On Nuttall's partition of a three-sheeted Riemann surface and limit zero distribution of Hermite–Padé polynomials

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The Second International Conference "Quasilinear Equations, Inverse Problems and their Applications"
in memory of Gennadi Markovich Henkin
(Dolgoprudny, Russia, September 12–15, 2016)

Main Subject of the Talk. Multivalued Analytic Functions

Let f be a multivalued analytic function on $\overline{\mathbb{C}}$ with a finite set $\Sigma_f = \Sigma = \{a_1, \dots, a_p\}$ of branch points, i.e. $f \in \mathscr{A}(\overline{\mathbb{C}} \setminus \Sigma)$ but f is not a (single valued) meromophic function in $\overline{\mathbb{C}} \setminus \Sigma$.

Notation $\mathscr{A}^\circ(\overline{\mathbb{C}} \setminus \Sigma) := \mathscr{A}(\overline{\mathbb{C}} \setminus \Sigma) \setminus \mathscr{M}(\overline{\mathbb{C}} \setminus \Sigma)$.

Let fix a point $z_0 \notin \Sigma$, and let $\mathbf{f} = (\mathbf{f}, z_0)$ be a germ of $\mathbf{f}$ at the point z_0, i.e., power series (p.s.) at $z = z_0$

$$\mathbf{f}(z) = \sum_{k=0}^{\infty} c_k (z - z_0)^k. \tag{1}$$

In other words

$$(\mathbf{f}, z_0) \cong (z_0, \{c_k\}_{k=0}^{\infty}). \tag{2}$$

Analytic Continuation. General Concepts

Let we are given a germ $\mathbf{f} = (z_0, \{c_k\}_{k=0}^{\infty})$ of the multivalued analytic function $f \in \mathscr{A}^{\circ}(\overline{\mathbb{C}} \setminus \Sigma)$.

All the global properties of f can be recovered from these local data, i.e. from a given germ $\mathbf{f}$.

Problem of "recovering" some of global data from the local ones.

An example.

Cauchy–Hadamard Formulae

for the radius of convergence $R = R(f)$ of given p.s. $\mathbf{f}$. Let

$$\frac{1}{R} = \varlimsup_{k \to \infty} |c_k|^{1/k}.$$

Then p.s. $\mathbf{f}$ converges for $|z - z_0| < R$.

Analytic Continuation. General Concepts

Fabry Ratio Theorem (1896)

Let $\mathbf{f} \in \mathscr{H}(z_0)$,

$$\mathbf{f}(z) = \sum_{k=0}^{\infty} c_k (z - z_0)^k.$$

Let

$$\frac{c_k}{c_{k+1}} \to s, \quad k \to \infty, \quad s \in \mathbb{C}^* := \mathbb{C} \setminus \{0\}.$$

Then $R = |s|$ *and s is a singular point of* $\mathbf{f}(z)$, $|z - z_0| < R$ *on the circle* $|z - z_0| = R$.

Analytic Continuation. Padé Approximants

Let $z_0 = 0$, $\mathbf{f} \in \mathscr{H}(0)$.

For fixed $n, m \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ we seek for two polynomials $P_{n,m}$, $Q_{n,m}$, $\deg P_{n,m} \leqslant n$, $\deg Q_{n,m} \leqslant m$, $Q_{n,m} \not\equiv 0$, and such

$$(Q_{n,m}\mathbf{f} - P_{n,m})(z) = O\left(z^{n+m+1}\right), \quad z \to 0.$$

In generic case

$$\mathbf{f}(z) - \frac{P_{n,m}}{Q_{n,m}}(z) = O\left(z^{n+m+1}\right). \quad z \to 0, \tag{3}$$

From (3) it follows

$$\frac{P_{n,m}}{Q_{n,m}}(z) = c_0 + c_1 z + \cdots + c_{n+m} z^{n+m} + O\left(z^{n+m+1}\right), \quad z \to 0.$$

Rational function $[n/m]_{\mathbf{f}}(z) := P_{n,m}(z)/Q_{n,m}(z)$ is called the Padé approximant of type (n, m) to p.s. $\mathbf{f}$ at the point $z = 0$.

