Fokas Methods applied to a Boundary Valued Problem for Conjugate Conductivity Equations

Conference in memory of Gennadi Henkin

Joint work with Slah Chaabi and Franck Wielonsky (Université Aix-Marseille)

Moscow, the 15th of september, 2016

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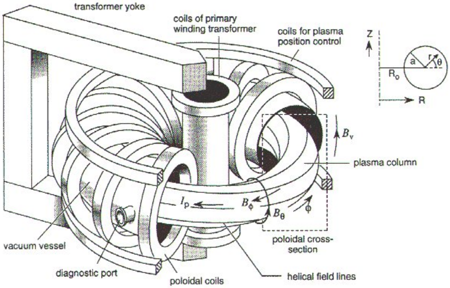
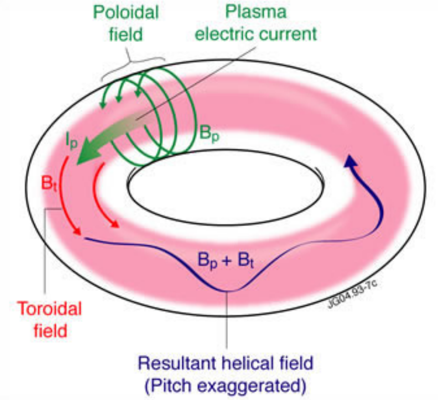
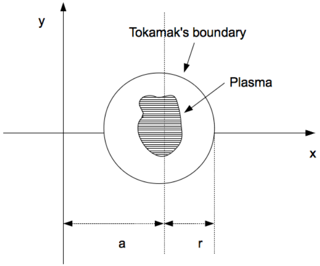
- Is it possible to obtain a relation between u and $\partial_{\vec{n}} u$ on $\partial\Omega$ without computing u in Ω ?

One other motivation : What is a Tokamak ?





Mathematical Formulation



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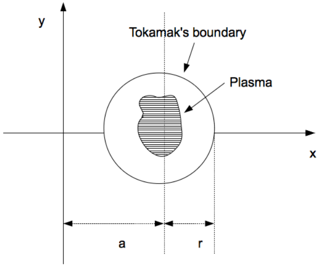

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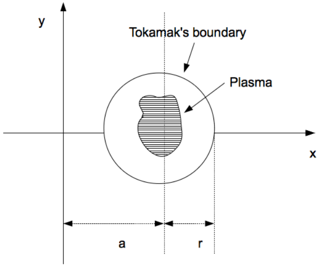

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- Generalized axisymetrical potential for $\alpha \in \mathbb{R}$:

$$\operatorname{div}(x^{\alpha}\nabla u) = 0 \quad \Leftrightarrow \quad \Delta u + \frac{\alpha}{x}\frac{\partial u}{\partial x} = 0.$$

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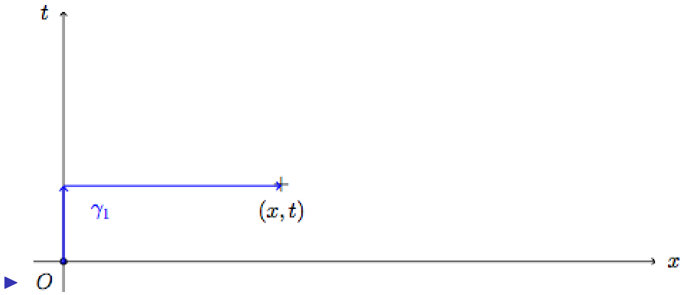
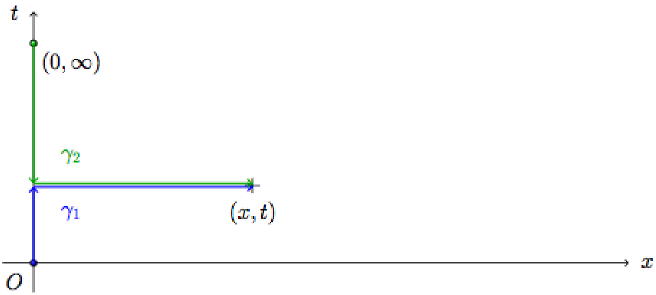
- Integration of ν between points of the boundary and (x,t).



