QUASILINEAR EQUATIONS, INVERSE PROBLEMS AND THEIR APPLICATIONS
Moscow Institute of Physics and Technology, Dolgoprudny
12 Sept. 2016 - 15 Sept. 2016

Acoustic tomography of scalar and vector inhomogeneities based on the Novikov-Agaltsov algorithm

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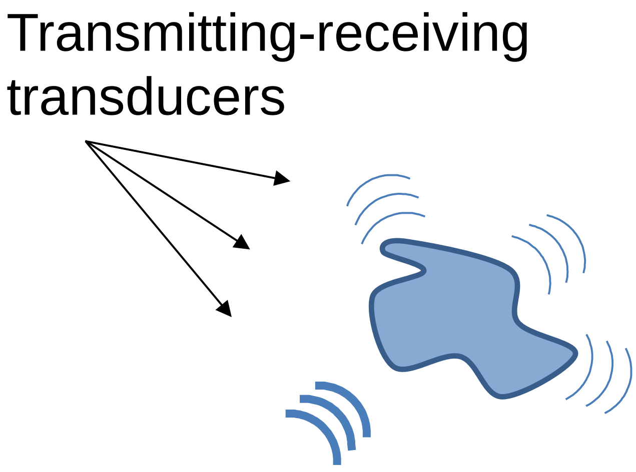
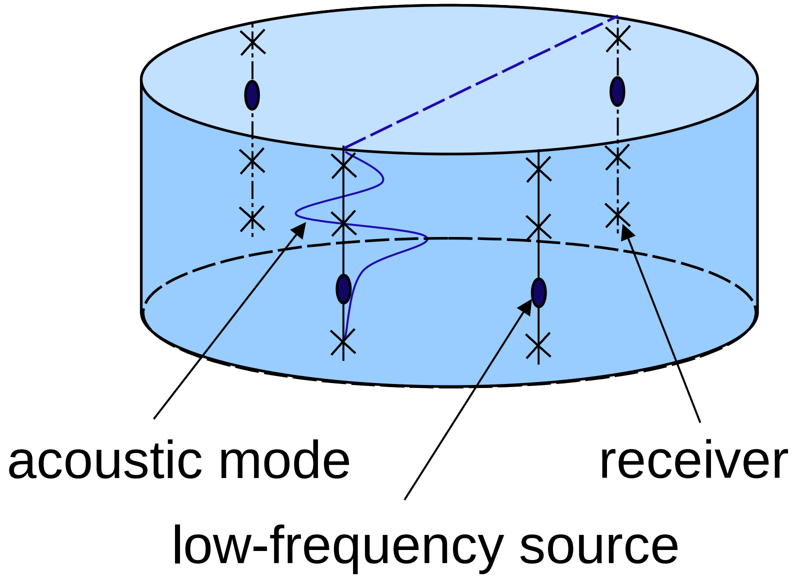
Acoustic tomography is a powerful tool for studying natural media that are transparent to acoustic waves; it is employed when direct measurement of medium characteristics is difficult or impossible.

Medical diagnostic, ocean tomography, geophysical researches are the main areas of application of acoustic tomography.

Ultrasound medical tomography



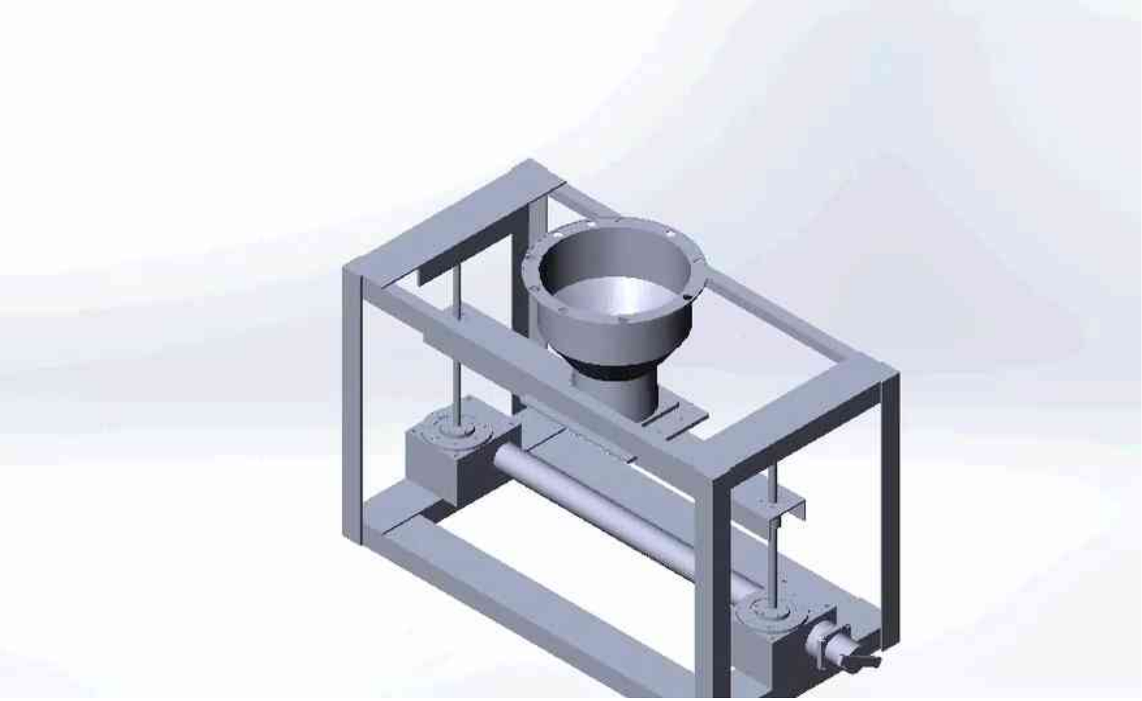
Ocean mode tomography



Acoustic Tomograph developing at Faculty of Physics, Acoustics Department, MSU,

for the reconstruction of sound speed, absorption and blood flow in the soft biological tissues (first of all in the breast) for the cancer diagnostic purposes.

Top view



General view

The circular rotating antenna with the uniquely distributed 26 emitting-receiving transducers are equivalent to a fixed multi-element antenna with 256 transducers.

Motivation

There are known methods how to solve the problem of acoustic tomography. Most of them are **approximate**. The linear approximation is generally applied with iteration procedures and regularizations. The general perturbation theory is also considered.

There are quite **mathematically rigorous** (at least, for a rather wide class of scatterers) functional-analytical methods for solving the inverse problems, which were initially developed in quantum mechanics. Since the Schrödinger equation in the monochromatic (isoenergetic) case is the same as the Helmholtz equation up to notation, it gives the simple idea to apply these methods for acoustic applications.

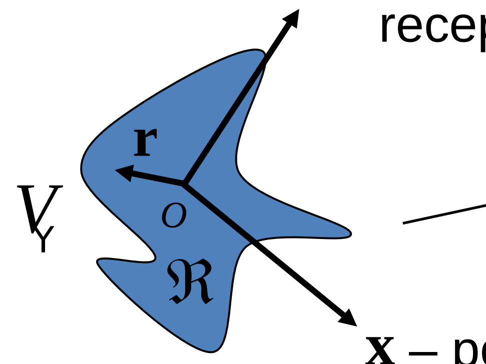
The main goal of report is to consider the application of functional-analytical [1-3] algorithm for the purpose of 2D acoustic tomography of both scalar and vector inhomogeneities.

This algorithm takes into account the multiple scattering processes and does not require either linearization of the model or iterations.

1.Novikov R.G. The inverse scattering problem on a fixed energy level for the two-dimensional Schrödinger operator // Journal of Functional Analysis. 1992. V. 103. N 2. P. 409–463.
2.Agaltsov A.D., Novikov R.G. Riemann–Hilbert problem approach for two-dimensional flow inverse scattering // J. Math. Phys. 2014. V. 55. N 10. P. 103502.
3.Agaltsov A.D. On the reconstruction of parameters of a moving fluid from the Dirichlet-to-Neumann map // Eurasian Journal of Mathematical and Computer Applications.

Statement of 2D problem

It is assumed that the investigated area is surrounded over perimeter by the quasi-point transducers emitting and receiving acoustic fields $u(\mathbf{r})$.
In the tomographic area there are an unknown **vector inhomogeneity**



$$\mathbf{A}(\mathbf{r},\omega) = \omega \frac{\boldsymbol{v}(\mathbf{r})}{c^2(\mathbf{r})} + i\nabla \ln \sqrt{\rho(\mathbf{r})}, \quad \mathbf{r} \in V_{\mathrm{Y}}$$

and an unknown **scalar inhomogeneity**

$$v(\mathbf{r},\omega) = \omega^2 \left(\frac{1}{c_0^2} - \frac{1}{c^2(\mathbf{r})} \right) - 2i\,\omega\, \frac{\alpha(\mathbf{r},\omega)}{c(\mathbf{r})},$$

ω – circular frequency, c_0 – background sound speed value.

$\boldsymbol{v}(\mathbf{r})$ – flows, $\rho(\mathbf{r})$ – density, $c(\mathbf{r})$ – sound speed,

$\alpha(\mathbf{r},\omega)$ – amplitude absorption coefficient.

How to reconstruct these quantities, if we know acoustic fields $u(\mathbf{y})$?

Reconstruction algorithm that uses data from a quasi-point transducers

Step 1. Calculation of operator $(\mathrm{F} - \mathrm{F}_0)(\mathbf{y}, \mathbf{x}; \omega)$

$$\left.\frac{\partial u}{\partial n}\right|_{\mathrm{Y}} = \hat{\mathrm{F}}(\omega)\left(u|_{\mathrm{Y}}\right), \left.\frac{\partial u_0}{\partial n}\right|_{\mathrm{Y}} = \hat{\mathrm{F}}_0(\omega)\left(u_0|_{\mathrm{Y}}\right), \nabla^2 u_0 + k_0^2 u_0 = 0, \;\; k_0 = \frac{\omega}{c_0}.$$

Step 2. Estimation of Faddeev generalized scattering amplitude $h^{\pm}(\omega)$ from $(\mathrm{F} - \mathrm{F}_0)(\mathbf{y}, \mathbf{x}; \omega)$.

Step 3. Reconstruction of estimates $\mathbf{A}^{\mathrm{div}}(\mathbf{r}, \omega)$, $v^{\mathrm{div}}(\mathbf{r}, \omega)$ from $h^{\pm}(\omega)$, via solving some Riemann–Hilbert problem on Faddeev eigenfunctions.