Analytic Continuation. Padé Approximants. Row Sequences

Padé Table for $\mathbf{f}$:

$$\left\{[n/m]_{\mathbf{f}}\right\}_{n,m=0}^{\infty}.$$

When $m \in \mathbb{N}_0$ is fixed we have the m-th row of Padé Table.
When $n = m$ we have the n-th diagonal PA sequence $[n/n]_{\mathbf{f}}$.

Let $m = 1$ then $Q_{n,1}(z) = z - \zeta_{n,1}$, where $\zeta_{n,1} = c_n/c_{n+1}$.

Fabry Theorem Interpretation

Let $m = 1$ and

$$\zeta_{n,1} \to s \in \mathbb{C}^*, \quad n \to \infty.$$

Then $\mathbf{f} \in \mathscr{H}(|z| < |s|)$ and s is a singular point of $\mathbf{f}(z), |z| < |s|$.

Padé Approximants (PA). Row Sequences

Theorem (Suetin, 1981)

Let $\mathbf{f} \in \mathscr{H}(0)$ *and* $m \in \mathbb{N}$ *is fixed.*
Suppose that for each $n \geqslant n_0$ *PA* $[n/m]_{\mathbf{f}}$ *has exactly* m *finite poles* $\zeta_{n,1}, \dots, \zeta_{n,m}$ *such that*

$$\zeta_{n,j} \to a_j \in \mathbb{C} \setminus \{0\}, \quad n \to \infty, \quad j = 1, \dots, m,$$

where

$$0 < |a_1| \leqslant \dots \leqslant |a_{\mu-1}| < |a_\mu| = \dots = |a_m| = R.$$

Then
1) $\mathbf{f}(z)$ *has meromorphic continuation into* $|z| < R$*, all the points* $a_1, \dots, a_{\mu-1}$ *are the only poles of* $\mathbf{f}$ *in* $|z| < R$*;*
2) all the points $a_\mu, \dots, a_m$ *are singular points of* $\mathbf{f}$ *on the* $|z| = R$*.*

Pólya Theorem

Let $\mathbf{f} \in \mathscr{H}(\infty)$, $\mathbf{f}(z) = \sum_{k=0}^{\infty} \frac{c_k}{z^{k+1}}$, $\mathbf{f}$ is a germ of $f \in \mathscr{A}^\circ(\overline{\mathbb{C}} \setminus \Sigma)$.

Denote

$$A_n(\mathbf{f}) := \begin{vmatrix} c_0 & c_1 & \dots c_{n-1} \\ c_1 & c_2 & \dots c_n \\ \dots & \dots & \dots \\ c_{n-1} & c_n & \dots c_{2n-2} \end{vmatrix}.$$

For a compact set $K \subset \mathbb{C}$ denote by $\Omega(K) \ni \infty$ the infinite component of $\overline{\mathbb{C}} \setminus K$. Let $d(K)$ be the transfinite diameter of K.

Pólya Theorem (1929)

Let $\mathbf{f} \in \mathscr{H}(\Omega(K))$ *where* $K \subset \mathbb{C}$ *is a compact set. Then*

$$\varlimsup_{n\to\infty} \left|A_n(\mathbf{f})\right|^{1/n^2} \leqslant d(K) = \operatorname{cap}(K).$$

Analytic Continuation. Some Conclusions

In each case we should know the infinite vector $\mathbf{c} = (c_0, \dots, c_k, \dots)$ of all Taylor coefficients of the given germ $\mathbf{f}$.
Any finite set $\mathbf{c}_N = (c_0, \dots, c_N)$ is not enough for the conclusions about any global property of $f \in \mathscr{A}^\circ(\overline{\mathbb{C}} \setminus \Sigma)$.
To be more precise, we should know an infinite tail $\mathbf{c}^N = (c_{N+1}, c_{N+2}, \dots)$ of $\mathbf{c}$.
Main Question: In what way can we use the local data

$$\mathbf{f}(z) = \sum_{k=0}^{\infty} \frac{c_k}{z^{k+1}}$$

to discover some of the global properties of $f \in \mathscr{A}^\circ(\overline{\mathbb{C}} \setminus \Sigma)$?

Stahl Theory (1985–1986) for Diagonal PA. Stahl Compact Set S

Let $\mathbf{f} \in \mathscr{H}(\infty)$ be a germ of $f \in \mathscr{A}^\circ(\overline{\mathbb{C}} \setminus \Sigma)$.