$$\mu_1(x,t,k) = e^{ikx-k^2t}\int_{\gamma_1}\nu$$



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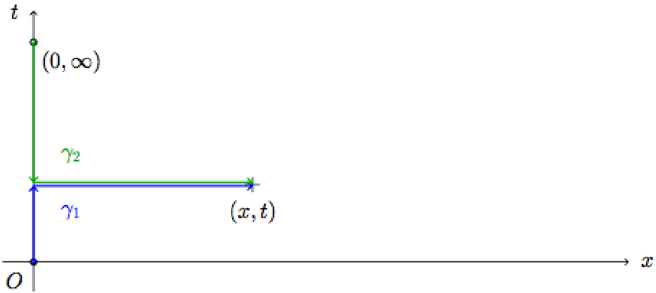
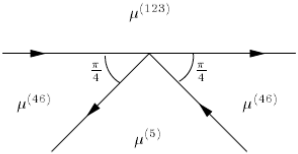
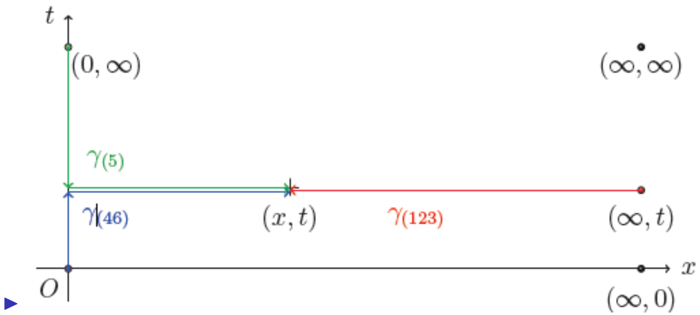


$$\mu_{46}(x,t,k) = e^{ikx-k^2t}\int_{\gamma_1}\nu$$

$$\mu_{5}(x,t,k) = e^{ikx-k^2t}\int_{\gamma_2}\nu$$





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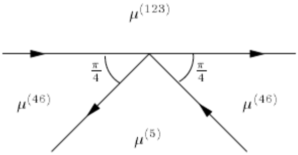
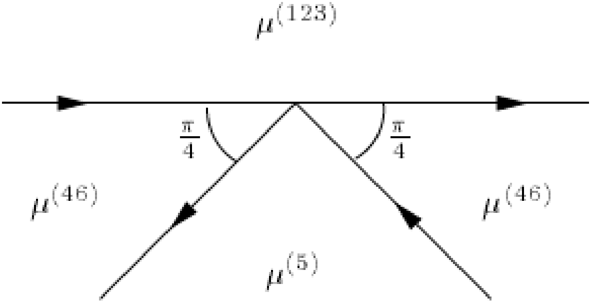


Riemann-Hilbert Problems



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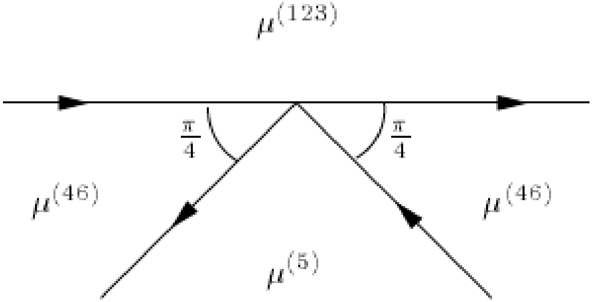

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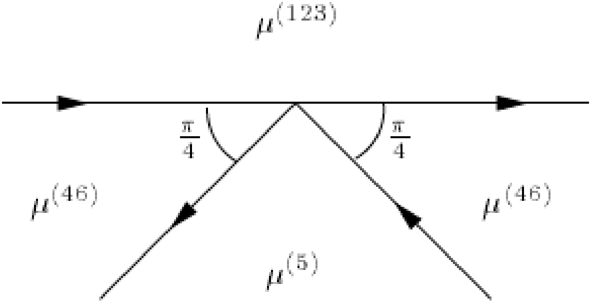

