Symplectic non-squeezing for the discrete nonlinear Schrödinger equation

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“Quasilinear equations, inverse problems and their applications”
dedicated to the memory of Gennadi Henkin
Dolgoprudny, Russia, September 12–15, 2016.
Joint work with Alexandre Sukhov
Let $\mathbb{B}^n$ be the unit ball in $\mathbb{C}^n$; then $\mathbb{D} = \mathbb{B}^1 \subset \mathbb{C}$ is the unit disc. $\mathbb{B}^n(r)$ is the ball of radius $r$. 

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Let $r, R > 0$. Suppose there is a symplectic embedding $F : B^n(r) \to D(R) \times \mathbb{C}^{n-1}$. Then $r \leq R$. 

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- Bourgain (1994-95) proved the result for cubic NLS.

Consider time $t$ flow $F : u(0) \mapsto u(t)$ of the equation

$$iu_t + u_{xx} + |u|^p u = 0, \quad x \in \mathbb{R}/\mathbb{Z}, \quad t > 0.$$ 

Then $F$ is a symplectic transformation of $L^2(0, 1)$, $0 < p \leq 2$. Bourgain proved the non-squeezing property for $p = 2$. For other values of $p$ the question is open.
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We prove a non-squeezing result for a symplectic transformation $F$ of the Hilbert space assuming that the derivative $F'$ is bounded in Hilbert scales. We apply our result to discrete nonlinear Schrödinger equations.
Let $H$ be a complex Hilbert space with fixed orthonormal basis $(e_n)_{n=1}^\infty$. Let $(\theta_n)_{n=1}^\infty$ be a sequence of positive numbers such that $\theta_n \to \infty$ as $n \to \infty$, for example, $\theta_n = n$. For $s \in \mathbb{R}$ we define $H_s$ as a Hilbert space with the following norm:

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The family $(\mathbb{H}_s)$ is called the Hilbert scale corresponding to the basis $(e_n)$ and sequence $(\theta_n)$. We have $\mathbb{H}_0 = \mathbb{H}$. For $s > r$, the space $\mathbb{H}_s$ is dense in $\mathbb{H}_r$, and the inclusion $\mathbb{H}_s \subset \mathbb{H}_r$ is compact.
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**Example.** $\mathbb{H} = L^2(0, 1)$ with the standard Fourier basis, $\theta_n = (1 + n^2)^{1/2}$, $n \in \mathbb{Z}$. Then $\mathbb{H}_s$ is the standard Sobolev space.
Let $\mathbb{B}(r) = \mathbb{B}^\infty(r)$ be the ball of radius $r$ in $\mathbb{H}$.

**Theorem**

Let $r, R > 0$. Let $F : \mathbb{B}(r) \to \mathbb{D}(R) \times \mathbb{H}$ be a symplectic embedding of class $C^1$. Suppose there is $s_0 > 0$ such that for every $|s| < s_0$ the derivative $F'(z)$ is bounded in $\mathbb{H}_s$ uniformly in $z \in \mathbb{B}(r)$. Then $r \leq R$. 
Consider the following system of equations

\[ iu'_n + f(|u_n|^2)u_n + \sum_k a_{nk}u_k = 0. \]  

\( (1) \)

Here \( u(t) = (u_n(t))_{n \in \mathbb{Z}}, u_n(t) \in \mathbb{C}, t \geq 0. \)
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We assume that \( f : \mathbb{R}_+ \to \mathbb{R} \) and its derivative are continuous on the positive reals, furthermore,

\[ \lim_{x \to 0} f(x) = \lim_{x \to 0} \lfloor xf'(x) \rfloor = 0. \]

