Inverse problems in models of distribution of resources

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Contents

The Houthakker–Johansen model

- $x = (x_1, \dots, x_n)$ – technology;
- $\mu(dx)$ – non-negative measure describing the distribution of powers over technologies;
- $l = (l_1, \dots, l_n)$ – vector of available production factors;
- $u(x)$ – technology loading coefficient;
- $F(l)$ – production function, relating amounts of product to amounts of resources used in production process

The problem of distribution of resources in the Houthakker–Johansen model

$$\begin{cases} \displaystyle\int\limits_{\mathbb{R}^n_+} u(x)\,\mu(dx) \to \max, \\ \displaystyle\int\limits_{\mathbb{R}^n_+} xu(x)\,\mu(dx) \le l, \\ 0 \le u(x) \le 1. \end{cases} \tag{1}$$

The generalized Neumann–Pearson lemma

1. If $l \geq 0$ then the problem (1) has a solution.
2. If $u_0(x)$ is a solution to problem (1) then there exist Lagrange multipliers $p_0 \geq 0$, $p = (p_1, \ldots, p_n) \geq 0$, not simultaneously equal to zero, such that

$$u_0(x) = \begin{cases} 0 \text{ for almost all } x \text{ w.r.t. } \mu \text{ such that } p_0 < px; \\ 1 \text{ for almost all } x \text{ w.r.t. } \mu \text{ such that } p_0 > px; \end{cases}$$

$$p_j\left(l_j - \int\limits_{\mathbb{R}^n_+} x_j u_0(x)\, \mu(dx)\right) = 0, \quad j = 1, \ldots, n.$$

3. If $p_0 > 0$, $p = (p_1, \ldots, p_n) \geq 0$, $l = \int_{\mathbb{R}^n_+} x\theta(p_0 - px)\, \mu(dx)$ then $u(x) = \theta(p_0 - px)$ is a solution to (1).

Duality of production and profit functions

- Profit function

$$\Pi(p, p_0) = \int_{\mathbb{R}_+^n} (p_0 - px)_+ \mu(dx), \tag{2}$$

- Production function $F(l)$ is concave, non-decreasing and continuous on $\mathbb{R}_+^n$.

$$\Pi(p, p_0) = \sup_{l \geq 0} (p_0 F(l) - pl),$$

$$F(l) = \frac{1}{p_0} \inf_{p \geq 0} (\Pi(p, p_0) + pl).$$

Aggregation

- Let $F_0(X^0)$ be a positively homogeneous, concave, positive, continuous on $\mathbb{R}^n_+$ utility function;
- Let $q_0(p)$ be the price index:

$$q_0(p) = \inf_{\{X^0 \geq 0 | F_0(X^0) > 0\}} \frac{pX^0}{F_0(X^0)},$$

$$F_0(X^0) = \inf_{\{p \geq 0 | q_0(p) > 0\}} \frac{pX^0}{q_0(p)}.$$

- $X^j = (X^j_1, \ldots, X^j_m)$ — amounts of products of other industries used in j-th industry;
- $l^j = (l^j_1, \ldots, l^j_n)$ — amounts of raw resources used in j-th industry;
- $F_j(X^j, l^j)$ — production function of j-th industry.

The problem of distribution of resources (the nonlinear input-output model)

$l = (l_1, \ldots, l_n)$ — total amounts of available raw resources.

$$\begin{cases} F_0(X^0) \to \max, \\ F_j(X^j, l^j) \geq \sum_{i=0}^{m} X_j^i, \quad j = 1, \ldots, m, \\ \sum_{j=1}^{m} l^j \leq l, \\ X^0 \geq 0,\ X^1 \geq 0,\ \ldots,\ X^m \geq 0, \\ l^1 \geq 0,\ \ldots,\ l^m \geq 0. \end{cases} \tag{3}$$

Equilibrum market mechanisms

Let $l > 0$. Then vectors X^0, …, X^m, l^1, …, l^m satisfying the restrictions of problem (3) solve this problem if and only if there exist $p_0 > 0$, $p = (p_1, \dots, p_m) \geq 0$, $s = (s_1, \dots, s_n) \geq 0$ such that

- $X^0 \in \operatorname{Arg\,max}\{p_0 F_0(\tilde{X}) - p\tilde{X} \mid \tilde{X} \geq 0\}$;
- $(X^j, l^j) \in \operatorname{Arg\,max}\{p_j F_j(\tilde{X}, \tilde{l}) - p\tilde{X} - s\tilde{l} \mid \tilde{X} \geq 0,\ \tilde{l} \geq 0\}$, $j = 1$, …, m;
- $p_j\big(F_j(X^j, l^j) - \sum_{i=0}^{m} X_j^i\big) = 0, \quad j = 1, \dots, m$;
- $s_j\big(l_j - \sum_{i=0}^{m} l_j^i\big) = 0, \quad j = 1, \dots, m$.