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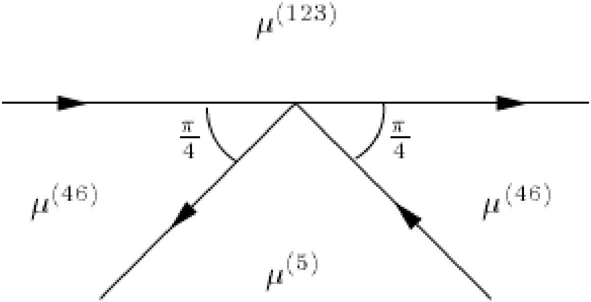

- $\mu(x, t, k) = \mathcal{O}\left(\frac{1}{k}\right)$
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- $\mu = \frac{1}{2\pi i}\int_L \frac{\phi(k')dk'}{k'-k}$ (Plemelj Formula)

Lax Pairs and closed differential form for the GASP equation.

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- $L_\alpha(u) = 0 \Leftrightarrow L_{2-\alpha}(x^{\alpha-1}u) = 0$.

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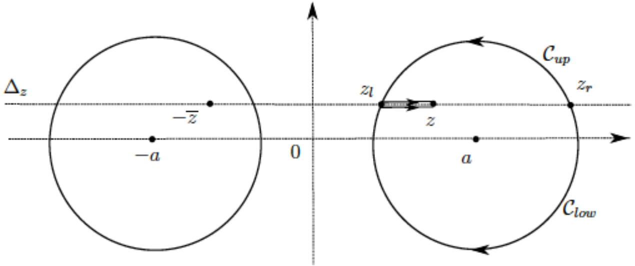

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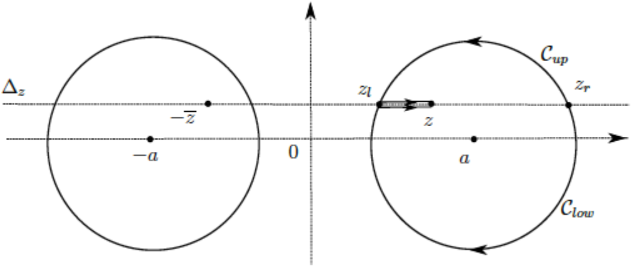

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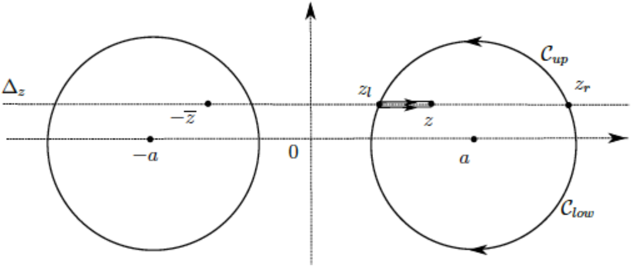

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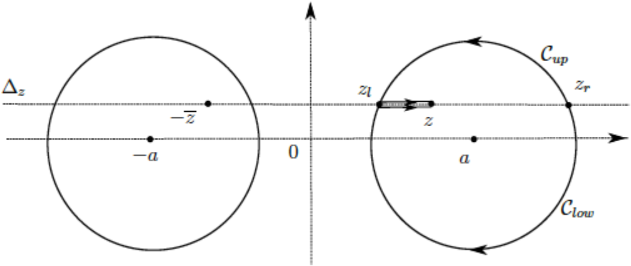

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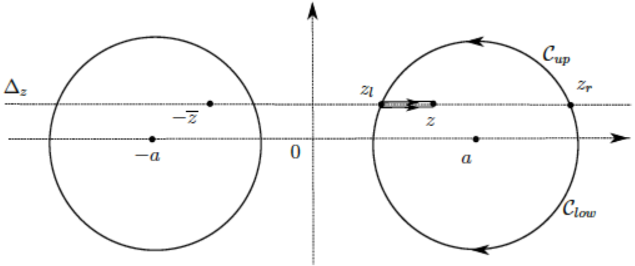

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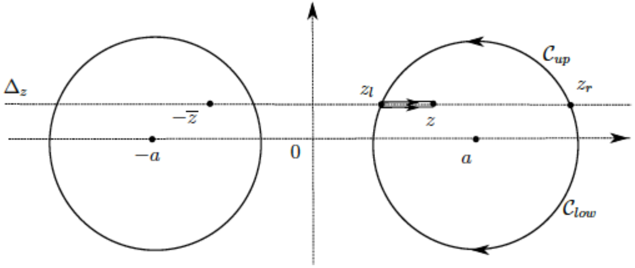
- 

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- $\phi \sim_{k\to+\infty} \frac{u(z)-u(z_r)}{k^{2m-1}}$

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- m integrations by parts give $\widetilde{\phi}_{z_r, -\overline{z}_r}$, then a_r.

