ:

- Let $\Psi_0(z)$ be "the concave hull" of the function

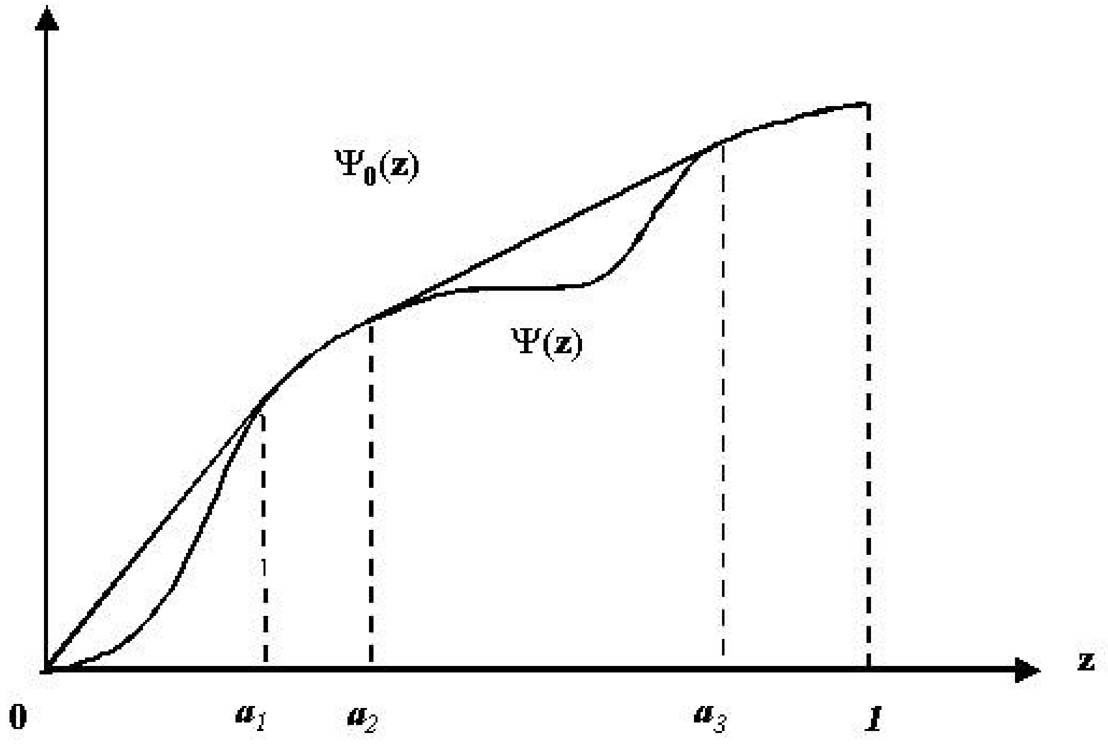
$\forall$ $\Psi(z) = {}_0\int^z dx/\phi(x) = \Phi(0) - \Phi(z)$,

- $E = \{z: \Psi(z) < \Psi_0(z), 0 \leq z \leq 1\} = \cup\sigma_l$,

$\forall$ σ_l is an (open) interval in [0,1].

- Proposition. For every $\sigma = (a, b) \subset E$ there exists a wave train with overfall b-a. If $\sigma \subset (0,1)$ then the speed of the wave train is equal to
- $$c = \phi(a) = \phi(b) = (b-a) / {}_a\int^b dx/\phi(x).$$

$(o,a_{1)}, (a_1, a_2) \subset \mathbf{E}$



Asymptotic structure of solutions-1

Let E´ = [0,1]\E , E does not contain interior

isolated points. Let σ = (a, b) $\subset$ E

Define diffusion functions

$$\Psi_\sigma(n/t) = \begin{cases} a & \text{for } n < \phi(a)t - At^{1/2}, \\ \phi^{(-1)}(n/t) & \text{for } \phi(a)t + At^{1/2} \leq n \leq \phi(b)t - At^{1/2}, \\ b & \text{for } n > \phi(b)t + At^{1/2}. \end{cases}$$

- **Let F_σ be the wave train for the interval σ.**

Asymptotic structure of solutions-2
Henkin, Polterovich (1999) - hypothesis

For a set $\{(n,t): F_n(t) \in \sigma\}$, solutions look like F^σ if

$\sigma \subset E$, and like diffusion Ψ^σ if $\sigma \subset E'$.

Let

$F^*_n(t, d_\sigma, \sigma \subset E) = \sum_{\sigma \subset E} F^\sigma(n - c_\sigma t + d_\sigma) +$