Aggregated macro-description

- $\Pi_j(s,p) = \sup_{\tilde{X}\geq 0, \tilde{l}\geq 0}\left(p_j F_j(\tilde{X},\tilde{l}) - p\tilde{X} - s\tilde{l}\right)$ – profit function of j-th industry;
- $F^A(l)$ – aggregated profuction function.

Variational principle (dual problem)

- $\Pi^A(s,p_0) = \max\left\{\sum_{j=1}^m \Pi_j(s,p) \mid p \geq 0,\ s \geq 0,\ q_0(p) \geq p_0\right\}$;
- $\Pi^A(s,p_0)$ – aggregated profit function.

Inverse problem statement

$$\Pi^A(s,p_0) = \sup_{l\geq 0}\big(p_0 F^A(l) - sl\big),$$
$$F^A(l) = \frac{1}{p_0}\inf_{s\geq 0}\big(\Pi^A(s,p_0) + sl\big).$$

Find a non-negative measure $\mu_A(dx)$ supported in $\mathbb{R}_+^n$ and such that

$$\Pi^A(s,p_0) = \int_{\mathbb{R}_+^n} \left(p_0 - sx\right)_+ \mu_A(dx).$$

Relation to integral geometry problems

$$\frac{\partial^2}{\partial p_0^2}\Pi^A(s,p_0) = \int\limits_{sx=p_0} \mu_A(dx),$$

$$\int\limits_{\mathbb{R}^n_+} e^{-sx}\mu(dx) = \int\limits_0^{+\infty} e^{-\tau} d_\tau\left(\frac{\partial \Pi^A(s,\tau)}{\partial\tau}\right).$$

Theorem (G. M. Henkin, A. A. Shananin)

Suppose that a measure $\mu(dx)$ satisfies the conditions

- $\int_{\mathbb{R}^n_+} e^{-A|x|}|\mu|(dx) < \infty$ *for some* $A > 0$, (4)
- $\int_{\mathbb{R}^n_+} (p_0 - sx)_+\mu(dx) = 0$ *for all* $p_0 > 0$, $s \in K$, *where* K *is an open cone in* $\mathbb{R}^n_+$.

Then $\mu(dx) \equiv 0$.

Characterization theorem (G. M. Henkin, A. A. Shananin)

A function $\Pi(s, p_0)$ can be represented in the form

$$\Pi(s, p_0) = \int_{\mathbb{R}^n_+} (p_0 - sx)_+ \mu(dx), \quad (s, p_0) \in \mathbb{R}^{n+1}_+,$$

where $\mu(dx)$ is a non-negative measure supported in $\mathbb{R}^n_+$ and satisfying condition (4) if and only if

1. $\Pi(s, p_0)$ is a positively homogeneous convex function on $\mathbb{R}^{n+1}_+$ and for fixed $s \in \mathbb{R}^n_+$ the measure $\frac{\partial^2}{\partial \tau^2}\Pi(s, \tau)$ decays exponentially as $\tau \to +\infty$;
2. function $G(s) = \int_0^{+\infty} e^{-\tau} d_\tau \left(\frac{\partial \Pi(s,\tau)}{\partial \tau}\right)$ belongs to $C^\infty(\mathbb{R}^n_+)$ and for some open cone $\Gamma \subset \operatorname{int} \mathbb{R}^n_+$ and some $s \in \Gamma$ for all $\lambda > 0$, ξ^1, ..., $\xi^k \in \Gamma$, $k \geq 1$ the following inequality holds:

$$(-1)^k D_{\xi^1} \cdots D_{\xi^k} G(\lambda s) \geq 0, \quad D_\xi = \sum_j \xi_j \frac{\partial}{\partial s_j}, \quad \xi = (\xi_1, \ldots, \xi_n).$$

Example 1

Let $n = 2$, let F_{CES} be a CES production function:

$$F_{CES}(l_1, l_2) = \left(\alpha_1 l_1^{-\rho} + \alpha_2 l_2^{-\rho}\right)^{-\gamma/\rho}, \quad \alpha_1, \alpha_2 > 0,\ \rho \geq 1,\ 0 < \gamma < 1.$$

Then the profit function equals

$$\Pi_{CES}(s_1, s_2, p_0) = \gamma^{\frac{\gamma}{1-\gamma}}(1-\gamma)p_0^{\frac{1}{1-\gamma}}\left(\alpha_1^{\frac{1}{1+\rho}} s_1^{\frac{\rho}{1+\rho}} + \alpha_2^{\frac{1}{1+\rho}} s_2^{\frac{\rho}{1+\rho}}\right)^{\frac{\gamma(1+\rho)}{\rho(1-\gamma)}}.$$

For $\rho > -1$ there exists a distribution of powers over technologies corresponding to these functions.