Let u be a solution to the equation $\Delta u + \alpha x^{-1}\partial_x u = 0$, $\alpha = -2m$, $m \in \mathbb{N}$, in the domain $\mathcal{D}$ with smooth tangential derivatives u_t and normal derivatives u_n on the boundary $\mathcal{C}$.

$$u(z) = -\frac{1}{\pi}\mathrm{Im}\int_{(z,z_r)} \big((k-z)(k+\bar{z})\big)^m J(z,k)dk + 2\mathrm{Re}\, a_r + u(z_r), \tag{1}$$

where a_r can be explicitly computed in terms of the tangential derivative along $\mathcal{C}$ of u_t and u_n, up to the order $m-1$, in z_r. Function $J(z,k)$ is given by

$$J(z,k) = -\int_{\mathcal{C}} W(z',k),$$

where $W(z,k)$ is the differential form

$$W(z,k) = \big((k-z)(k+\bar{z})\big)^{-m-1}\left((k+\bar{z})u_z(z)dz + (k-z)u_{\bar{z}}(z)d\bar{z}\right) \tag{2}$$

$$= \big((k-z)(k+\bar{z})\big)^{-m-1}\left((k-iy)u_t(z) + ixu_n(z)\right)ds, \tag{3}$$

with $z = x + iy$ and ds the unit length element on $\mathcal{C}$.

The case $\alpha = -1$.

- $W(z,k) = \dfrac{(k+\bar{z})u_z dz + (k-z)u_{\bar{z}} d\bar{z}}{\sqrt{(k-z)(k+\bar{z})}}$

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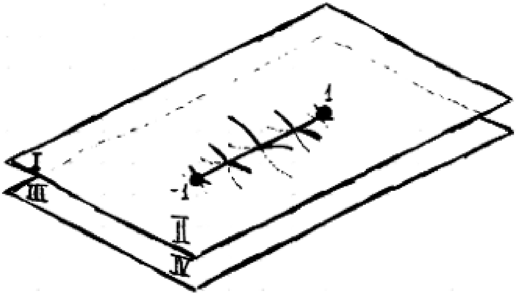

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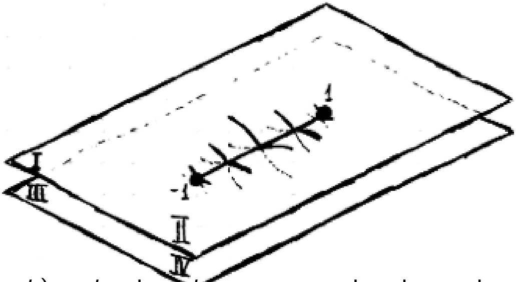

- $\lambda_1(z,k) \sim k$ when $k \to \infty_1$ on the sheet above $\mathbb{S}_{z,1}$
- $\lambda_2(z,k) \sim -k$ when $k \to \infty_2$ on the sheet below $\mathbb{S}_{z,2}$

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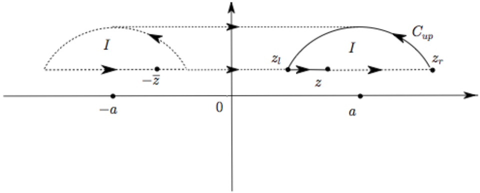