Theorem 1

Denote by $\mathscr{D} := \{G : G \text{ is a domain}, G \ni \infty, \mathbf{f} \in \mathscr{H}(G)\}$. Then
1) there exists a unique "maximal" domain $D = D(\mathbf{f}) \in \mathscr{D}$, i.e.,

$$\operatorname{cap}(\partial D) = \inf\Big\{\operatorname{cap}(\partial G) : G \in \mathscr{D}\Big\};$$

2) there exists a finite set $e = e(f)$, such that the compact set
$S := \partial D \setminus e = \bigcup\limits_{j=1}^{q} S_j$, and possesses the following S-property

$$\frac{\partial g_S(z,\infty)}{\partial n^+} = \frac{\partial g_S(z,\infty)}{\partial n^-}, \quad z \in S^\circ = \bigsqcup_{j=1}^{q} S_j^\circ,$$

S_j° is the open arc of S_j, $g_S(z,\infty)$ is Green's function for $D(\mathbf{f})$.

Stahl Theory (1985–1986) for Diagonal PA. Convergence of PA

Theorem 2

For the diagonal PA $[n/n]_{\mathbf{f}} = P_n/Q_n$ *of* $\mathbf{f}$ *we have as* $n \to \infty$

$$[n/n]_{\mathbf{f}}(z) \xrightarrow{\text{cap}} \mathbf{f}(z), \quad z \in D = D(\mathbf{f}); \tag{4}$$

the rate of convergence in (4) *is given by*

$$\left|\mathbf{f}(z) - [n/n]_{\mathbf{f}}(z)\right|^{1/n} \xrightarrow{\text{cap}} e^{-2g_S(z,\infty)} < 1, \quad z \in D, \quad n \to \infty;$$

the following representation holds true in $D(\mathbf{f})$*:*

$$\mathbf{f}(z) \overset{\text{cap}}{=} [N_0/N_0](z) + \sum_{n=N_0}^{\infty} \frac{A_n}{(Q_n Q_{n+1})(z)}, \quad z \in D(\mathbf{f}).$$

Stahl Theory (1985–1986) for Diagonal PA. Structure of S

Theorem 3

Compact set S consists of the trajectories of a quadratic differential

$$S = \left\{ z \in \mathbb{C} : \operatorname{Re} \int_a^z \sqrt{\frac{V_{p-2}}{A_p}(\zeta)}\, d\zeta = 0 \right\},$$

$$A_p(z) = \prod_{j=1}^{p'} (z - a_j^*),\ \{a_1^*, \dots, a_{p'}^*\} \subset \Sigma = \{a_1, \dots, a_p\},\ p' \leqslant p,$$

$$V_{p-2}(z) = \prod_{j=1}^{p'-2} (z - v_j),$$ *v_j are the Chebotarëv points of S.*

In general $\Sigma \setminus \{a_1^*, \dots, a_{p'}^*\} \neq \varnothing$. Thus these points are "invisible" for PA. Stahl terminology: points of $\{a_1^*, \dots, a_{p'}^*\}$ are "active" branch points of f, the other points are "inactive" branch points of f.

Classical PA. Stahl's Theory: Numerical Examples

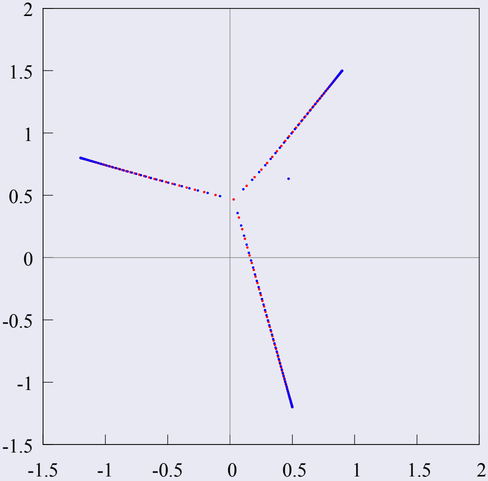

Figure 1: Zeros and poles of the PA $[130/130]_{\mathbf{f}}$ for $f(z) = \big(z - (-1.2 + 0.8i)\big)^{1/3}\big(z - (0.9 + 1.5i)\big)^{1/3}\big(z - (0.5 - 1.2i)\big)^{-2/3}$.