Steps of this algorithm have been discussed previously :

*R.G. Novikov, Phys. Lett. A **238**, 73 (1998).*

R.G. Novikov, M.Santacesaria, International Math. Res. Notices, doi:10.1093/imrn/rns025 (2012).

A.D. Agaltsov, R.G. Novikov, J. Math. Phys., 55, 10 (2014),
V. A. Burov, N. V. Alekseenko, and O. D. Rumyantseva, Acoustical Physics, 55, 6 (2009),
V. A. Burov, A. S. Shurup, D. I. Zotov and O. D. Rumyantseva, Acoustical Physics, 59, 3 (2013)

Main relationships

Simultaneous reconstruction of $c(\mathbf{r}),\ \boldsymbol{v}(\mathbf{r}),\ \alpha(\mathbf{r},\omega),\ \rho(\mathbf{r})$.

$$\begin{cases} \mathbf{F}(\mathbf{r}) \equiv \dfrac{1}{\omega}\operatorname{rot}\mathbf{A}^{\mathrm{div}}(\mathbf{r},\omega) = \operatorname{rot}\dfrac{\boldsymbol{v}(\mathbf{r})}{c^2(\mathbf{r})}, \\ Q(\mathbf{r},\omega) \equiv v^{\mathrm{div}} - \mathbf{A}^{\mathrm{div}}(\mathbf{r},\omega)\cdot\mathbf{A}^{\mathrm{div}}(\mathbf{r},\omega) = \\ = f_1 - \omega^2 f_2 + i\omega f_3 - 2i\omega\left(\dfrac{\omega}{\omega_0}\right)^{\zeta(\mathbf{r})}\dfrac{\alpha_0(\mathbf{r})}{c(\mathbf{r})}, \quad \alpha(\mathbf{r},\omega) = \left(\dfrac{\omega}{\omega_0}\right)^{\zeta(\mathbf{r})}\alpha_0(\mathbf{r}). \end{cases}$$

<u>There are no needs to estimate</u> $\nabla\phi^{\mathrm{div}}$.

$$\operatorname{Re} Q(\mathbf{r},\omega) \to \boxed{f_1 = \sqrt{\rho(\mathbf{r})}\,\nabla\frac{1}{\sqrt{\rho(\mathbf{r})}},} \quad \boxed{f_2 = \frac{1}{c^2(\mathbf{r})} - \frac{1}{c_0^2} + \underline{\frac{\boldsymbol{v}(\mathbf{r})}{c^2(\mathbf{r})}\cdot\frac{\boldsymbol{v}(\mathbf{r})}{c^2(\mathbf{r})}} \cong \frac{1}{c^2(\mathbf{r})} - \frac{1}{c_0^2},}$$

$$\operatorname{Im} Q(\mathbf{r},\omega) \to \boxed{f_3 = \nabla\cdot\frac{\boldsymbol{v}(\mathbf{r})}{c^2(\mathbf{r})} - \frac{\boldsymbol{v}(\mathbf{r})\cdot\nabla\ln\rho(\mathbf{r})}{c^2(\mathbf{r})},}\ \zeta(\mathbf{r}),\ \alpha_0(\mathbf{r}).$$

Agaltsov A.D. Eurasian Journal of Mathematical and Computer Applications. 2016.

Particular case

Reconstruction of $c(\mathbf{r}),\ \rho(\mathbf{r}),\ \alpha(\mathbf{r},\omega)$, without flows $\boldsymbol{v}(\mathbf{r})$.

V. A. Burov, A.L. Konyushkin and O. D. Rumyantseva, Acoustical Physics, 43, 4 (1997)

$$\begin{cases} \operatorname{Re} Q(\mathbf{r},\omega_1) = f_1 - \omega_1^2 f_2, \\ \operatorname{Re} Q(\mathbf{r},\omega_2) = f_1 - \omega_2^2 f_2, \\ \dots \end{cases} \Rightarrow f_1,\ f_2 \Rightarrow c(\mathbf{r}),\ \rho(\mathbf{r})$$

$$\operatorname{Im} Q(\mathbf{r},\omega_j) = 2\omega_j \left(\frac{\omega_j}{\omega_0}\right)^{\zeta(\mathbf{r})} \frac{\alpha_0(\mathbf{r})}{c(\mathbf{r})}, \Rightarrow \ln\left(\frac{\omega_j}{\omega_k}\right)\left[\zeta(\mathbf{r})+1\right] = \ln\left(\frac{\operatorname{Im} Q(\mathbf{r},\omega_j)}{\operatorname{Im} Q(\mathbf{r},\omega_k)}\right),$$

LSM estimation of $\zeta(\mathbf{r})$.

$$\Rightarrow \zeta(\mathbf{r}) \Rightarrow \alpha_0(\mathbf{r}) = \frac{c(\mathbf{r})}{2\,\omega_j} \operatorname{Im} Q(\mathbf{r},\omega_j) \left(\frac{\omega_0}{\omega_j}\right)^{\zeta(\mathbf{r})}$$

General case

Simultaneous reconstruction of $c(\mathbf{r}),\ \boldsymbol{v}(\mathbf{r}),\ \alpha(\mathbf{r},\omega),\ \rho(\mathbf{r})$.

$$\begin{cases} \operatorname{Re} Q(\mathbf{r},\omega_1) = f_1 - \omega_1^2\, f_2, \\ \operatorname{Re} Q(\mathbf{r},\omega_2) = f_1 - \omega_2^2\, f_2, \\ \ldots \end{cases}$$

$$\Rightarrow \quad f_1,\ f_2 \quad \Rightarrow \quad c(\mathbf{r}),\ \rho(\mathbf{r})$$

Procedure for the reconstruction of sound speed and density is the same.

$$\begin{cases} \operatorname{Im} Q(\mathbf{r},\omega_1) = \omega_1\, f_3 - 2\,\omega_1 \left(\dfrac{\omega_1}{\omega_0}\right)^{\zeta(\mathbf{r})} \dfrac{\alpha_0(\mathbf{r})}{c(\mathbf{r})}, \\ \operatorname{Im} Q(\mathbf{r},\omega_2) = \omega_2\, f_3 - 2\,\omega_2 \left(\dfrac{\omega_2}{\omega_0}\right)^{\zeta(\mathbf{r})} \dfrac{\alpha_0(\mathbf{r})}{c(\mathbf{r})}, \\ \ldots \end{cases}$$

There are some difficulties with simultaneous reconstruction of

$\boldsymbol{v}(\mathbf{r}),\ \zeta(\mathbf{r}),\ \alpha_0(\mathbf{r}),$

in multi frequency regime.

Reconstruction of attenuation power index $\zeta(\mathbf{r})$, if $\boldsymbol{v}(\mathbf{r}) \neq 0$

$$\frac{\dfrac{1}{\omega_2}\operatorname{Im} Q(\mathbf{r},\omega_2) - \dfrac{1}{\omega_1}\operatorname{Im} Q(\mathbf{r},\omega_1)}{\dfrac{1}{\omega_3}\operatorname{Im} Q(\mathbf{r},\omega_3) - \dfrac{1}{\omega_1}\operatorname{Im} Q(\mathbf{r},\omega_1)} = \frac{\left(\dfrac{\omega_2}{\omega_1}\right)^{\zeta(\mathbf{r})} - 1}{\left(\dfrac{\omega_3}{\omega_1}\right)^{\zeta(\mathbf{r})} - 1},$$

$$\zeta \in (0, \infty)$$

$$\omega_1 < \omega_2 < \omega_3$$

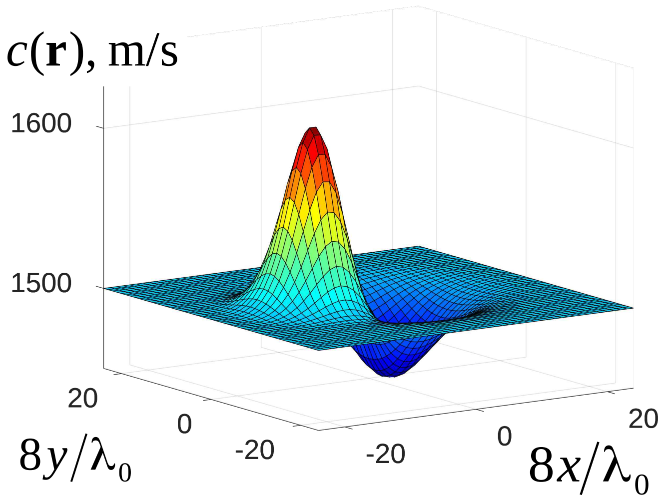
Multi frequency generalization **(??)**:

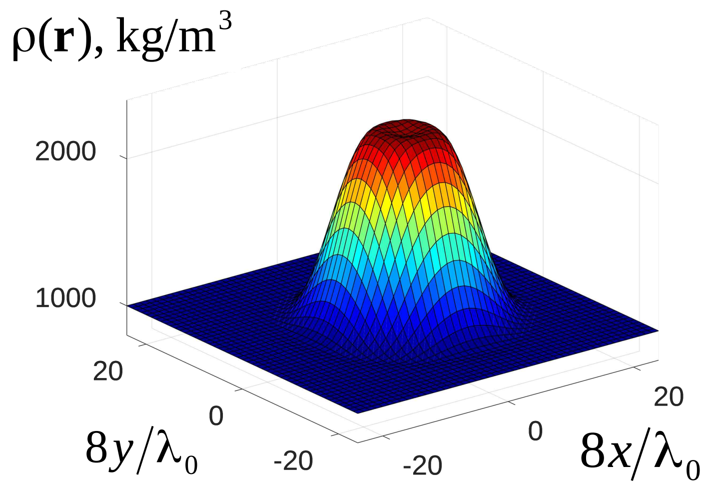
$$\prod_{\{j1,\ j2,\ j3\}} \frac{\dfrac{1}{\omega_{j2}}\operatorname{Im} Q(\mathbf{r},\omega_{j2}) - \dfrac{1}{\omega_{j1}}\operatorname{Im} Q(\mathbf{r},\omega_{j1})}{\dfrac{1}{\omega_{j3}}\operatorname{Im} Q(\mathbf{r},\omega_{j3}) - \dfrac{1}{\omega_{j1}}\operatorname{Im} Q(\mathbf{r},\omega_{j1})} = \prod_{\{j1,\ j2,\ j3\}} \frac{\left(\dfrac{\omega_{j2}}{\omega_{j1}}\right)^{\zeta(\mathbf{r})} - 1}{\left(\dfrac{\omega_{j3}}{\omega_{j1}}\right)^{\zeta(\mathbf{r})} - 1}$$