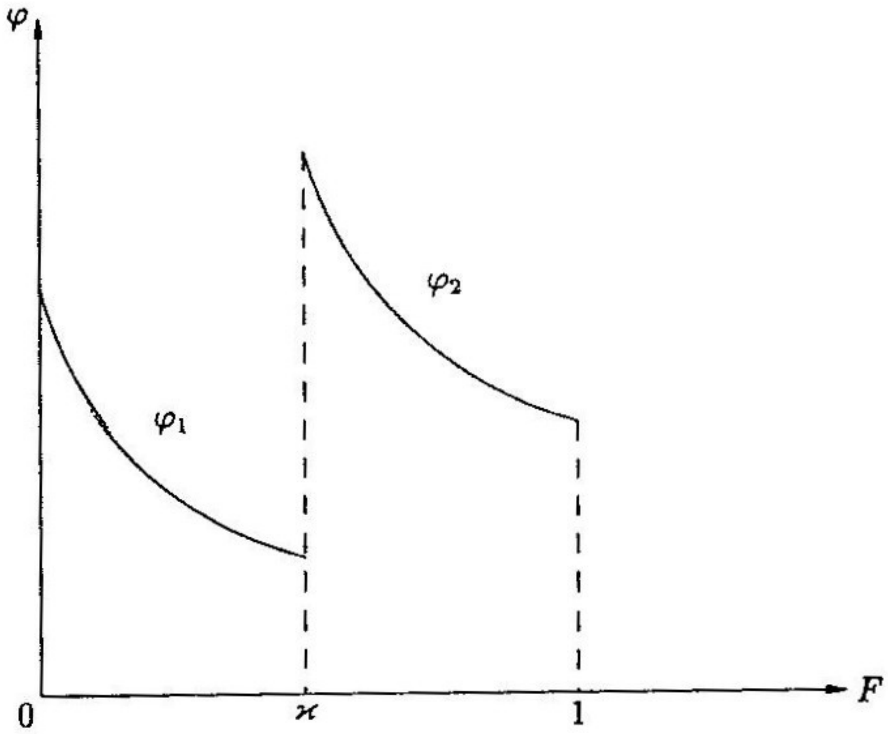
$\sum_{\sigma \subset E'} \Psi^\sigma(n/t) - \sum_{\sigma \subset [0,1]} a_\sigma,$

where a_σ- left endpoint of σ.

Hypothesis. There exists $d_\sigma(t)$: $d_\sigma(t)/t \to o$ as $t \to \infty$, and

$\sup_n | F_n(t) - F^*_n(t, d_\sigma(t), \sigma \subset E) | \to 0$ as $t \to \infty$.

At first, it was proved for the following φ:



Asymptotic structure of solutions-2

- **Henkin, Shananin (2004)**
- **Henkin, Shananin, Tumanov (2005)**
- **Henkin (2006)**
- $d_\sigma(t) = q_\sigma \ln t + o_\sigma(\ln t)$.

Asymptotic structure of solutions-3

Theorem 8 (Henkin, 2006).
Let $\phi\,(\cdot)$ be a positive twice continuously differentiable function on [0,1]; ϕ' may have only isolated zeros that are not coincide with endpoints of intervals σ. If $F(n,t)$ is a solution of a Cauchy problem (1), (13), and $t \to \infty$. Then for arbitrary A>0

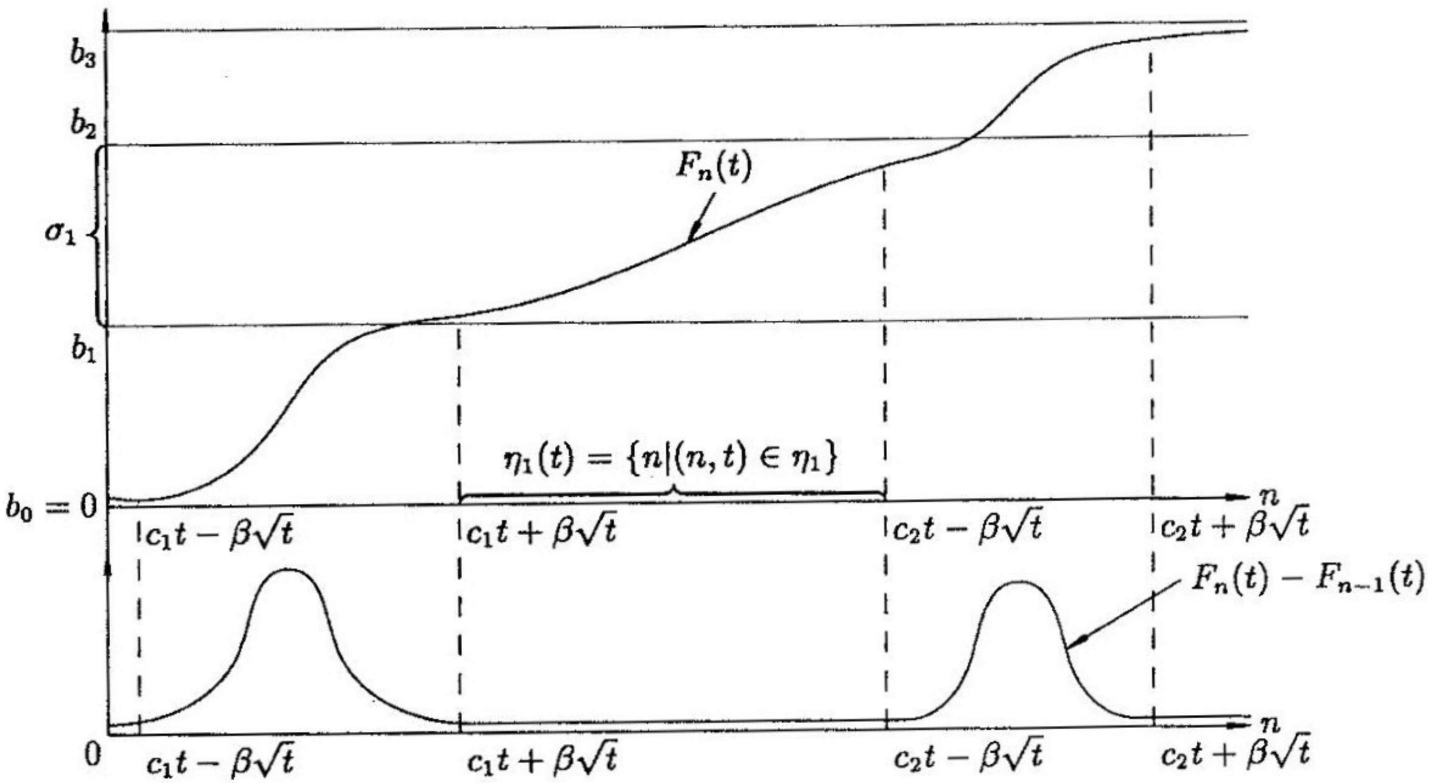
$$F(n,t) \to F^{\sigma}(n - c_{\sigma}t - d_{\sigma}(t)), \text{ if } \quad -At^{1/2} < n - c_{\sigma}t < At^{1/2},\ \sigma \subset E$$

$$F(n,t) \to \Psi^{\sigma}(n/t), \text{ otherwise,}$$

uniformly with respect to n.