Example 2

- Let $m = 2,\ n = 2,\ F_0(X_1^0, X_2^0) = \min(X_1^0, X_2^0)$,

$$\mu_1(dx) = k_0\delta(x - z), \qquad z = (z_1, z_2),$$
$$\mu_2(dx) = k_1\delta(x - y^1) + k_2\delta(x - y^2), \qquad y^j = (y_1^j, y_2^j),\ j = 1, 2,$$

where $k_1 + k_2 > k_0,\ y_1^1 > y_1^2,\ y_2^1 > y_2^2$. Then

$$\Pi^A(s,p_0) = \max\Big\{(k_0 - k_2)_+\Big(p_0 - s(z+y^1)\Big)_+ + \min(k_0,k_2)\Big(p_0 - s(z+y^2)\Big)_+,$$
$$\min(k_0,k_1)\Big(p_0 - s(z+y^1)\Big)_+ + (k_0 - k_2)_+\Big(p_0 - s(z+y^2)\Big)_+\Big\}.$$

- Denote $K_1 = \left\{s \in \mathbb{R}_+^2 \mid sy^2 \le sy^1\right\},\ K_2 = \left\{s \in \mathbb{R}_+^2 \mid sy^1 \le sy^2\right\}$.

Then $\quad \Pi^A(s, p_0) = \max_j \Pi_j(s, p_0), \quad \Pi_j(s, p_0) = \int_{\mathbb{R}_+^2} (p_0 - sx)_+ \mu_j(dx),$

$$\Pi^A(s, p_0) = \Pi_j(s, p_0) \quad \text{for } s \in K_j; \qquad \mathbb{R}_+^n = \cup_{j=1}^n K_j,$$
$$G(s) = \max_j G_j(s), \qquad G_j(s) = \int_0^{+\infty} e^{-\tau} d_\tau\Big(\frac{\partial \Pi_j(s,\tau)}{\partial \tau}\Big).$$

Stable correspondances (A. V. Karzanov, A. A. Shananin)

Let $X = \{x^1, \ldots, x^m\} \subset \mathbb{R}^n_+$, $Y = \{y^1, \ldots, y^m\} \subset \mathbb{R}^n_+$ and $C \subset \mathbb{R}^n_+$ be a cone.

Definition. A bijection $\gamma\colon X \to Y$ is called a C-stable correspondance if for any x^i, $x^j \in X$, $p \in C$ the inequality $px^i < px^j$ implies $p\gamma(x^i) \leq p\gamma(x^j)$.

Theorem

A bijection $\gamma\colon X \to Y$ is a C-stable correspondance if and only if for any x^i, $x^j \in X$, $x^i \neq x^j$

- *if $x^j - x^i \in C^*$ then $\gamma(x^j) - \gamma(x^i) \in C^*$;*
- *if $x^j - x^i \notin C^*$, $x^i - x^j \notin C^*$ then there exist such $\lambda \geq 0$, $\mu \geq 0$, $\lambda + \mu > 0$ that $\lambda(x^j - x^i) = \mu(\gamma(x^j) - \gamma(x^i))$.*

A model of industry with substitution of production factors at the micro-level.

Let $f(u)$ be a positively homogeneous of first order, concave, continuous function on $\mathbb{R}_+^n$, positive on $\operatorname{int}\mathbb{R}_+^n$. A technology is given by a vector $x = (x_1, \dots, x_n)$.

A production function at the micro-level: $f\left(\frac{u_1}{x_1}, \dots, \frac{u_n}{x_n}\right)$.

Examples:

- The Leontieff function with constant proportions $f(u) = \min(u_1, \dots, u_n)$ corresponds to the production function at the micro-level in the Houthakker–Johansen model.
- CES-function

$$f(u) = \left(u_1^{-\rho} + \cdots + u_n^{-\rho}\right)^{-1/\rho} = u_1 \oplus_\rho \cdots \oplus_\rho u_n, \quad \rho \geq -1.$$