Classical PA. Stahl's Theory: Numerical Examples

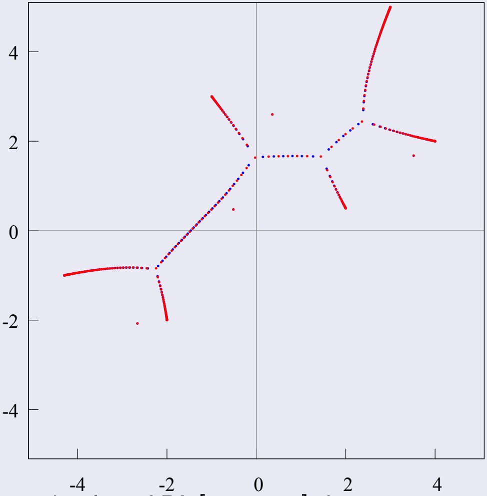

Figure 2: Zeros and poles of PA $[267/267]_{\mathbf{f}}$ for $f(z) = \{(z + (4.3 + 1.0i))(z - (2.0 + 0.5i))(z + (2.0 + 2.0i))(z + (1.0 - 3.0i))(z - (4.0 + 2.0i))(z - (3.0 + 5.0i))\}^{-1/6}$.

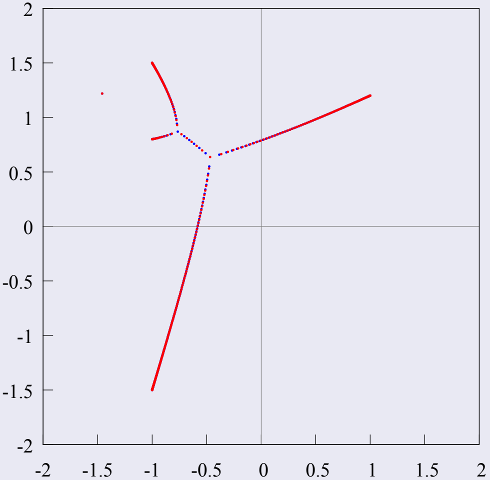

Figure 3: Zeros and poles of PA $[300/300]_{\mathbf{f}}$ for

$$f(z)=\left(\frac{z-(-1.0+i\cdot .8)}{z-(1.0+i\cdot 1.2)}\right)^{1/2}+\left(\frac{z-(-1.0+i\cdot 1.5)}{z-(-1.0-i\cdot 1.5)}\right)^{1/2}.$$

Classical PA. Stahl's Theory: Numerical Examples

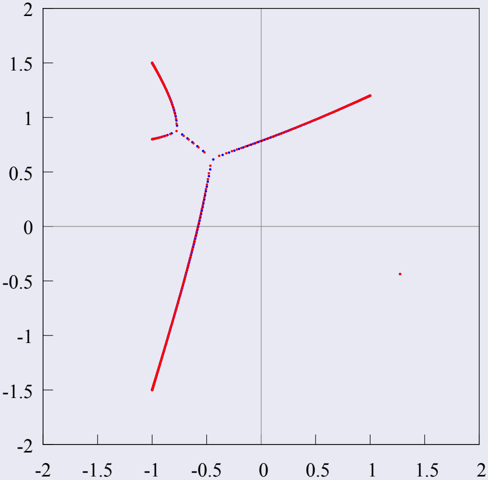

Figure 4: Zeros and poles of PA $[300/300]_{\mathbf{f}}$ for

$$f(z)=\log\left(\frac{z-(-1.0+i\cdot .8)}{z-(1.0+i\cdot 1.2)}\right)+\log\left(\frac{z-(-1.0+i\cdot 1.5)}{z-(-1.0-i\cdot 1.5)}\right).$$

PA for Hyperelliptic Functions. Nuttall's Approach (1975–1977)

Let $f \in \mathbb{C}(z,w)$, $w^2 = \prod_{j=1}^{2g+2}(z-e_j)$. Then all the branch points $e_1,\dots,e_{2g+2}$ are active.