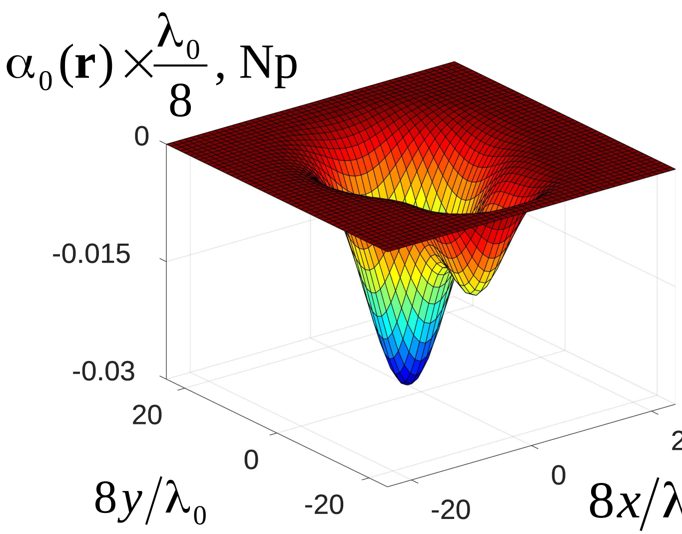
$$\omega_{j1} < \omega_{j2} < \omega_{j3}$$

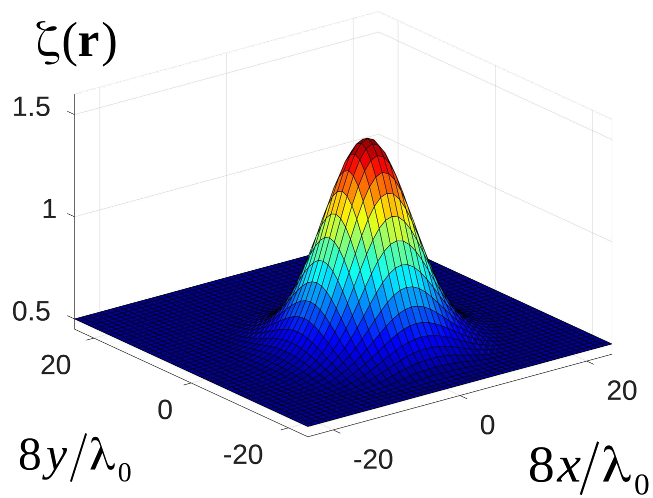
Reconstruction of $c(\mathbf{r})$, $\rho(\mathbf{r})$, $\alpha_0(\mathbf{r})$, $\zeta(\mathbf{r})$ *without flows* $\boldsymbol{v}(\mathbf{r})$

Parameters of the medium



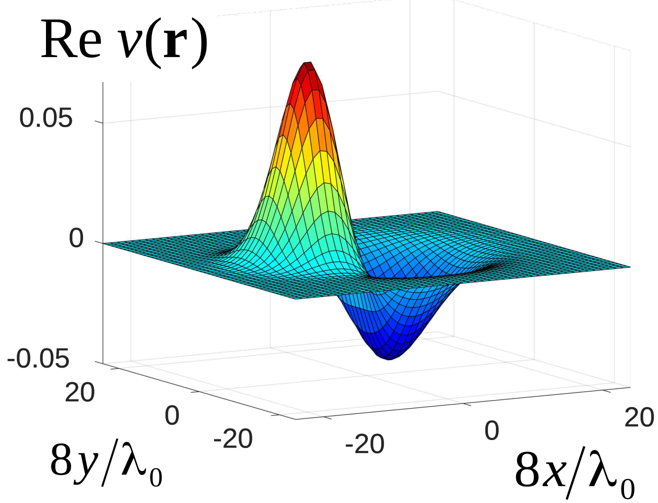


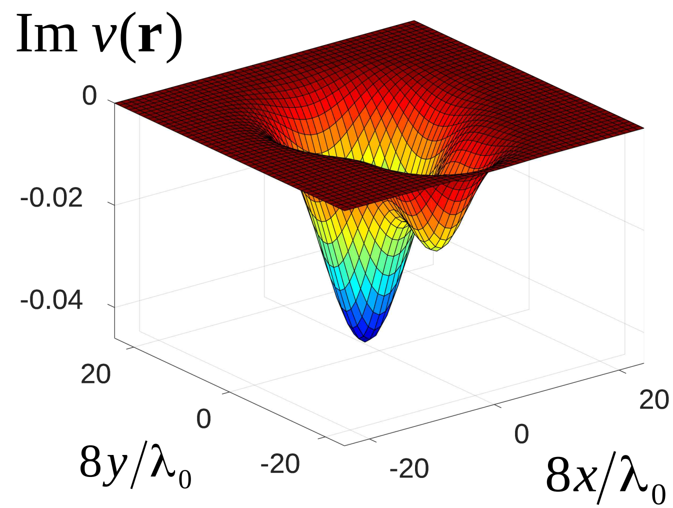




Scalar components of scatterer

$$\operatorname{Re} v(\mathbf{r},\omega) = \omega^2 \left[\frac{1}{c_0^2} - \frac{1}{c^2(\mathbf{r})} \right] \qquad \operatorname{Im} v(\mathbf{r},\omega) = -2\omega \frac{\alpha(\mathbf{r},\omega)}{c(\mathbf{r})}.$$





This is scatterer of medium strength:

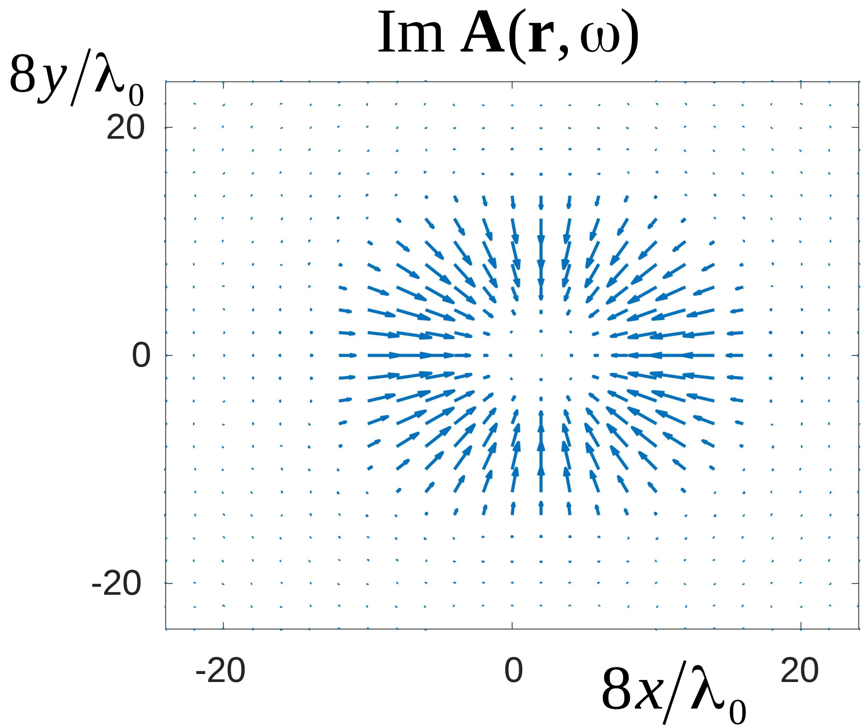
Additional phase shift along OX axis: for positive velocity contrast (Re $v>0$) $\Delta\psi_{pos} \approx 0.29\pi$, for negative velocity contrast (Re $v<0$) $\Delta\psi_{neg} \approx -0.19\pi$; amplitude attenuation is ≈ 1.99 (times).

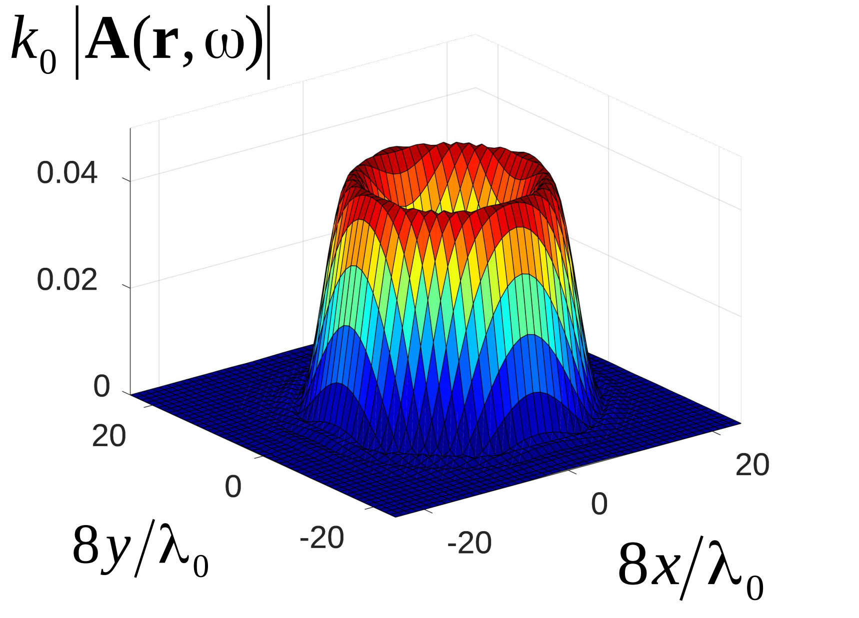
The similar parameters along OY axis: $\Delta\psi_{pos} \approx 0$, $\Delta\psi_{neg} \approx -0.22\,\pi$; amplitude attenuation is ≈ 2.84 (times).