The problem of distribution of resources in presence of substitution of production factors at the micro-level

$$\begin{cases} \int\limits_{\mathbb{R}^n_+} \min\left(1, f\left(\frac{u_1(x)}{x_1}, \dots, \frac{u_n(x)}{x_n}\right)\right) \mu(dx) \to \max\limits_u, \\ \int\limits_{\mathbb{R}^n_+} u_j(x)\mu(dx) \le l_j, \quad j = 1, \dots, n, \\ u(x) = \left(u_1(x), \dots, u_n(x)\right) \ge 0. \end{cases} \tag{5}$$

$$\text{Put } \; q(p) = \inf_{\{u \ge 0 | f(u) > 0\}} \frac{pu}{f(u)},$$

$$p \circ x = (p_1 x_1, \dots, p_n x_n), \quad \pi(x, p, p_0) = \left(p_0 - q(p \circ x)\right)_+.$$

Study of problem (5)

- If $l \geq 0$ then the problem (5) has a $\mu(dx)$-integrable solution,
- A distribution of resources $u(x) = \big(u_1(x), \ldots, u_n(x)\big)$ satisfying the restrictions of problem (5) is optimal only if (and for $l > 0$ if) there exist such $p_0 \geq 0$, $p = (p_1, \ldots, p_n) \geq 0$, not simultaneously equal to zero, that
  1. $p_i \left(\int\limits_{\mathbb{R}^n_+} u_i(x)\, \mu(dx) - l_i \right) = 0, \quad i = 1, \ldots, n;$
  2. $u(x) = 0$ a.e. w.r.t. $\mu(dx)$ on $\big\{x \geq 0 \mid p_0 < q(p \circ x)\big\}$;
  3. $F(u(x)) = 1$ and $p_0 - pu(x) = \pi(p_0, p, x)$ a.e. w.r.t. $\mu(dx)$ on $\big\{x \geq 0 \mid p_0 > q(p \circ x)\big\}$.

Duality of production and profit functions in the model with substitution of production factors at the micro-level

- Profit function

$$\Pi(p, p_0) = \int\limits_{\mathbb{R}^n_+} \big(p_0 - q(p \circ x)\big)_+ \mu(dx). \tag{6}$$

- Production function $F(l)$ is concave, non-decreasing, continuous on $\mathbb{R}^n_+$.

$$\Pi(p, p_0) = \sup_{l \geq 0} \big(p_0 F(l) - pl\big),$$

$$F(l) = \frac{1}{p_0} \inf_{p \geq 0} \big(\Pi(p, p_0) + pl\big).$$

Example 3

Let $m = 2$, $n = 2$, $q(p_1, p_2) = p_1^{\nu} p_2^{1-\nu}$, $0 < \nu < 1$,

$$\mu_j(dx) = x_1^{\alpha_1^j - 1} x_2^{\alpha_2^j - 1} dx_1 dx_2, \quad \alpha_i^j > 1, \quad i, j = 1, 2.$$

Then $\Pi_j(s, p_j) = A_j \dfrac{p_j^{\alpha_1^j + \alpha_2^j + 1}}{s_1^{\alpha_1^j} s_2^{\alpha_2^j}}, \quad A_j > 0, \quad j = 1, 2,$

$$\Pi^A(s, p_0) = B \frac{p_0^{\alpha_1^A + \alpha_2^A + 1}}{s_1^{\alpha_1^A} s_2^{\alpha_2^A}}, \quad B > 0,$$

where $\alpha_j^A = \dfrac{\nu(\alpha_1^2 + \alpha_2^2 + 1)\alpha_j^1 + (1-\nu)(\alpha_1^1 + \alpha_2^1 + 1)\alpha_j^2}{\nu(\alpha_1^2 + \alpha_2^2 + 1) + (1-\nu)(\alpha_1^1 + \alpha_2^1 + 1)}, \quad j = 1, 2.$

Then $\mu^A(dx) = b x_1^{\alpha_1^A - 1} x_2^{\alpha_2^A - 1}$ exists, where $b > 0$.

Characterization of transform (6) (A. D. Agaltsov)

Denote

$$c = (c_1, \dots, c_n) \in \operatorname{int} \mathbb{R}^n_+, \; z = (z_1, \dots, z_n),$$
$$x^{z-I} = x_1^{z_1 - 1} \cdots x_n^{z_n - 1}, \; x^{-z} = x_1^{-z_1} \cdots x_n^{-z_n},$$
$$\rho_q = \Gamma(z_1) \cdots \Gamma(z_n) \Gamma(z_1 + \cdots + z_n) \Big/ \int\limits_{\mathbb{R}^n_+} x^{z-I} e^{-q(x)} dx_1 \cdots dx_n.$$

Then

$$G(s) = \int\limits_{\mathbb{R}^n_+} e^{-sx} \mu(dx) = (2\pi i)^{-n} \int\limits_{c+i\mathbb{R}^n} s^{-z} \rho_q(z) \left(\int\limits_{\mathbb{R}^n_+} p^{z-I} \Pi(p, 1) dp \right) dz,$$
$$dp = dp_1 \cdots dp_n, \quad dz = dz_1 \cdots dz_n.$$

The problem of estimation of elasticity of subsitution of production factors at the micro-level

Input: $\{p^t, p_0^t, y^t \mid t = 1, \ldots, T\}$, where p^t is the vector of prices of production factors, p_0^t is the price of product, y^t is the amount of product at the moment t.