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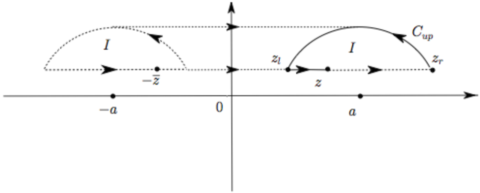
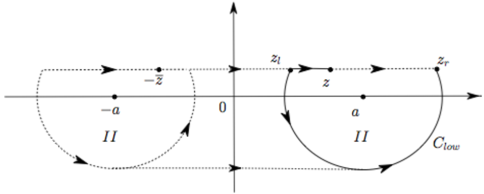


- Sheet 2



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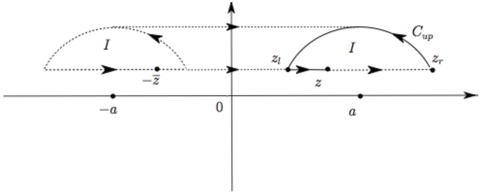

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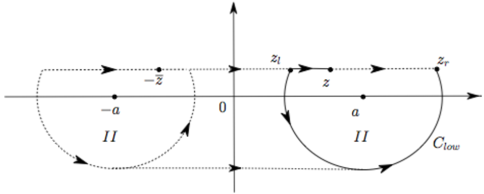
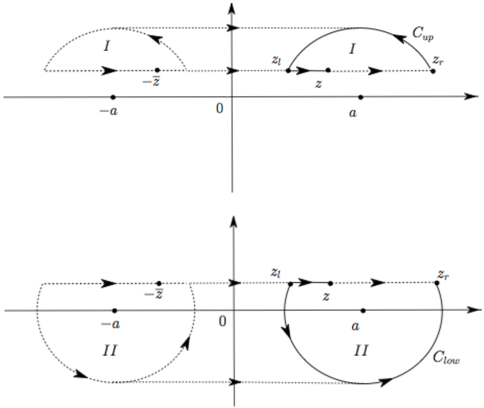
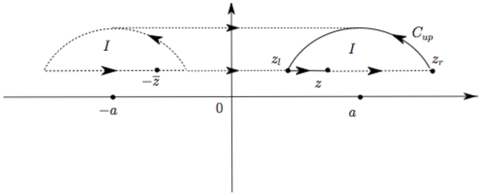
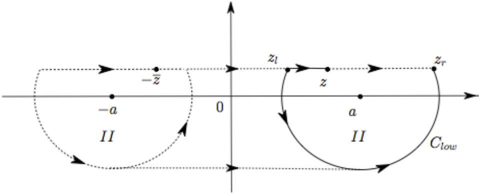


- $\phi(z,\infty_1) = -\phi(z,\infty_2)$.



C_{up}
I
I
z_l
z_r
$-\bar{z}$
z
$-a$
0
a
z_l
z_r
$-\bar{z}$
z
$-a$
0
a
II
II
C_{low}

- 

- 

$$\phi(z,k) = \frac{1}{4i\pi}\int_{\mathcal{C}_{up}\cup-\bar{\mathcal{C}}_{up}} J(z,k')\left(\frac{\lambda(z,k)}{\lambda_1(z,k')}+1\right)\frac{dk'}{k'-k}$$

$$+\frac{1}{4i\pi}\int_{\mathcal{C}_{low}\cup-\bar{\mathcal{C}}_{low}} J(z,k')\left(\frac{\lambda(z,k)}{\lambda_2(z,k')}+1\right)\frac{dk'}{k'-k}, \quad (4)$$

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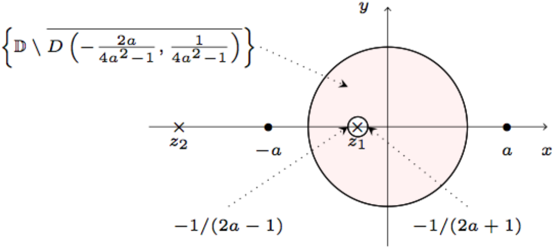

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- $\sum_{\ell=0}^{m-1} a_{n-\ell} \beta_\ell^n = 0, \; n \geq 2m-1$.
- For $N \in \{0, \ldots, m-2\}$ and $n = 2m-1+N$,

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$$\forall \xi \in \mathbb{D} \setminus \overline{D\left(-\frac{2a}{4a^2-1}, \frac{1}{4a^2-1}\right)}.$$

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- Thank you very much for your attention !