Let $\pi\colon \mathfrak{R}_2 \to \overline{\mathbb{C}}$ be the two-sheeted Riemann surface (RS) of function w, $\mathbf{z}=(z,w)\in\mathfrak{R}_3$ be a point on $\mathfrak{R}_2$, $\pi(\mathbf{z})=z$. Let $\{\infty^{(1)},\infty^{(2)}\}=\pi^{-1}(\infty)$

$$G(\mathbf{z}) := -\int_{e_1}^{\mathbf{z}} \frac{z^g+\cdots}{w}\,dz$$

be the canonical Abelian integral of 3-rd kind, which periods are all pure imaginary, i.e.,

$$\begin{aligned} G(\mathbf{z}) &= -\log z + \text{regular part}, \quad &\mathbf{z}\to\infty^{(1)},\\ G(\mathbf{z}) &= \log z + \text{regular part}, \quad &\mathbf{z}\to\infty^{(2)}. \end{aligned}$$

PA for Hyperelliptic Functions. Nuttall's Approach (1975–1977)

The inequality

$$G(z^{(1)}) < G(z^{(2)}) \tag{5}$$

define the global partition of $\mathfrak{R}_2$ onto two open sheets $\mathfrak{R}^{(1)} \ni z^{(1)}$ and $\mathfrak{R}^{(2)} \ni z^{(2)}$ such that:

1) both of sheets are domains, $\pi(\mathfrak{R}^{(1)}) = \pi(\mathfrak{R}^{(2)}) = D = D(\mathbf{f})$ is Stahl domain for each $f \in \mathbb{C}(z, w)$,

2) for the boundary set $\Gamma = \partial\mathfrak{R}^{(1)} = \partial\mathfrak{R}^{(2)}$ we have $\pi(\Gamma) = S$.

$$g_S(z, \infty) = -\operatorname{Re} G(z^{(1)}), \quad z \notin S.$$

The partition (5) is called Nuttall's partition of a two-sheeted RS $\mathfrak{R}_2$. When $\mathbf{f} \in \mathscr{H}(\infty)$ is a germ of a hyperelliptic function, all Stahl's theorems follows from Nuttall's partition (5) (Nuttall, 1975–1977).

Stahl's Theory. Active and Inactive Points. Model Class

Let $\varphi(z) := z + \sqrt{z^2-1}$, $z \in \overline{\mathbb{C}} \setminus [-1,1]$, is the inverse of Zhukowsky function; here $\sqrt{z^2-1}/z \to 1$ as $z \to \infty$. Thus $|\varphi(z)| > 1$ when $z \notin \Delta := [-1,1]$.

Let

$$f(z) := \prod_{j=1}^{m} f(z; A_j, \alpha_j) = \prod_{j=1}^{m} \left(A_j - \frac{1}{\varphi(z)}\right)^{\alpha_j}, \tag{6}$$

where $m \geqslant 2$, $A_j \in \mathbb{C}$, $|A_j| > 1$, $\alpha_j \in \mathbb{C} \setminus \mathbb{Z}$, $\sum\limits_{j=1}^{m} \alpha_j = 0$.

Then $f \in \mathscr{A}^{\circ}(\overline{\mathbb{C}} \setminus \Sigma)$, $\Sigma = \{\pm 1, a_1, \dots, a_m\}$, $a_j := (A_j + 1/A_j)/2$.
Let $\mathscr{Z}$ be the class of all functions f of type (6).
For each germ $\mathbf{f} \in \mathscr{H}(\infty)$ of $f \in \mathscr{Z}$ we have Stahl compact set $S = [-1,1]$ and $D = D(\mathbf{f}) = \overline{\mathbb{C}} \setminus [-1,1]$. From Stahl Theory it follows that the points ± 1 are the only active points. All the points $\{a_1, \dots, a_m\}$ are inactive singular points.
Question: Is it possible to recover $\{a_1, \dots, a_m\}$ from the germ $\mathbf{f}$?

Active and Inactive Points. Hermite–Padé Polynomials

For a germ $\mathbf{f} \in \mathscr{H}(\infty)$ of $f \notin \mathbb{C}(z)$ let define Padé polynomials

$$(P_{n,0} + P_{n,1}\mathbf{f})(z) = O\left(\frac{1}{z^{n+1}}\right), \quad z \to \infty. \tag{7}$$

Then $[n/n]_{\mathbf{f}} = -P_{n,0}/P_{n,1}$.
If f is not a hyperelliptic function, let define Hermite–Padé polynomials $Q_{n,j}$, $j = 0, 1, 2$, of degree n for $\mathbf{f}, \mathbf{f}^2$ as

$$(Q_{n,0} + Q_{n,1}\mathbf{f} + Q_{n,2}\mathbf{f}^2)(z) = O\left(\frac{1}{z^{2n+2}}\right), \quad z \to \infty. \tag{8}$$

The construction (8) of HP polynomials is based just on the same germ **f** as the construction (7) but involves $\mathbf{f}, \mathbf{f}^2$ instead of **f**.
Does the construction of HP has any advantages before the construction (7) as $n \to \infty$?