Let $q(p) = (p_1^{-\rho} + \cdots + p_n^{-\rho})^{-1/\rho}$, where $\rho \geq -1$.

Problem: Find such ρ that there exists a non-negative measure $\mu(dx)$ satisfying

$$\int_{\mathbb{R}_+^n} \theta\Big(p_0^t - \big((p_1^t x_1)^{-\rho} + \cdots + (p_n^t x_n)^{-\rho}\big)^{-1/\rho}\Big)\mu(dx) = y^t, \; t = 1, \ldots, T. \quad (7)$$

Study of the moment problem (7)

Hypersurfaces $\left((p_1^t x_1)^{-\rho} + \cdots + (p_n^t x_n)^{-\rho}\right)^{-1/rho} = p_0^t$, $t = 1, \ldots,$ T divide $\mathbb{R}_+^n$ into a finite number of regions $\{V\}$. For each region V compose a boolean vector (spectrum of the region) $b(V) = (b_1(V), \ldots, b_T(V))$, where

$$b_t(V) = \begin{cases} 1, \text{ if } p_0^t > \left((p_1^t x_1)^{-\rho} + \cdots + (p_n^t x_n)^{-\rho}\right)^{-1/\rho} \text{ for } x \in \operatorname{int} V, \\ 0, \text{ if } p_0^t < \left((p_1^t x_1)^{-\rho} + \cdots + (p_n^t x_n)^{-\rho}\right)^{-1/\rho} \text{ for } x \in \operatorname{int} V. \end{cases}$$

Denote by $B\big((p^1, p_0), \ldots, (p^T, p_0^T)\big)$ the spectrum of the partition, i.e. the set of vectors $b(V)$ as V runs over all regions of partition of $\mathbb{R}_+^n$ by hypersurfaces $\left((p_1^t x_1)^{-\rho} + \cdots + (p_n^t x_n)^{-\rho}\right)^{-1/\rho} = p_0^t$, $t = 1, \ldots, T$.

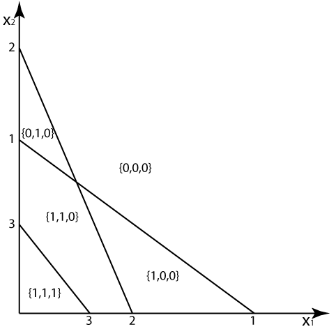
Proposition. The moment problem (7) has a solution if and only if the vector $(y^1, \ldots, y^T)$ belongs to the convex conical hull of the spectrum $B\big((p^1, p_0^1), \ldots, (p^T, p_0^T)\big)$.

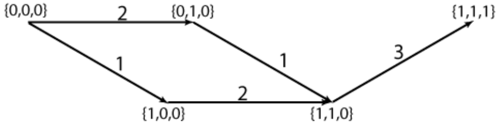
Rhombus tilings

Consider the case $n = 2$. Denote $e_t = \left(1; t - \left[\frac{T+1}{2}\right]\right), t = 1, \ldots, T$.

For each region V define a point $\xi(V) = \sum_{t=1}^{T} b_t(V) e_t$. Connect points corresponding to neighbor regions by a segment. The resulting figure is the rhombus tiling corresponding to the partition.

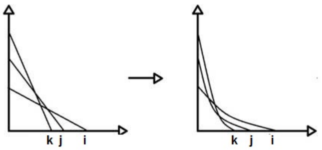
Points of intersection of curves of the partition correspond to rhombi of the tiling.





Deformations of partitions and flips of rhombus tilings

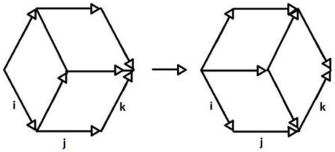
Any three curves intersect at the same point at most once as ρ runs over $[-1,0)\cup(0,+\infty)$. The spectrum of the partition changes according to the flip operation.



Theorem (Leclerc B., Zelevinsky A.)

Any two complete rhombus tilings can be achieved one from another using a finite number of flip operations.

Addition (Molchanov E. G.): the theorem is valid for rhombus tilings with the same top and bottom boundaries.



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