Vector component of scatterer

$$\mathbf{A}(\mathbf{r},\omega) = i\nabla \ln \sqrt{\rho(\mathbf{r})}, \qquad \mathrm{rot}\ \mathbf{A}(\mathbf{r},\omega) = 0$$

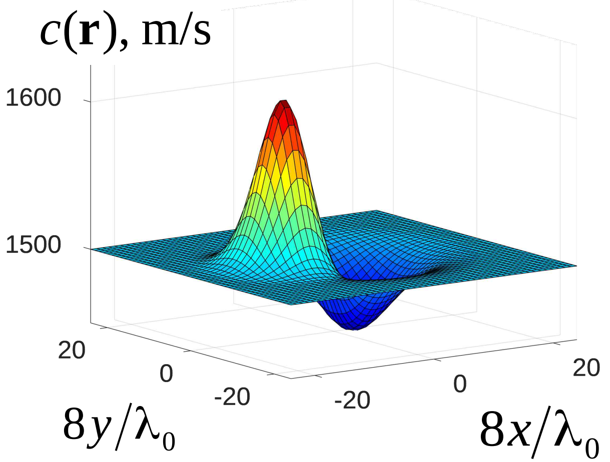
$$\boxed{\boldsymbol{v}(\mathbf{r}) \equiv 0}$$

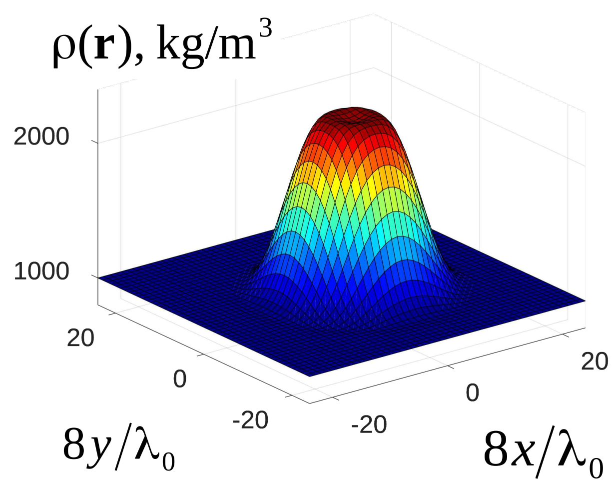




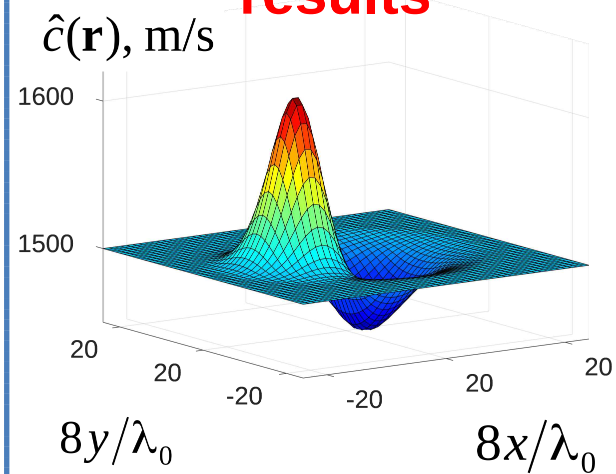
Reconstruction results obtained by using data on 2 frequencies, without noise