Henkin proved a similar theorem for Burgers equation as well.
(Maximum and comparison principles + localized conservations laws).

Asymptotic structure of solutions-4
($d_\sigma(t) = 0$)



Comparison with Burgers Equation

Our equation with an arbitrary "step of discretization":

$$\partial F(x,t)/\partial t + \phi(F(x,t))[F(x,t)/\partial x - F(x-\varepsilon,t)/\partial x]/\varepsilon = 0 \quad (*)$$

Burgers Equation

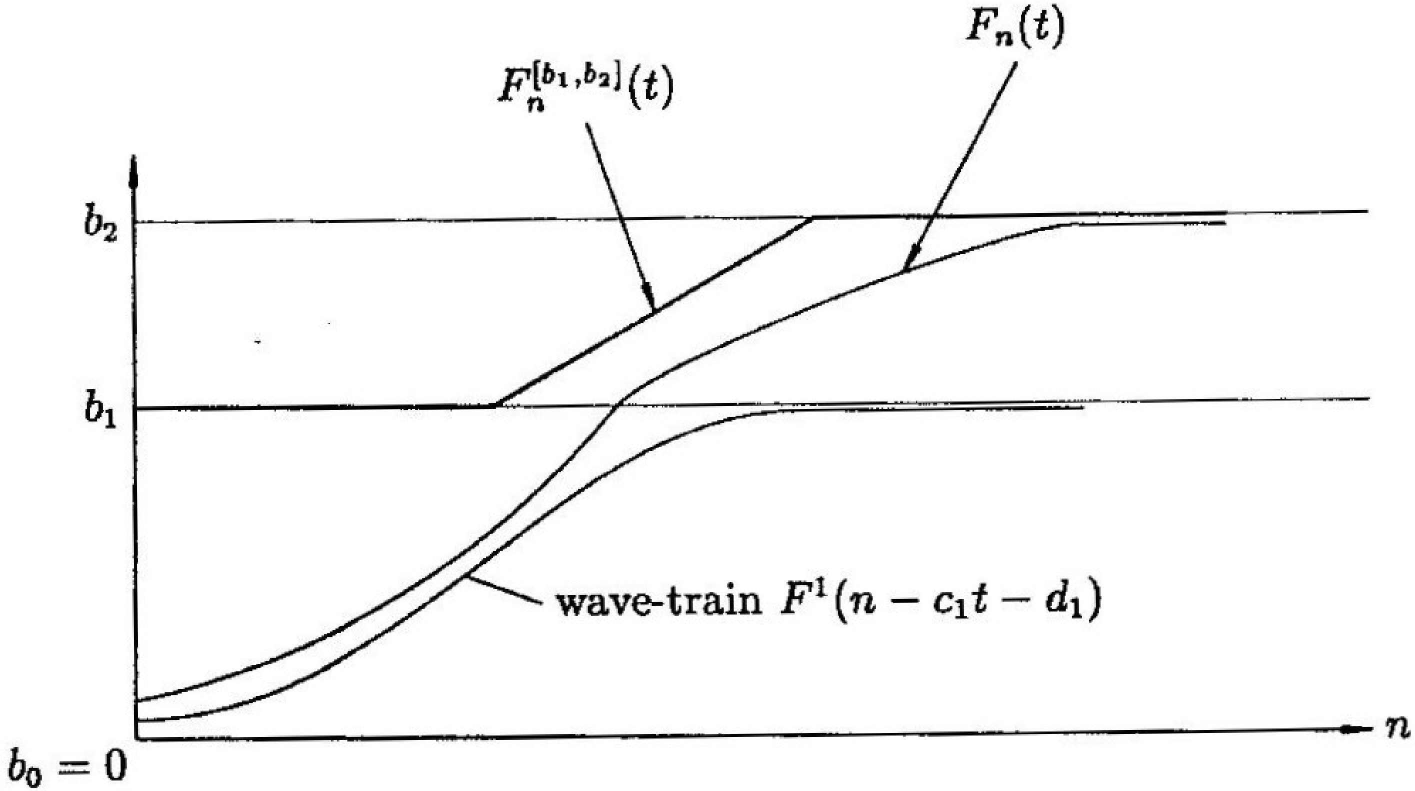
$$\partial F/\partial t + \phi(F)(\partial F/\partial x) = \varepsilon\,(\partial^2 F/\partial x^2),\ \varepsilon \geq 0,\ \ x \in \Re \qquad (**)$$

At first sight (*) looks like a discretization of (**) under $\varepsilon = +0$. But solutions of (*) do not reveal shock wave behavior as (**) do.

Using second-order Tailor expansion, one gets from (*) :

$$\partial F/\partial t + \phi(F)(\partial F/\partial x) = (\varepsilon/2)\phi(F)\,(\partial^2 F/\partial x^2)\ (***).$$

Solutions of (*) and (***) behave quite similarly; speeds of wave trains are equal (Rykova, 2004).