Hermite–Padé Polynomials. Cubic Functions

Let w be given by the equation

$$w^3 + r_2(z)w^2 + r_1(z)w + r_0(z) = 0, \quad r_0, r_1, r_2 \in \mathbb{C}(z).$$

Let $\pi\colon \mathfrak{R}_3 \to \overline{\mathbb{C}}$ be the three-sheeted RS of the w, $\mathbf{z} = (z, w)$, $\pi(\mathbf{z}) = z$. Suppose $\infty \notin \Sigma_w$, $\pi^{-1}(\infty) = \{\infty^{(1)}, \infty^{(2)}, \infty^{(3)}\}$.
Let $\mathscr{U}(\mathbf{z})$ be a unique Abelian integral of 3-rd kind on $\mathfrak{R}_3$ which periods are pure imaginary and such that

$$\mathscr{U}(\mathbf{z}) = -2\log z + \text{regular part}, \quad \mathbf{z} \to \infty^{(1)},$$
$$\mathscr{U}(\mathbf{z}) = \log z + \text{regular part}, \quad \mathbf{z} \to \infty^{(j)}, \quad j = 2, 3.$$

Let define the open subsets $\mathfrak{R}^{(1)}, \mathfrak{R}^{(2)}$ and $\mathfrak{R}^{(3)}$ of $\mathfrak{R}_3$ by

$$\operatorname{Re}\mathscr{U}(z^{(1)}) < \operatorname{Re}\mathscr{U}(z^{(2)}) < \operatorname{Re}\mathscr{U}(z^{(3)}), \quad z^{(j)} \in \mathfrak{R}^{(j)}. \tag{9}$$

The (9) is similar to $G(z^{(1)}) < G(z^{(2)})$ for a $\mathfrak{R}_2$.

Hermite–Padé Polynomials. Cubic Functions

Nuttall's Partition of $\mathfrak{R}_3$

When 1-st sheet $\mathfrak{R}^{(1)}$ is a domain the partition

$$\operatorname{Re}\mathscr{U}(z^{(1)}) < \operatorname{Re}\mathscr{U}(z^{(2)}) < \operatorname{Re}\mathscr{U}(z^{(3)}), \quad z^{(j)} \in \mathfrak{R}^{(j)}.$$

is called Nuttall's Partition of $\mathfrak{R}_3$ (with respect to $\mathbf{z} = \infty^{(1)}$).

Nuttall's Conjecture (1984)

Nuttall's Partition exists for every $\mathfrak{R}_3$.

In general the Conjecture is still open problem (up to some trivial cases).

Hermite–Padé Polynomials. Cubic Functions

Theorem (Komlov, Kruzhilin, Palvelev, Suetin, 2016)

There exists a natural subclass $\mathscr{C}$ of cubic functions such that:
1) for each $w \in \mathscr{C}$ Nuttall's Conjecture holds true,
2) let $f \in \mathbb{C}(z, w) \setminus \mathbb{C}(z)$ and $\mathbf{f} \in \mathscr{H}(\infty^{(1)})$, then as $n \to \infty$

$$\frac{Q_{n,1}}{Q_{n,2}}(z) \xrightarrow{\text{cap}} -\left\{\mathbf{f}(z^{(1)}) + \mathbf{f}(z^{(2)})\right\}, \quad \mathbf{z} \in \mathfrak{D},$$

$$\frac{Q_{n,0}}{Q_{n,2}}(z) \xrightarrow{\text{cap}} \mathbf{f}(z^{(1)})\mathbf{f}(z^{(2)}), \quad \mathbf{z} \in \mathfrak{D},$$

where domain $\mathfrak{D} := \overline{\mathfrak{R}^{(1)}} \cup \mathfrak{R}^{(2)}$ and $\mathbf{f}$ is a (single valued) meromorphic function in $\mathfrak{D}$.