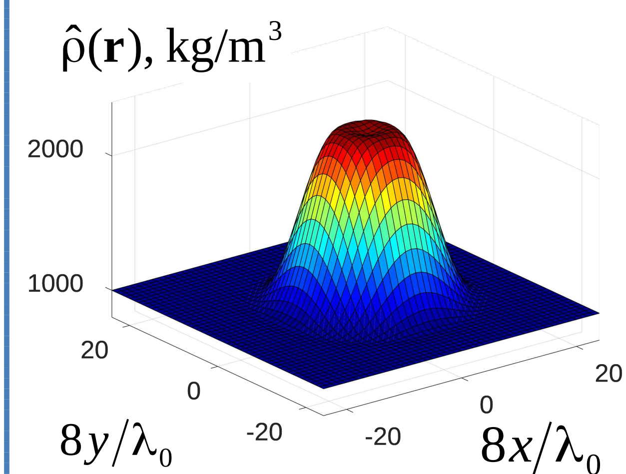
Model



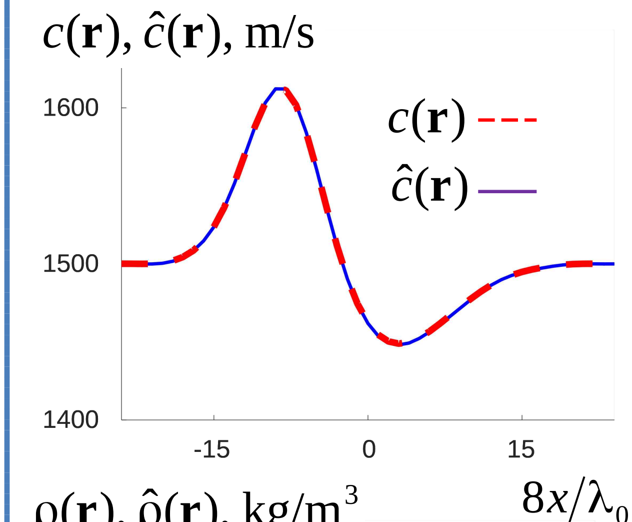


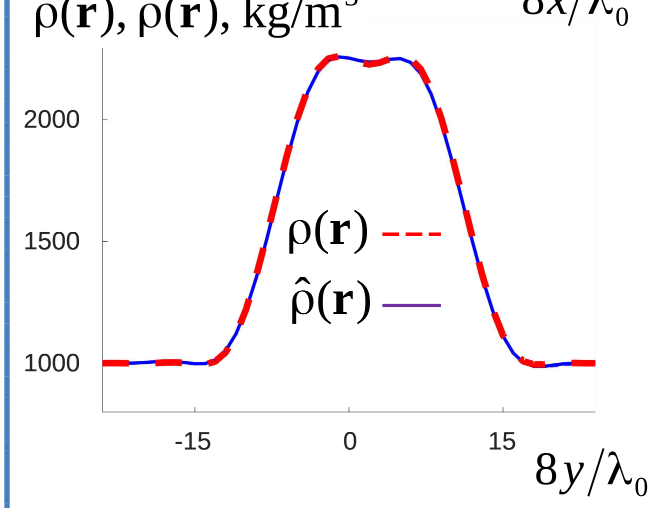
Reconstruction results





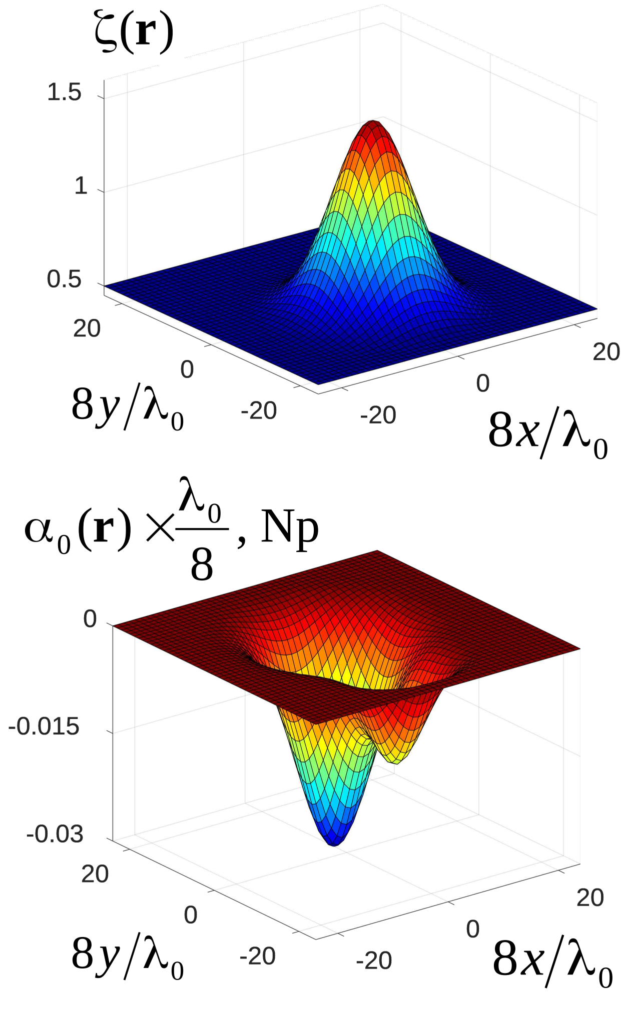
Crossections of model and result



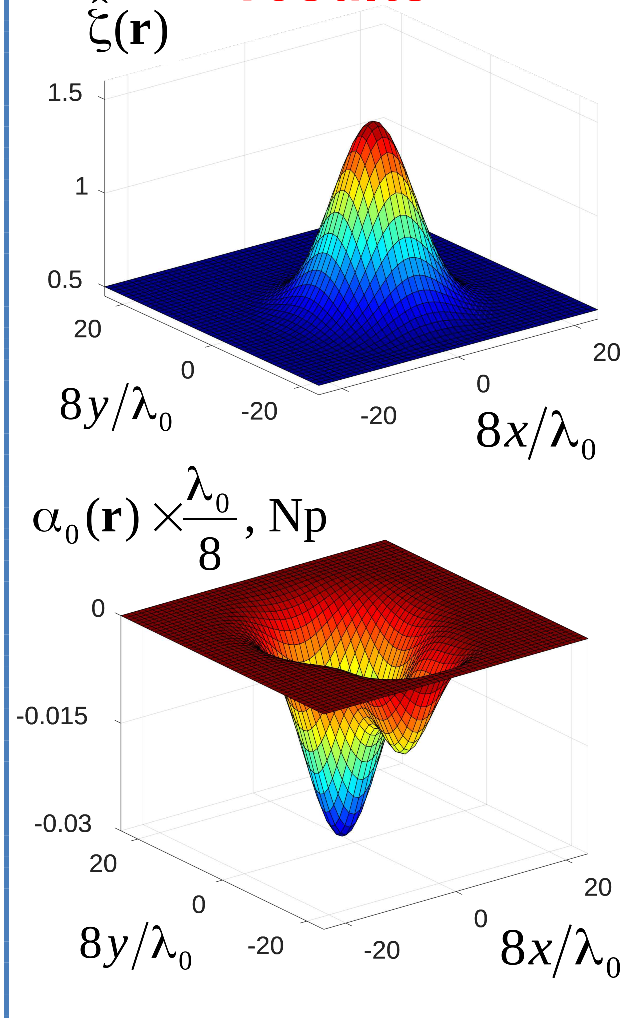


Reconstruction results obtained by using data on 2 frequencies, without noise

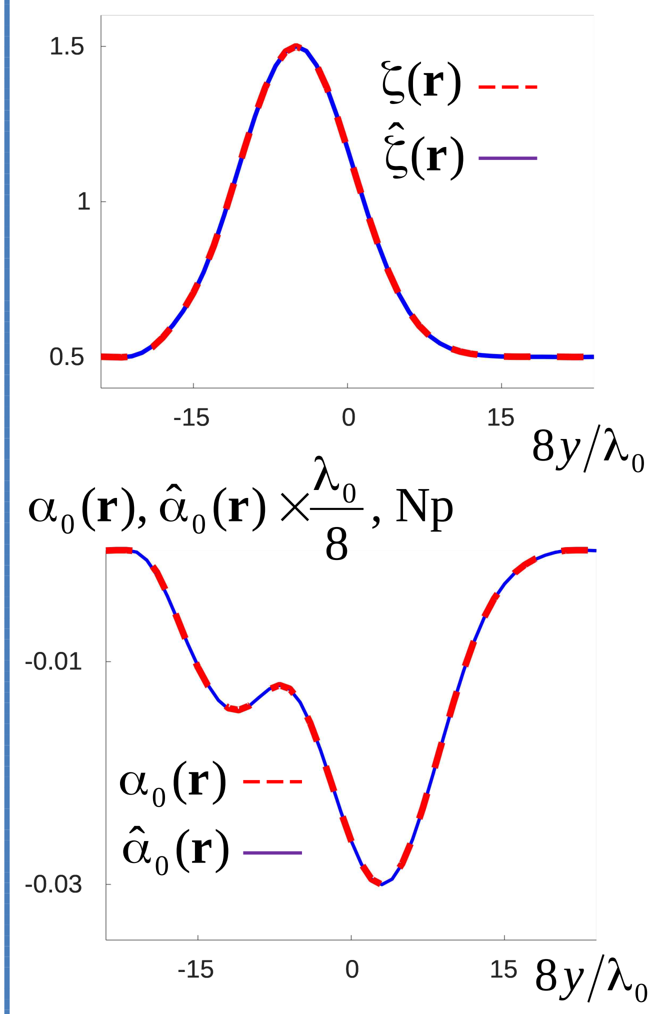
Model



Reconstruction results



Crossections of model and result

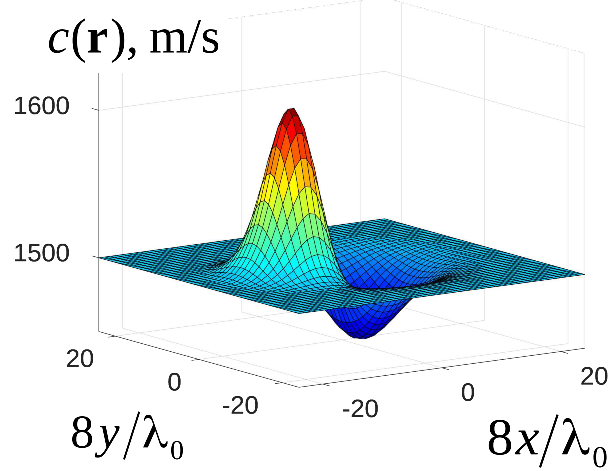


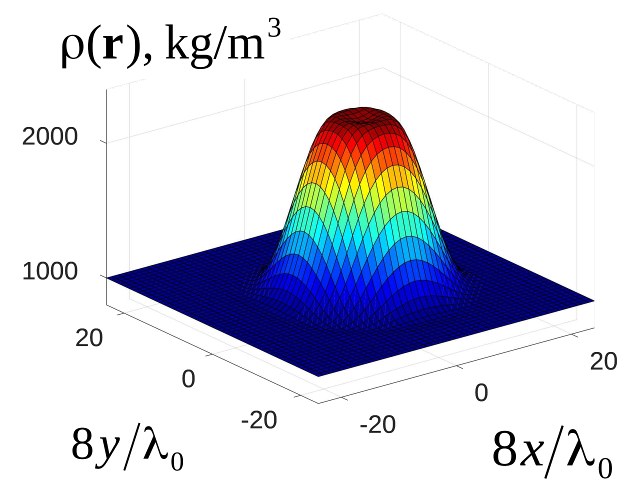
Influence of noise on reconstruction results

Normally distributed noise with rms amplitude deviation

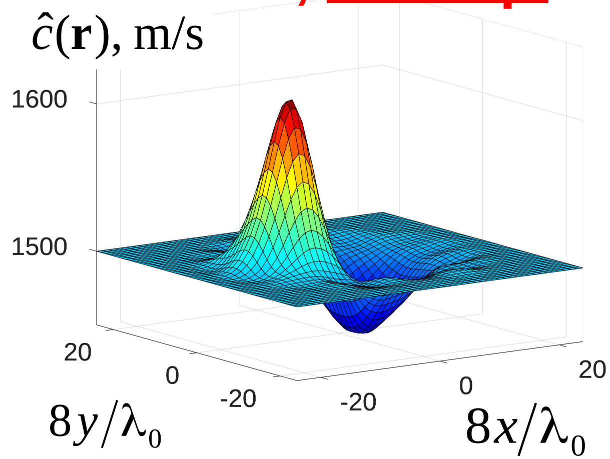
$$0.03 \star \sqrt{\int_Y d\mathbf{x} \int_Y d\mathbf{y} \left| G_{sc}(\mathbf{y},\mathbf{x};\, \omega_j) \right|^2 \Big/ \int_Y d\mathbf{x} \int_Y d\mathbf{y}}$$