$F_n(t)$
$F_n^{[b_1,b_2]}(t)$
b_2
b_1
wave-train $F^1(n - c_1t - d_1)$
n
$b_0 = 0$

Two-dimensional case

- m, n are levels of two efficiency parameters,
- $m, n = 0,1,\dots$
- f_{mn} – the proportion of firms at a level (m,n).
- $F_{mn} = {}_{k=1}\Sigma^{m}\ {}_{r=1}\Sigma^{n} f_{kr}$ – distribution function;
- $F_m^{(1)} = {}_{k=1}\Sigma^{m}\ {}_{r=1}\Sigma^{\infty}\ f_{kr}$;
- $F_n^{(2)} = {}_{k=1}\Sigma^{\infty}\ {}_{r=1}\Sigma^{n} f_{kr}$.

Two-dimensional case: assumptions

- **A firm can transit from the state (m,n) into one of two neighboring higher levels: (m+1,n) and (m,n+1).**
- **The proportion of firms per unit of time moving from the state (m,n) to the state (m+1,n) is proportional to the fraction of firms being in the state (m,n), and the proportion coefficient is positive and non-decreasing in the fraction of firms which are more advanced according to the first indicator. A similar hypothesis is admitted for the transition from (m,n) to (m,n+1).**

Two-dimensional case-2

$$dF_{mn}/dt = \phi_1(F^{(1)}_m)(F_{(m-1)n} - F_{mn}) +$$
$$+ \phi_2(F^{(2)}_n)(F_{m(n-1)} - F_{mn}),$$

where $F^{(1)}_m = \sup_n F_{mn}$, $F^{(2)}_n = \sup_m F_{mn}$ –marginal distributions.

- Boundary and initial conditions:

$F_{on}(t) \equiv 0$, $F_{m0}(t) \equiv 0$,

$F_{mn}(0) = \Sigma_{j \leq m,\, k \leq n}\ f_{jk}(0)$, $f_{jk}(0) \geq 0$,

$F_{mn}(0) = 1$, $m \geq m_0$, $n \geq n_0$,

where m_0, n_0 – given integer numbers.

Two-dimensional case-3

- A **wave train is a product of two wave trains for ϕ_1 and ϕ_2. Any solution converges to a wave train appropriately shifted.**

Unsolved Problems

Jumps over one level are possible:

$$dF_n/dt = (\varphi_1(F_n) + \varphi_2(F_n)) (F_{n-1} - F_n) + \varphi_2(F_{n-1}) (F_{n-2} - F_{n-1}),$$

where $\varphi_1(F_n)$, $\varphi_2(F_n)$ are speeds of transition from level n to the level n+1 and the level n+2 correspondingly.

Kolmogorov – Petrovsky – Piskunov

- **Firms jump from a level on any other level with larger efficiency, and the probabilities of all transitions due to imitation are proportional to the fractions of more advanced firms.**
- $dF_n/dt = -\alpha\,(F_n - F_{n-1}) - \beta F_n(1-F_n)$.
- **This is a semidiscrete variant of Kolmogorov – Petrovsky – Piskunov's Equation:**

$$\forall \partial F/\partial t - \varepsilon\,(\partial^2 F/\partial x^2) = V(F)$$

Local Imitation
Tashlitskaya, Shananin (2001)

Firms are able to imitate only technologies of the firms from the next higher efficiency level. Then the imitation component becomes $\beta(F_{n+1} - F_n)(F_n - F_{n-1})$, and we have:

$$dF_n/dt = -(\alpha + \beta(F_{n+1} - F_n))(F_n - F_{n-1}).$$

Finite initial conditions: $F_n(0) = 1,\ n \geq N$.

The case $\alpha = 0$: Langmuir's Chain-1

A change of variables

$T = \beta t$, $c_n(t) = F_{N+1-n} - F_{N-n}$

leads to the following system

$$dc_1/dt = c_1 c_2,$$

$$dc_n/dt = c_n(c_{n+1} - c_{n-1}), \ n = 2,\ldots,N-1,$$

$$dc_N/dt = -c_N c_{N-1},$$

$$c_n(0) = \gamma_n > 0, \ n = 1,\ldots,N$$

known as finite Langmuir's chain.

The case α =0: Langmuir's Chain-2

The stable stationary solutions of the chain have the following structure

$(y_1, 0, y_2, 0, \ldots, y_k, 0)$ if $N = 2k$,

$(y_1, 0, y_2, 0, \ldots, y_k, 0, y_{k+1})$ if $N = 2k+1$

THEOREM. (Tashlitskaya, Shananin).