For $w \in \mathscr{C}$, $f \in \mathbb{C}(z, w)$, $\mathbf{f} \in \mathscr{H}(\infty^{(1)})$ Stahl's domain $D(\mathbf{f}) = \pi(\mathfrak{R}^{(1)})$

$$\frac{P_{n,0}}{P_{n,1}}(z) \xrightarrow{\text{cap}} -\mathbf{f}(z^{(1)}), \quad z^{(1)} \in \mathfrak{R}^{(1)}.$$

Padé Polynomials $P_{100,0}$, $P_{100,1}$

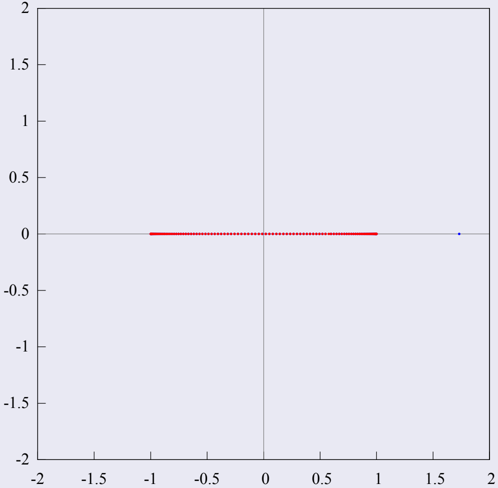

Figure 5: Numerical distribution of zeros and poles of PA $[100/100]_{\mathbf{f}}(z)$.

Hermite–Padé Polynomials $Q_{100,0}, Q_{100,1}, Q_{100,2}$

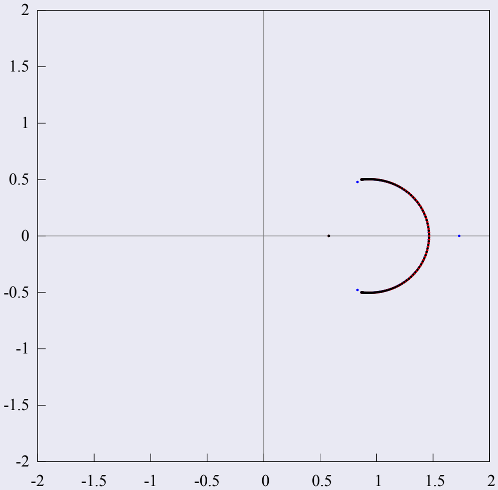

Figure 6: Distribution of zeros of HP polynomials $Q_{100,j}$, $j = 0, 1, 2$.

Numerical example

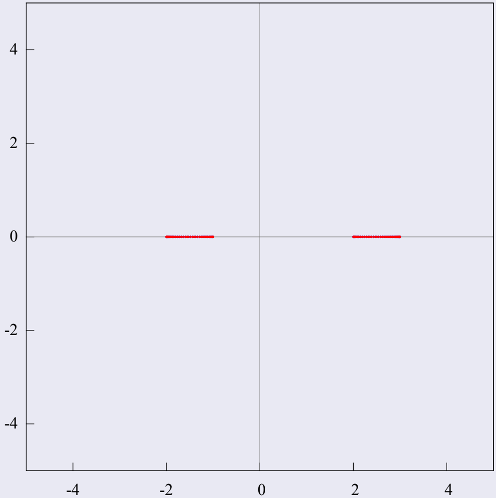

Figure 7: Numerical distribution of poles and zeros of PA $[60/60]_{\mathbf{f}}$.

Numerical example

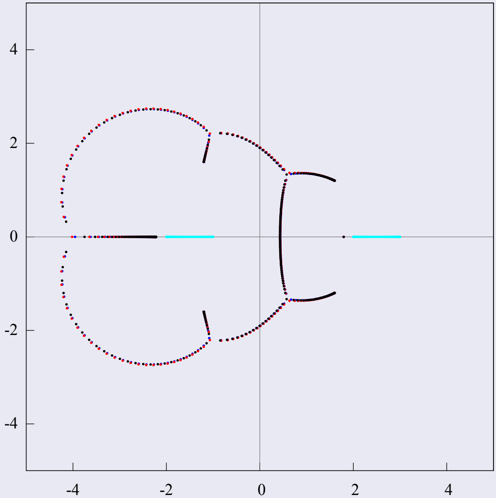

Figure 8: Numerical distribution of zeros $Q_{300,j}$ and R_{300}.

Thank you for your attention !