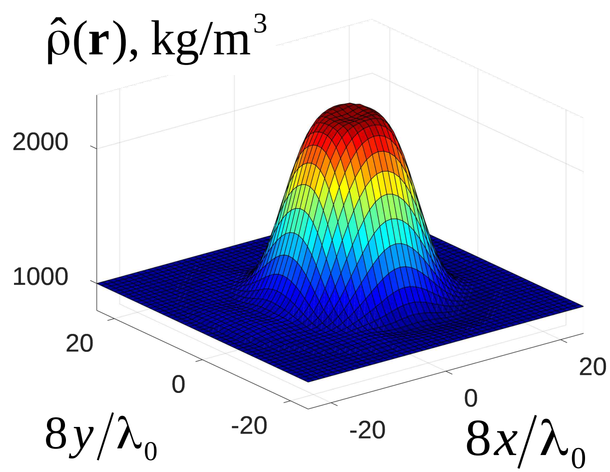
Model



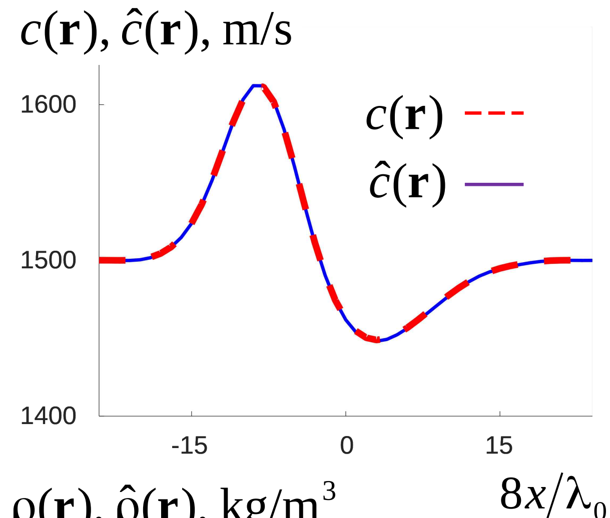


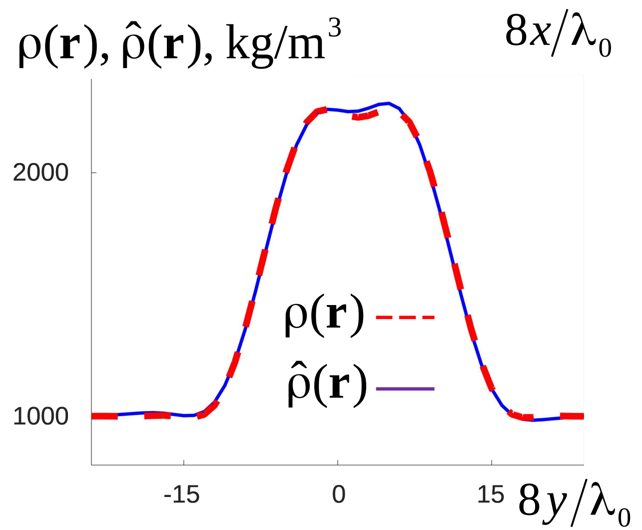
Reconstruction results, <u>11 freqs</u>





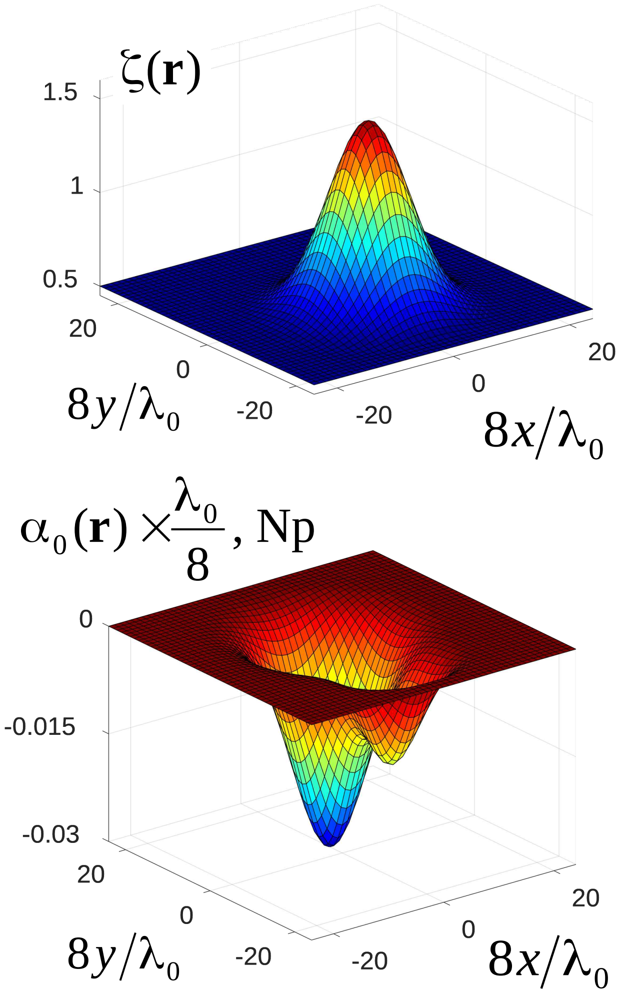
Crossections of model and result



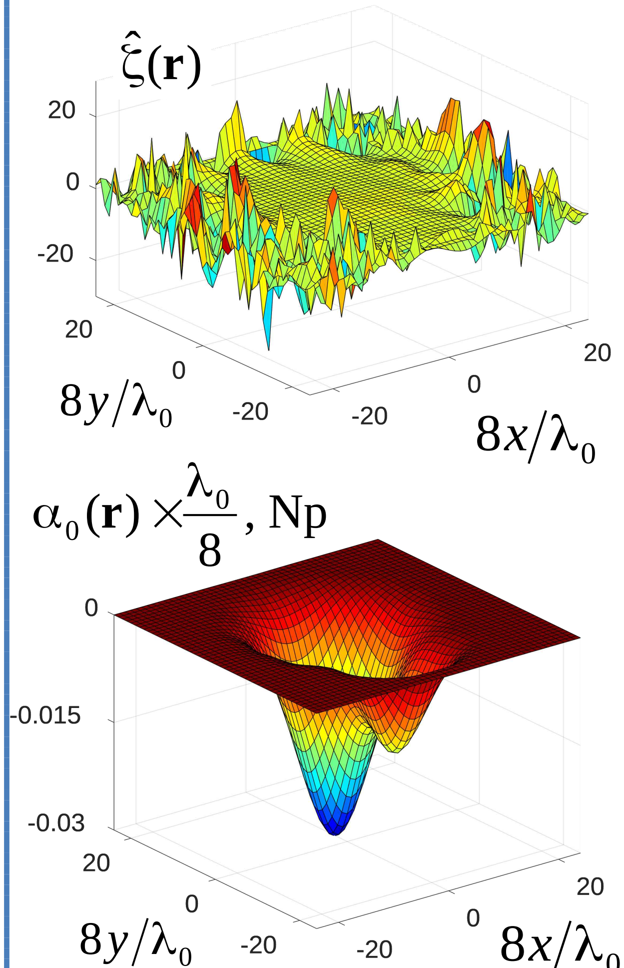


Reconstruction results obtained by using data on 11 frequencies, with noise

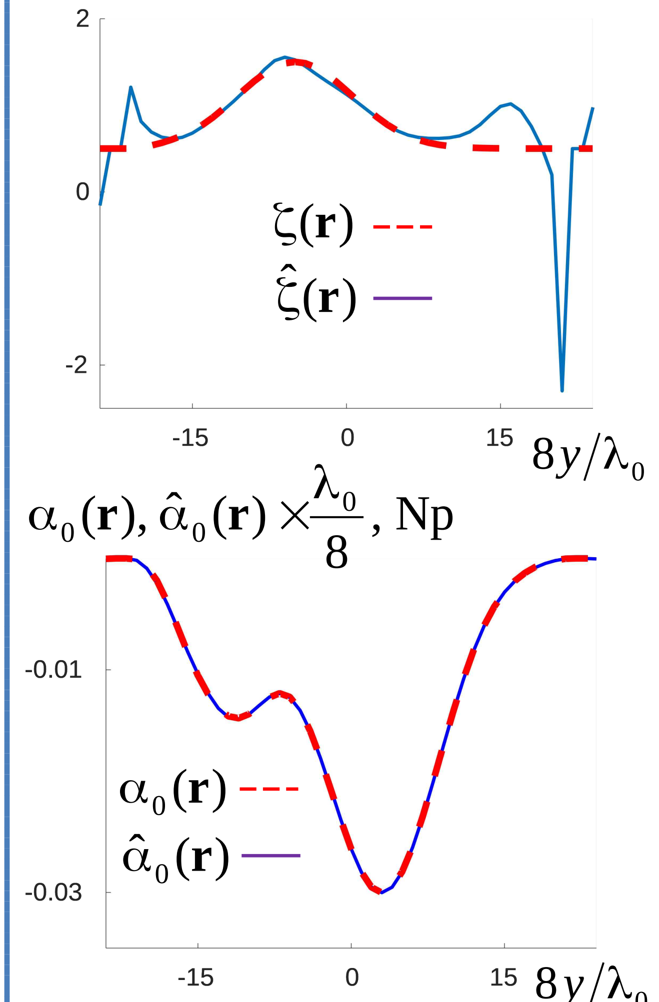
Model



Reconstruction results



Crossections of model and result



Improvements of $\zeta(\mathbf{r})$ reconstruction in multi frequency regime

$$\operatorname{Im} Q(\mathbf{r},\omega_j) = 2\omega_j \left(\frac{\omega_j}{\omega_0}\right)^{\zeta(\mathbf{r})} \frac{\alpha_0(\mathbf{r})}{c(\mathbf{r})}, \Rightarrow \ln\left(\frac{\omega_j}{\omega_k}\right)\left[\zeta(\mathbf{r})+1\right] = \ln\left(\frac{\operatorname{Im} Q(\mathbf{r},\omega_j)}{\operatorname{Im} Q(\mathbf{r},\omega_k)}\right),$$

LSM estimation of $\zeta(\mathbf{r})$.

If $\alpha_0(\mathbf{r}) \to 0$, then it is impossible to reconstruct $\zeta(\mathbf{r})$ since

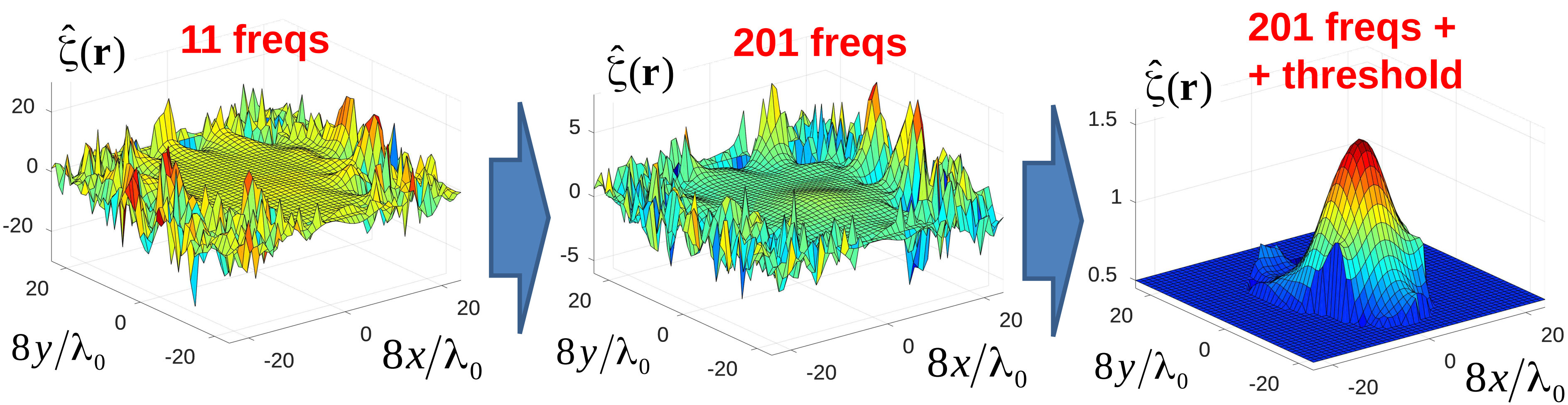
$$\frac{\operatorname{Im} Q(\mathbf{r},\omega_j)}{\omega_j} \cong \frac{\operatorname{Im} Q(\mathbf{r},\omega_k)}{\omega_k},$$ instabilities arrise in

To exclude such points $\mathbf{r}$, the threshold Π can be applied:

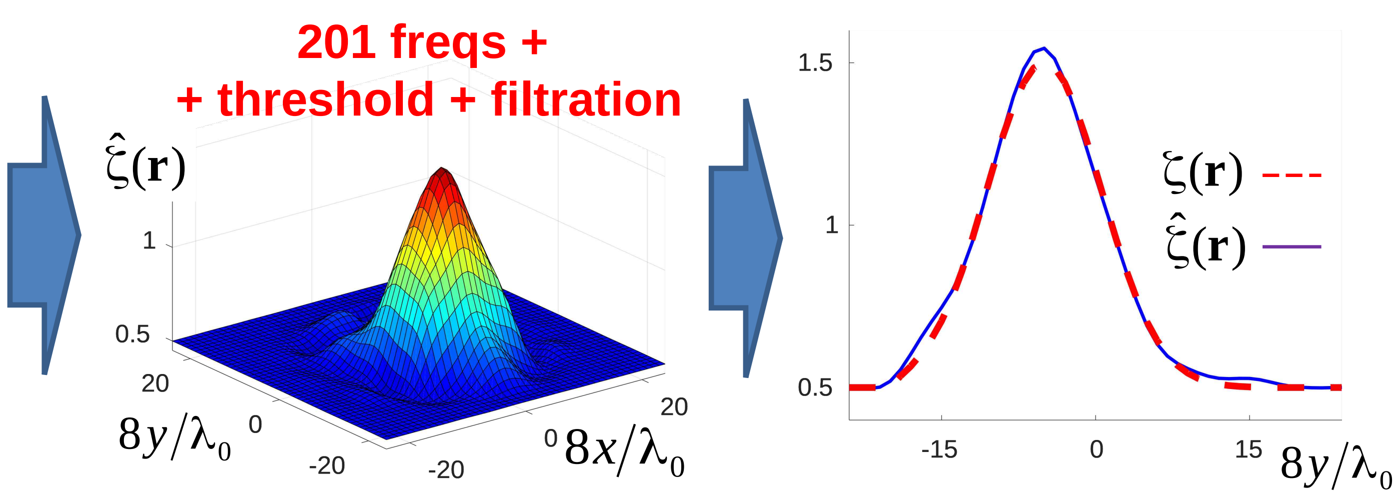
$$\left| \left\langle \frac{\operatorname{Im} Q(\mathbf{r},\omega_j)}{\omega_j} \right\rangle_j \right| \le \Pi \cong 2\frac{\overline{\alpha}_0^{water}}{c_0}$$

Acceptable reconstruction of $c(\mathbf{r})$ helps for acceptable reconstruction of $\alpha_0(\mathbf{r})$: $\alpha_0(\mathbf{r}) = \frac{c(\mathbf{r})}{2\,\omega_j} \operatorname{Im} Q(\mathbf{r},\omega_j)\left(\frac{\omega_0}{\omega_j}\right)^{\zeta(\mathbf{r})}$

Improvements of $\zeta(\mathbf{r})$ reconstruction in multi frequency regime



Filtration:
- space-spectrum,
- background values.



Reconstruction

of all scatterer's components

$c(\mathbf{r}),\ \rho(\mathbf{r}),\ \alpha_0(\mathbf{r}),\ \zeta(\mathbf{r}),\ \boldsymbol{v}(\mathbf{r})$

Reconstruction process

$$\begin{cases} \operatorname{Re} Q(\mathbf{r},\omega_1) = f_1 - \omega_1^2 \, f_2, \\ \operatorname{Re} Q(\mathbf{r},\omega_2) = f_1 - \omega_2^2 \, f_2, \\ \cdot \quad \cdot \quad \cdot \end{cases} \Rightarrow \quad f_1, \;\; f_2 \quad \Rightarrow \quad c(\mathbf{r}), \;\; \rho(\mathbf{r})$$

Procedure for the reconstruction of sound speed and density is the same as for $c(\mathbf{r}), \;\rho(\mathbf{r}), \;\alpha(\mathbf{r},\omega)$.