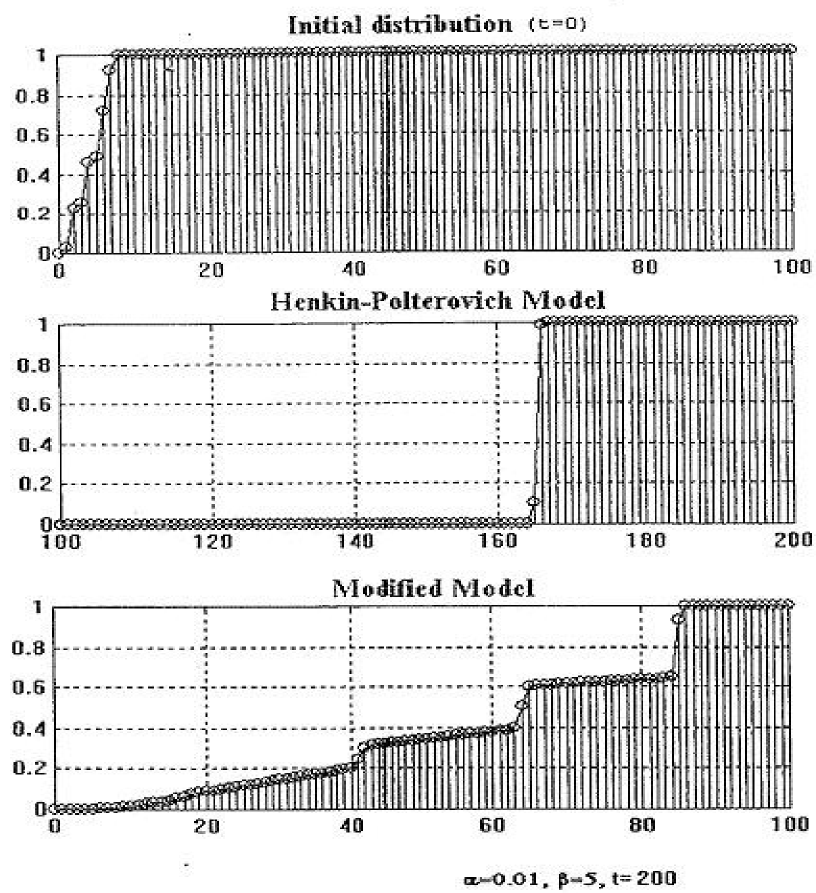
Solutions to the Cauchy problem for the

Langmuir finite chain converges, as $t \to \infty$, to

a stationary solution, which is determined uniquely by

initial data.

The case α =0: Langmuir's Chain-2a



Initial

distribution

H-P model

Modified Model

The case of small $\alpha > 0$: A perturbation of the Langmuir's Chain-3

Computations show three stages of evolution:

1. **The stage of formation of technology structures (the regime of Langmuir's – Volterr's chain, $F\beta >> \alpha$;**
2. **The stage of imitation – innovation interaction ($F\beta \sim \alpha$);**
3. **The stage of diffusion ($F\ \beta << \alpha$).**

Imitation from several more advanced levels

$dF_n /dt = \alpha (F_{n-1} - F_n) + \beta (F_k - F_n)) (F_{n-1} - F_n)$,

$k > n$

Computations (Savenkov, 2003):

If k=2, then every solution converges to a wave train that depends on initial conditions

Belenky' model-1

Speed of transition ψ from efficiency level n to level n+1 depends on a proportion of more advanced firms among all firms that are not worse than the firms of level n. This assumption entails the following equation

$$d\theta_n/dt = \psi(\theta_n/\theta_{n-1})(\theta_{n-1} - \theta_n),$$

where $\theta_n = 1 - F_n$.

Belenky' model-2

- **This equation**

$$d\theta_n/dt = \psi(\theta_n/\theta_{n-1})(\theta_{n-1} - \theta_n),$$

where $\theta_n=1-F_{n,}$,

may be reduced to our main equation

$$dF_n/dt = \varphi(F_n)\,(F_{n-1}- F_n)$$

by a substitution.

The theory is applicable.

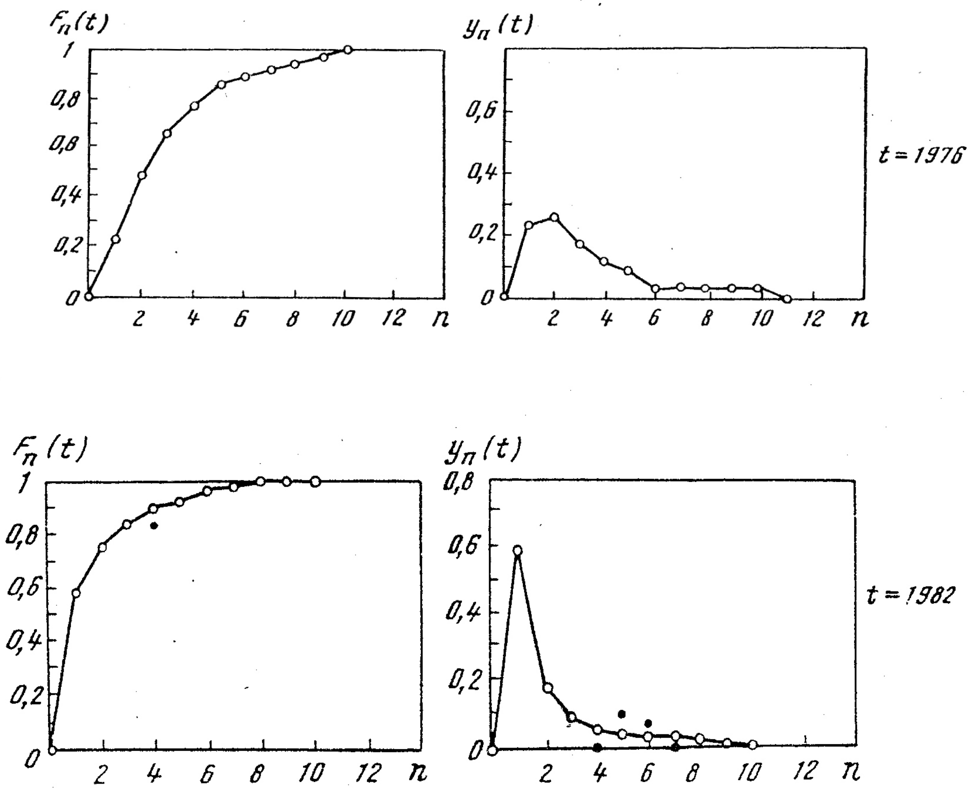
Unsolved Problems-2

Depreciation of capacities:

$$dF_n/dt = \varphi(F_n)\,(F_{n-1} - F_n) + \mu(F_{n+1} - F_n),$$

μ is a depreciation rate.

