Inverse scattering problem

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MIPT · 14 September 2016

Introduction

$$\psi_{xx} = \left(q(x) + k^2\right)\psi,$$
$$q = p', \quad p \in BV, \quad \text{supp } q \Subset \mathbb{R}$$
$$\psi(x,k) = e^{kx},\ x < x_1, \quad \psi(x,k) = a(k)e^{kx} + b(k)e^{-kx},\ x > x_N$$

Wronskian relation

$$\blacktriangleleft\ \langle \psi(x,k), \psi(x,-k) \rangle = \langle e^{kx}, e^{-kx} \rangle = -2k\ \blacktriangleright$$

Dirichlet problem in $K_+ = \{k \in \mathbb{C},\ \operatorname{Re} k \geq 0\}$:

$$a(k)a(-k) = 1 + b(k)b(-k),$$

$$a(-k) = \bar{a}(k), \quad b(-k) = \bar{b}(k),$$
$$|a(k)|^2 = 1 + |b(k)|^2 > 0, \quad k = i\xi,\ \xi \in \mathbb{R}$$

Main Equations

Matrix Jost solution

$$\Psi = \begin{pmatrix} \psi^+ & \psi^- \\ \psi_x^+ & \psi_x^- \end{pmatrix},$$

$$\Psi(x,-k)\sigma = \frac{1}{a(k)}\Psi(x,k)\begin{pmatrix} 1 & b(-k) \\ -b(k) & 1 \end{pmatrix}.$$

Darboux transformation

$$\frac{d}{dx}\Phi = \begin{pmatrix} 0 & 1 \\ \hat{q}(x)+k^2 & 0 \end{pmatrix}\Phi - \Phi\begin{pmatrix} 0 & 1 \\ q(x)+k^2 & 0 \end{pmatrix}, \quad \hat{\Psi} = \Phi(x,k)\Psi.$$

Lemma

$$\Psi(x,k) \times \left[\begin{pmatrix} 1 & 1 \\ k & -k \end{pmatrix}\begin{pmatrix} e^{kx} & 0 \\ 0 & e^{-kx} \end{pmatrix}\right]^{-1} \to \begin{pmatrix} 1 & 0 \\ p(x) & 1 \end{pmatrix},$$

$$|k| \to \infty, \ \ \operatorname{Re} k \geq 0.$$

Uniqueness proof...

Polygon

$$\Psi(x,k)|_{k=0}.$$

Bargmann (1949)

$$\Phi = \begin{pmatrix} f & -1 \\ -f^2+\lambda-\lambda_0 & f \end{pmatrix}, \quad f_x + f^2 = q(x) - \lambda_0.$$

δ-type potentials

$$q(x) = \sum_{j=1}^{N} \gamma_j \delta(x - x_j), \quad ||q|| = \sum |\gamma_j| \tag{1}$$

$$\begin{aligned} a(k) &= 1 + \frac{\gamma_1}{2k} + \frac{\gamma_2}{2k} + \frac{\gamma_1\gamma_2}{4k^2}[1 - e^{2kx_1}], \\ -b(k) &= e^{2kx_1}\frac{\gamma_1}{2k} + \frac{\gamma_2}{2k} + \frac{\gamma_1\gamma_2}{4k^2}[1 - e^{2kx_1}]. \end{aligned} \tag{2}$$

For real valued, compactly supported potentials Darboux transformations allow to kill all zeros of $2ka(k), \operatorname{Re} k > 0$ and the finite number of zeros of $2kb(k)$ outside the axis $k = i\xi,\ \xi \in \mathbb{R}$.

Paley-Wiener theorem

The potential q is non-integrable. Fourier-Stieltjes transformation of the δ-type potential (1) yields

$$\hat{q}(k) = \int e^{kx} dp(x) = \sum \gamma_j e^{kx_j},$$

$$\varphi(3k) = \mathrm{ch}(k)\varphi(k), \quad \varphi(k) = \int e^{kx} dp(x),$$

where p is the Cantor function.

Equation:

$$\varphi(\gamma k) = g(k)\varphi(k).$$

Solution:

$$\varphi(k) = \prod_{n=1}^{\infty} g(\frac{k}{\gamma^n}).$$

$$2k[a(k) - 1] = \int_{x_1}^{x_N} a^\dagger(\tau) e^{k\tau} d\tau \tag{3}$$

GLM equation

$$\psi(x,k) = e^{kx} + \int_{-\infty}^{x} K(x,y)e^{ky}dy, \quad \operatorname{Re} k \geq 0,$$

$$K(x,y) + F(x+y) + \int_{-\infty}^{x} K(x,s)F(s+y)ds = 0, \quad y < x,$$

$$\hat{F}(\xi) = \frac{b(k)}{a(k)} = (1+\beta_1 k)\frac{\exp 2kx_1 - 1}{\beta_1^2 k^2 + \exp 2kx_1 - 1}, \quad k = i\xi,$$

$$\beta_1 = \frac{2}{\gamma_1}, \quad \gamma_1 + \gamma_2 = 0.$$

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