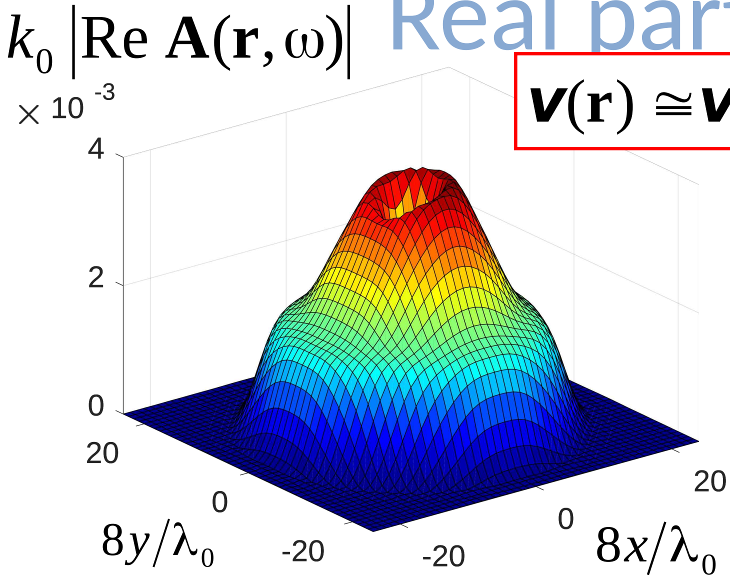
$$\boldsymbol{v}(\mathbf{r}) = \boldsymbol{v}^{\text{div}}(\mathbf{r}) + \boldsymbol{v}^{\text{rot}}(\mathbf{r}), \;\; \operatorname{div} \boldsymbol{v}^{\text{div}} \equiv 0, \;\; \operatorname{rot} \boldsymbol{v}^{\text{rot}} \equiv 0, \;\; \boldsymbol{v}^{\text{div}}(\mathbf{r}) = c^2(\mathbf{r}) \frac{\mathbf{A}^{\text{div}}(\mathbf{r},\omega_j)}{\omega_j}$$

If $\boldsymbol{v}(\mathbf{r}) \cong \boldsymbol{v}^{\text{div}}(\mathbf{r}), \quad \dfrac{|\boldsymbol{v}|}{c} << 1$ (for example, this is valid in ocean applications), then:

1. $\boldsymbol{v}(\mathbf{r}) \cong c^2(\mathbf{r}) \dfrac{\mathbf{A}^{\text{div}}(\mathbf{r},\omega_j)}{\omega_j}$,

2. $\alpha_0(\mathbf{r}) \cong \dfrac{c(\mathbf{r})}{2\,\omega_j} \operatorname{Im} Q(\mathbf{r},\omega_j) \left(\dfrac{\omega_0}{\omega_j} \right)^{\bar{\xi}}$, if $\left| \nabla \cdot \dfrac{\boldsymbol{v}(\mathbf{r})}{c^2(\mathbf{r})} - \dfrac{\boldsymbol{v}(\mathbf{r}) \cdot \nabla \ln \rho(\mathbf{r})}{c^2(\mathbf{r})} \right| << \dfrac{\alpha(\mathbf{r},\omega)}{c(\mathbf{r})}$

Real part of vector inhomogeneity with the vortex form



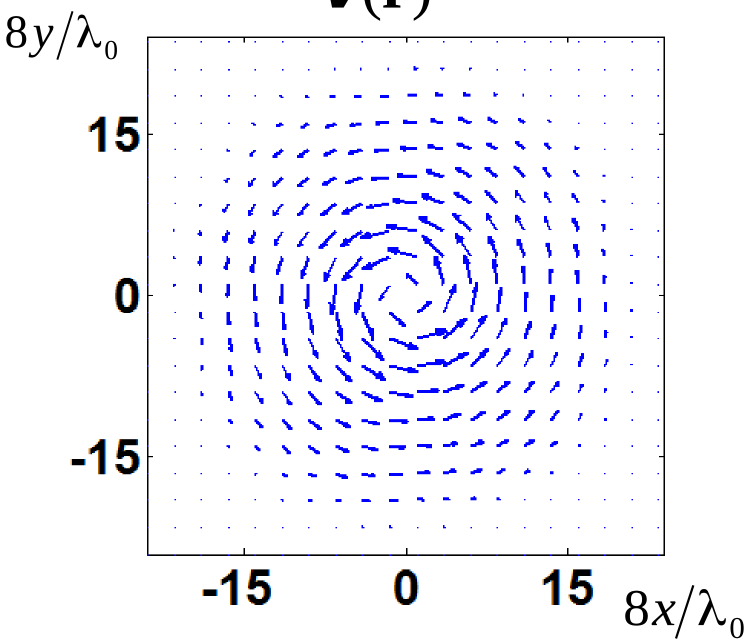
$\boldsymbol{v}(\mathbf{r}) \cong \boldsymbol{v}^{\mathrm{div}}(\mathbf{r})$

Normally distributed noise with rms amplitude deviation:

$$0.03\star\sqrt{\int_Y d\mathbf{x} \int_Y d\mathbf{y}\,\left|G_{\mathrm{sc}}(\mathbf{y},\mathbf{x};\,\omega_j)\right|^2 \Big/ \int_Y d\mathbf{x} \int_Y d\mathbf{y}}$$

Model

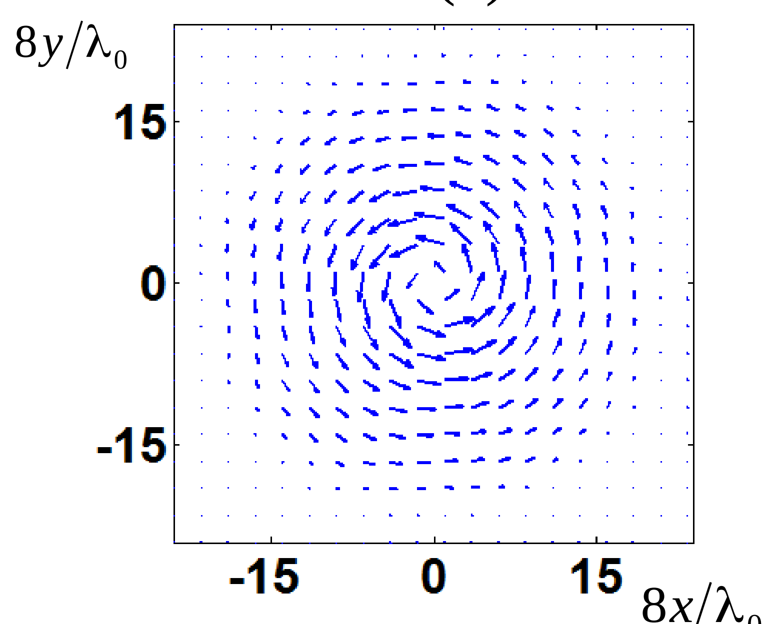
$\boldsymbol{v}(\mathbf{r})$



Reconstruction results

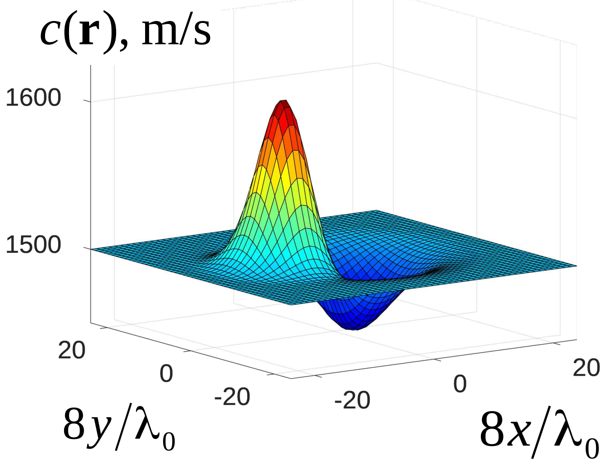
$\hat{\boldsymbol{v}}^{\mathrm{div}}(\mathbf{r})$

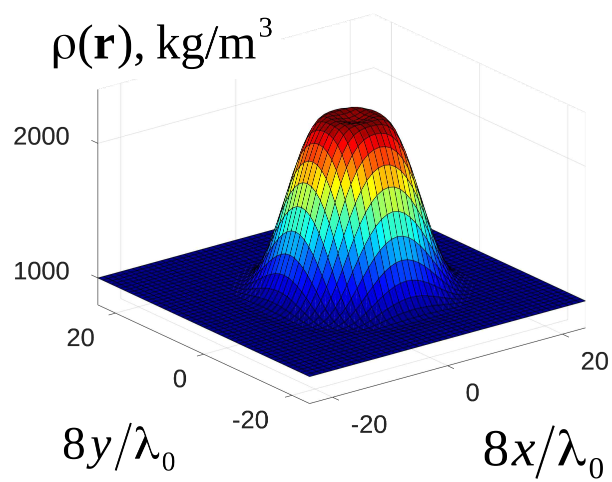
201 freqs



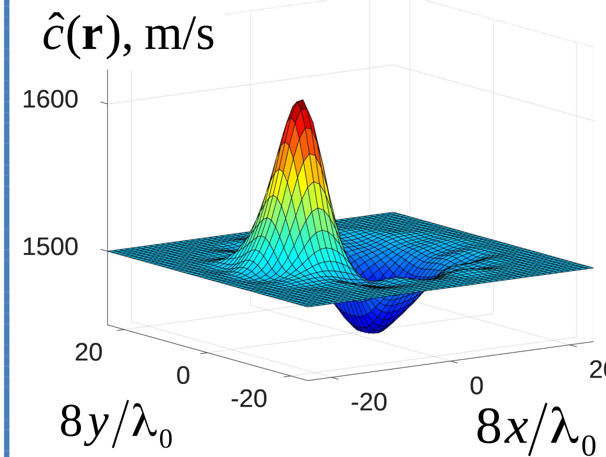
Reconstruction results obtained by using data on 201 frequencies, with noise