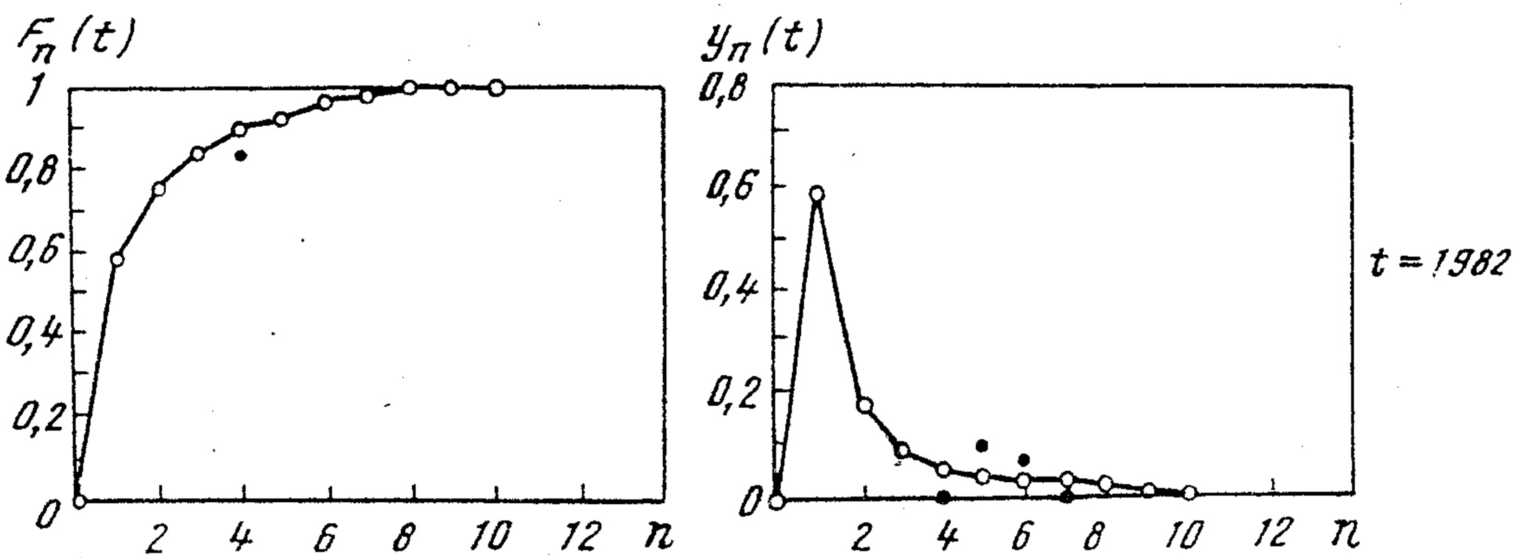
Ferrous Metallurgy in USSR
Levin, Spivak, Polterovich (1993)



t =1976

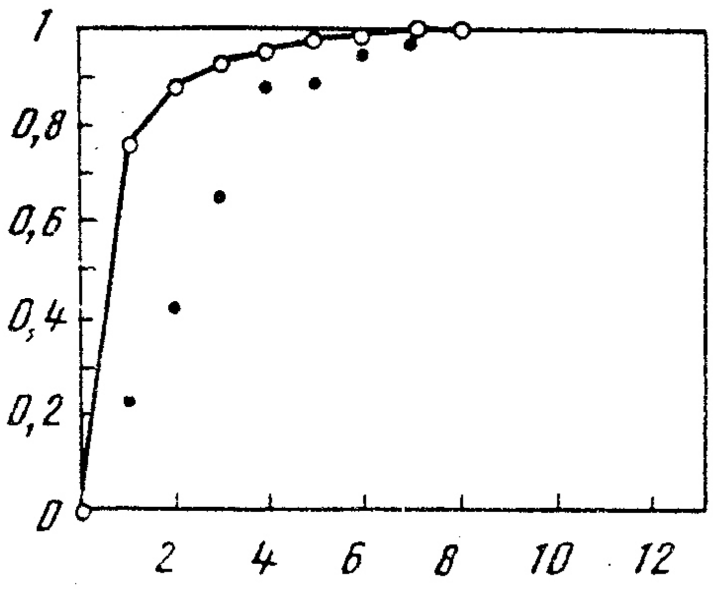
t =1982

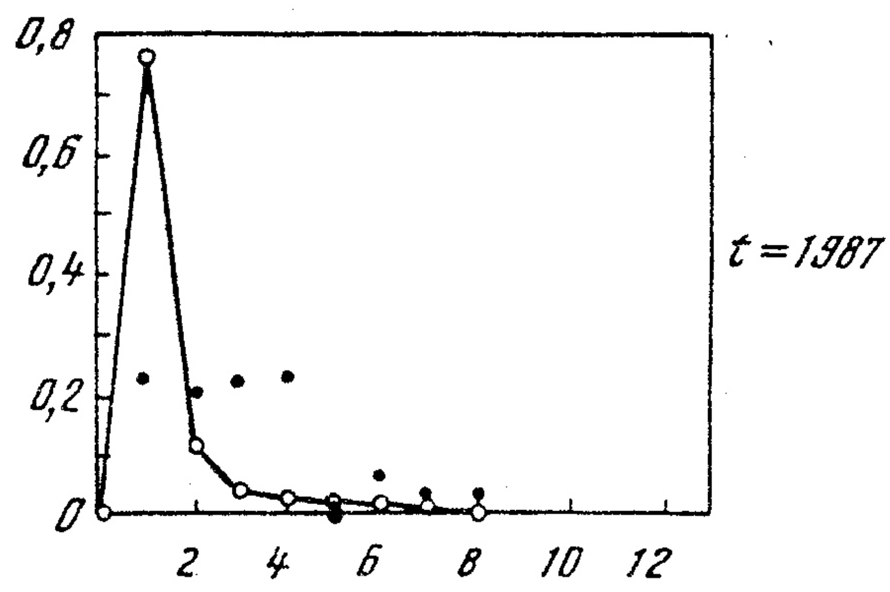
Ferrous Metallurgy in USSR



Ferrous Metallurgy in USSR

A reform occurred in 1982

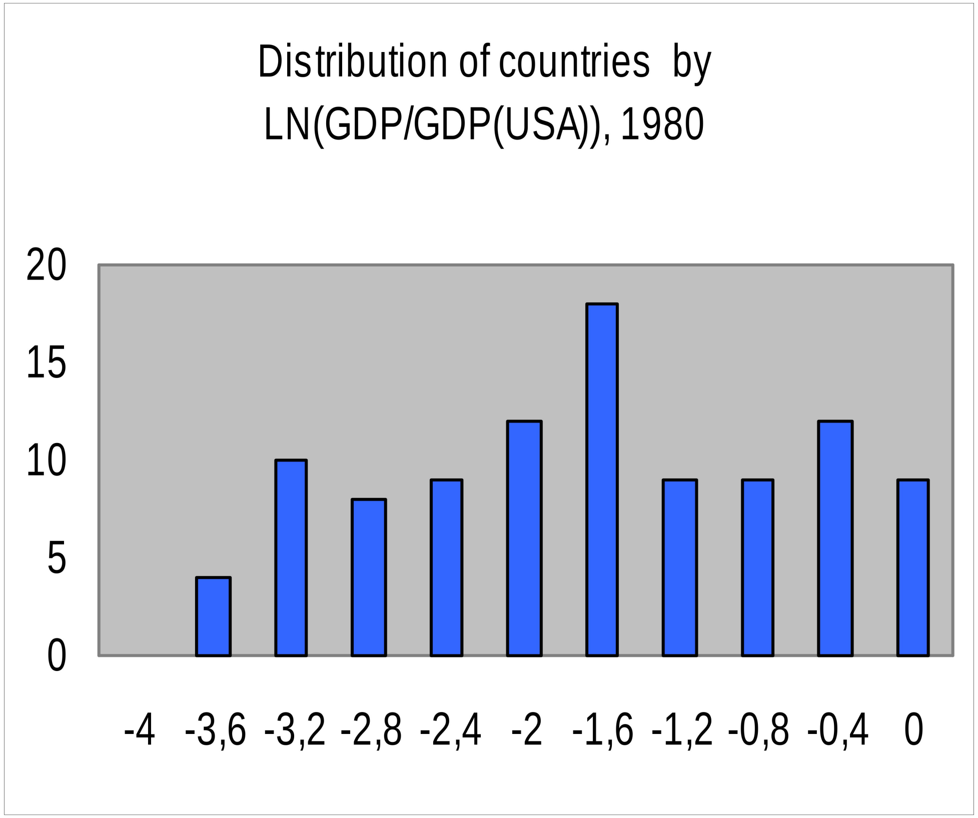




Unsolved Economic Problem: Evolution of distribution of countries by GDP (gross domestic product) per capita

- **Per capita GDP for Latin America and Caribbean countries decreased by an average 0.8 percent per year in the 1980s, and grew by mere 1.5 percent per year in the 1990s. In the Middle East and North Africa we observed the average fall of 1.0 percent per year in the 1980s and the average growth of 1.0 percent per year in the1990s. For 28 countries of East Europe and former USSR, the total loss of GDP amounted to 30% in the 1990s. In Sub-Sahara Africa there was a reduction if the GDP per capita.**