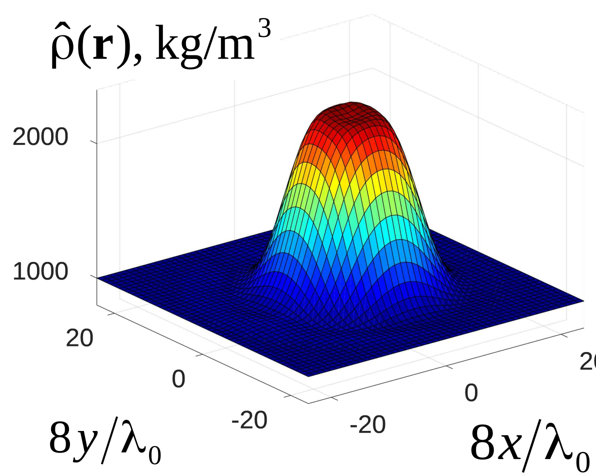
Model



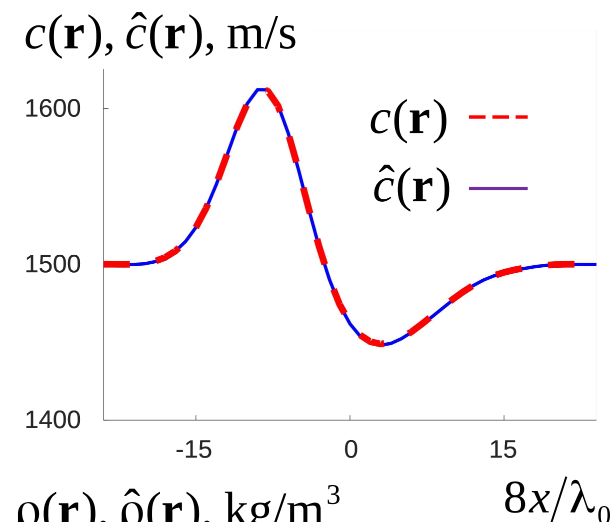


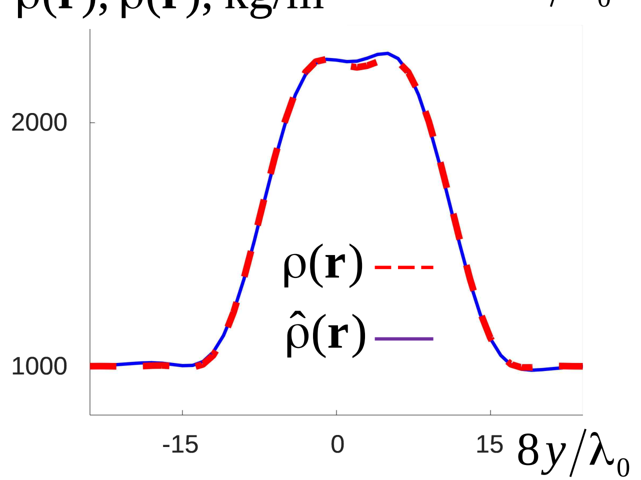
Reconstruction results





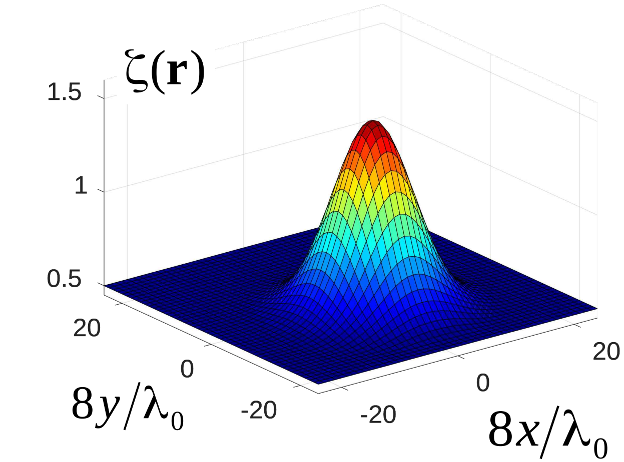
Crossections of model and result

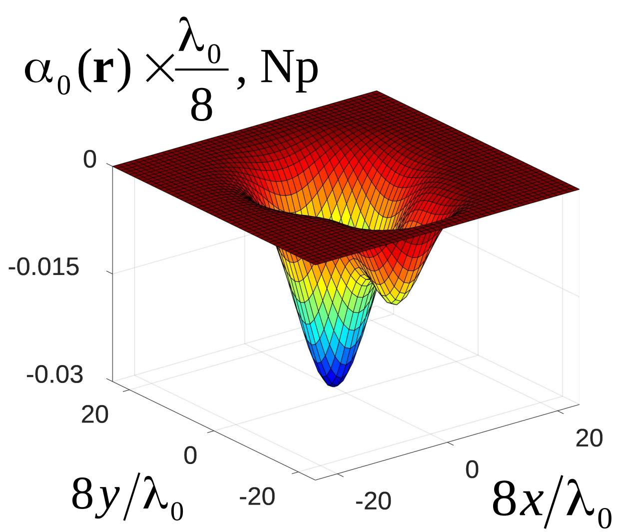




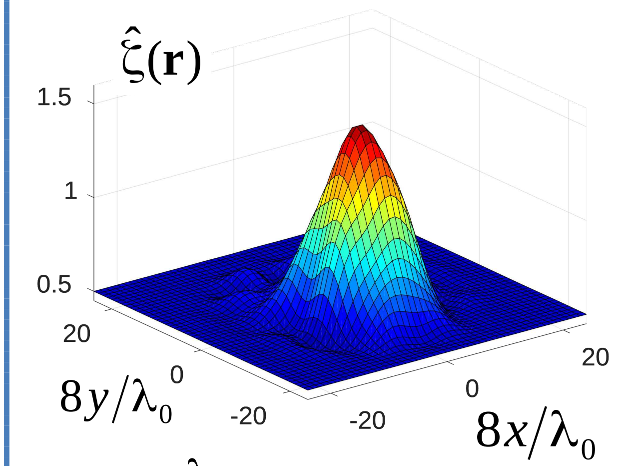
Reconstruction results obtained by using data on 201 frequencies, with noise

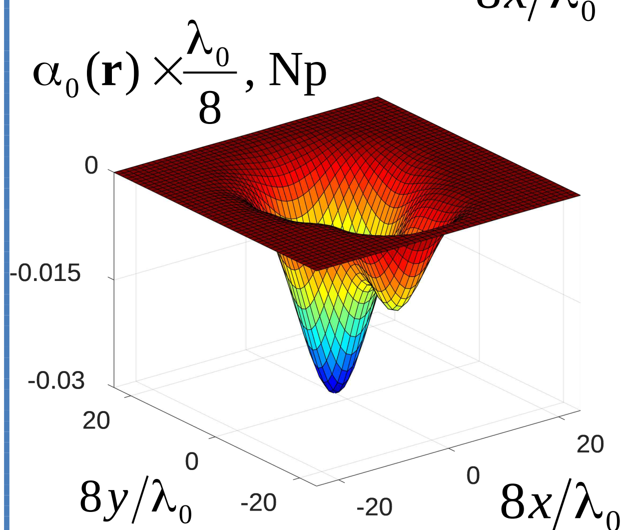
Model



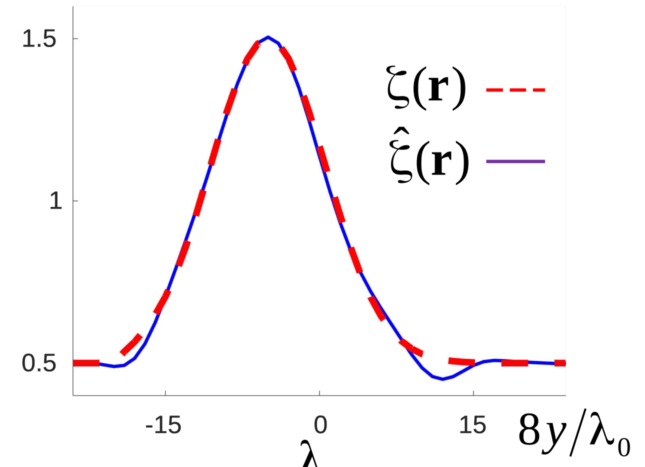


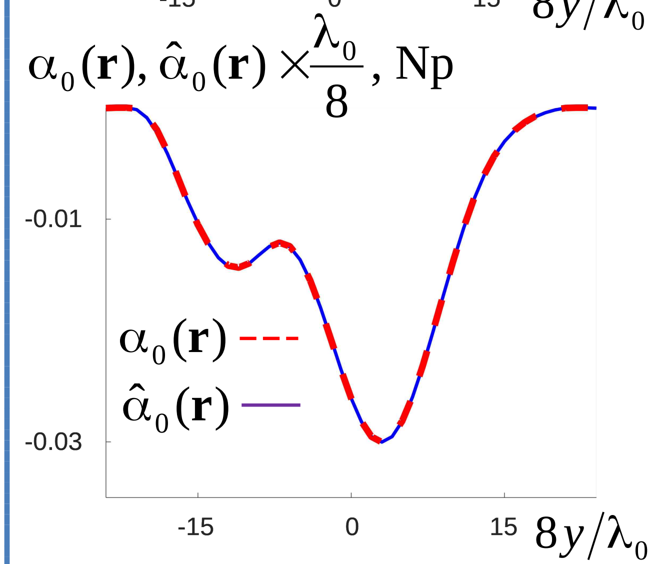
Reconstruction results





Crossections of model and result





Conclusions

1. The considered algorithm of the joint reconstruction of different scatterer's components based on the Novikov-Agaltsov algorithm, shows a high resolution and good noise stability, that makes it perspective for the development of practical schemes of acoustic tomography in different applications such as:

- reconstruction of $c(\mathbf{r}),\ \rho(\mathbf{r}),\ \alpha_0(\mathbf{r}),\ \zeta(\mathbf{r})$ in medical diagnostics, when influence of flows can be negligible,
- reconstruction of $c(\mathbf{r}),\ \rho(\mathbf{r}),\ \alpha_0(\mathbf{r}),\ \zeta(\mathbf{r}),\ \boldsymbol{v}(\mathbf{r})$ in ocean applications, when $\boldsymbol{v}(\mathbf{r}) \cong \boldsymbol{v}^{\mathrm{div}}(\mathbf{r}),\ |\boldsymbol{v}|/c \ll 1.$

2. Numerical modeling shows the better noise stability for the reconstruction of $c(\mathbf{r}),\ \rho(\mathbf{r}),\ \alpha_0(\mathbf{r}),\ \boldsymbol{v}^{\mathrm{div}}(\mathbf{r})$, while $\zeta(\mathbf{r}),\ \boldsymbol{v}^{\mathrm{rot}}(\mathbf{r})$ requires additional multi frequency scattering data and a priory information about reconstructed functions.

THANK YOU!