Distribution of countries by ln(GDP per capita/GDPper capita of USA), 1980

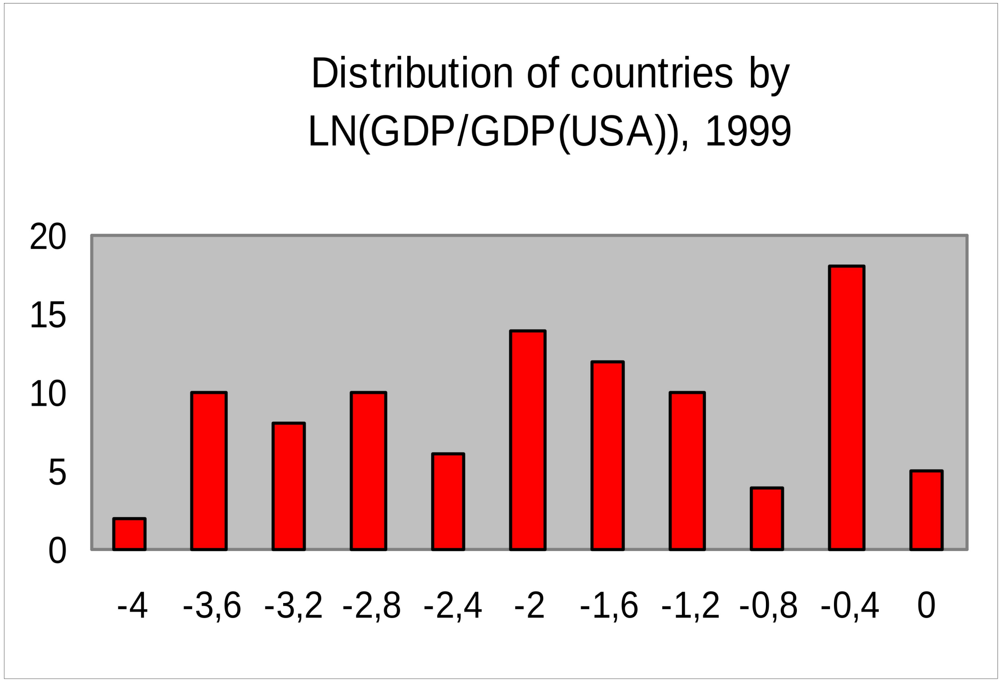


Tree peaks: ”Europe”,

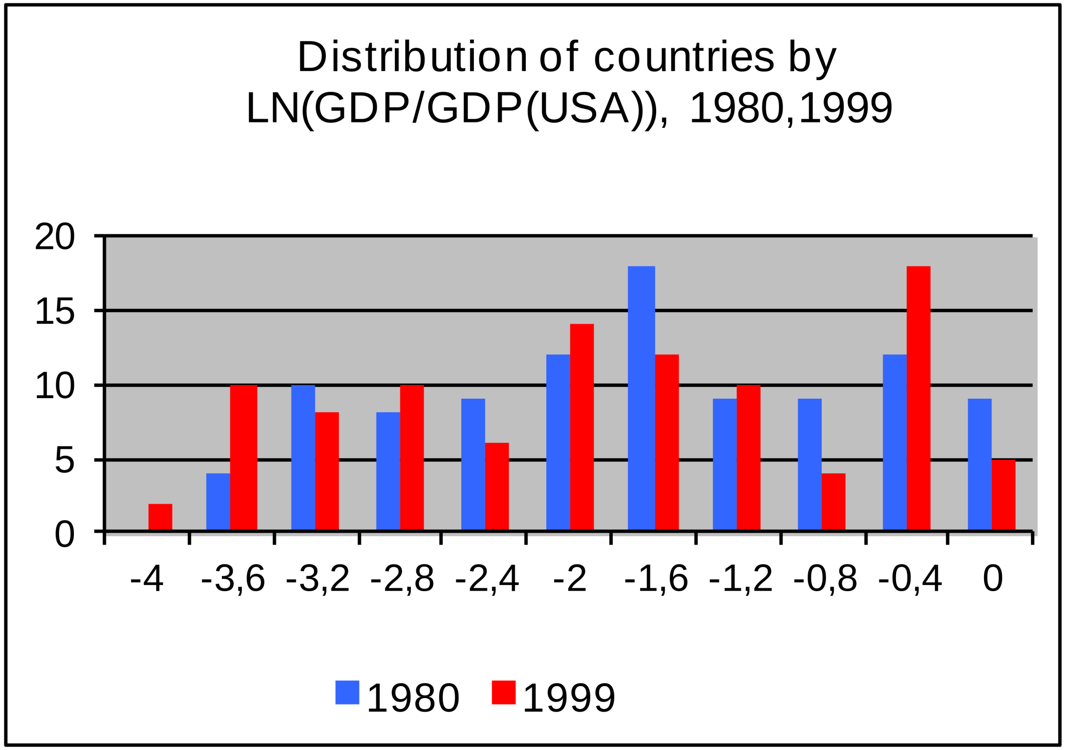
“Latin America” and

“Africa”

Distribution of countries by Ln(GDP per capita/GDPper capita of USA), 1999



Distribution of countries by Ln(GDP per capita/GDPper capita of USA), 1980 and1999



Distribution of countries by GDP per capita/GDPper capita of USA

- **Advanced industrial countries are growing at the same rate (Mankew, Romer, Weil (1992), Evans (1996)). Others?**
- **Aghion, Howitt (1998): imitation of the most advanced technology (not realistic).**
- **Guilmi, Gaffeo, Gallegati (2003): Countries with per capita income between 30% and 85% of the world average: Pareto distribution (data of 1960-2001).**
- **The problem remains open.**

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Imitation's Impact on the Dynamics of Long Wave Growth

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KLZ -1

- **Michael D. K¨onig, Jan Lorenz, Fabrizio Zilibotti. Innovation vs. imitation and the evolution of productivity distributions. October 2015. -KLZ**
- **Jan Lorenz, Fabrizio Zilibotti, Michael König. Distance to frontier, productivity distribution and travelling waves 19 November 2015.**

http://voxeu.org/article/distance-frontier-productivity-distribution-and-travelling-waves

KLZ -2

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- **Jan Lorenz, Fabrizio Zilibotti, Michael König. Distance to frontier, productivity distribution and travelling waves 19 November 2015.**

http://voxeu.org/article/distance-frontier-productivity-distribution-and-travelling-waves

- *Dynamics of the cumulative log-productivity distribution:*

$$\partial G_a(t)/\partial t = \begin{cases} G_a(t) - G_a(t)^2, \text{ if } a \le a^*(P), \\ (1 - G_{a^*}(t))G_a(t) - p(G_a(t) - G_{a-1}(t)), \text{ if } a > a^*(P). \end{cases}$$

Похож на дискр вар КПП

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