For example, one can take \( f(x) = x^p \) with real \( p > 0. \) The hypotheses on the function \( f \) are imposed in order for the flow of (1) to be \( C^1 \) smooth.
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Here \( A = (a_{nk}) \) is an infinite matrix independent of \( t. \)
Furthermore, \( A \) is a hermitian matrix, that is, \( a_{nk} = \overline{a_{kn}}. \) For simplicity we also assume that the entries \( a_{nk} \) are uniformly bounded and there exists \( m > 0 \) such that \( a_{nk} = 0 \) if \( |n - k| > m. \)
The equation (1) with \( f(x) = x \) is called the discrete self-trapping equation. The special case with \( a_{nk} = 1 \) if \( |n - k| = 1 \) and \( a_{nk} = 0 \) otherwise, is the discrete nonlinear (cubic) Schrödinger equation:

\[
iu'_n + |u_n|^2 u_n + u_{n-1} + u_{n+1} = 0.
\]

There are other discretizations of the Schrödinger equation, in particular, the Ablowitz-Ladik model that can be treated in a similar way.
Discrete non-linear Schrödinger equation

The equation (1) can be written in the Hamiltonian form:

\[ u'_n = i \frac{\partial H}{\partial u_n}. \]

The Hamiltonian \( H \) is given by

\[ H = \sum_n F(|u_n|^2) + \sum_{n,k} a_{nk} \overline{u_n} u_k, \]

here \( F' = f \) and \( F(0) = 0. \)
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The equation (1) preserves the $l^2(\mathbb{Z})$ norm

$$\|u\|_{l^2} = (\sum_n |u_n|^2)^{1/2}.$$ 

Hence, the flow $u(0) \mapsto u(t)$ of (1) is globally defined on $l^2(\mathbb{Z})$ and preserves the standard symplectic form $\omega = (i/2) \sum_n du_n \wedge d\overline{u_n}$. 
The equation (1) can be written in the Hamiltonian form:

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We verify that our main result applies to (1), hence, the non-squeezing property holds for the flow of (1).
The proof is based on (pseudo) holomorphic discs. A holomorphic disc $z : \mathbb{D} \rightarrow \mathbb{H}, \zeta \mapsto z(\zeta)$ satisfies the Cauchy-Riemann equation

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$$w_{\zeta} = A(w)\overline{w}_{\zeta}.$$ 

Here

$$A = QP^{-1}, \quad P = F_z, \quad Q = F_{\overline{z}}.$$
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Furthermore, if $F$ satisfies the hypotheses of the main theorem, then there is $0 < a < 1$ and $s_1 > 0$ such that for all $z \in \mathbb{B}(r)$ and $0 \leq s \leq s_1$ we have $\|A(F(z))\|_s < a$. 

Let $A$ be an operator valued function on $H$. We now don't assume that $A$ is obtained as above, but we do assume that $\|A\| < 1$. We call maps $z : D \rightarrow H$ satisfying the equation $z \zeta = A(z)z \zeta$ \textit{pseudo-holomorphic} or \textit{$A$-complex discs}. 

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Theorem (A)

Let $\Sigma = \mathbb{D} \times \mathbb{H}$. Let $A$ be a continuous operator-valued function on $\mathbb{H}$ such that $A(z) = 0$ for $z \notin \Sigma$. Then for some $p > 2$, for every point $z_0 \in \Sigma$ there exists an $A$-complex disc $f \in W^{1,p}(\mathbb{D}, \mathbb{H})$ such that $f(\mathbb{D}) \subset \Sigma$, $f(\partial \mathbb{D}) \subset \partial \Sigma$, $f(0) = z_0$, and $\text{Area}(f) = \pi$. Here $\text{Area}(f) = \int_{\mathbb{D}} f^* \omega$. 

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WLOG $R = 1$. Suppose $r > 1$. 

Let $F: B(r) \to \Sigma$ be a symplectic embedding, $F^* \omega = \omega$.

WLOG, shrinking $r$ if necessary, assume $F$ extends to a neighborhood of $B(r)$.

Let $A = QP - 1$, $P = Fz$, $Q = Fz$.

Then $\|A\|_s < a_0$, $0 < s < s_1$. Extend $A$ to $H$ satisfying the hypotheses of Theorem (A).
Proof of Non-Squeezing

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Then $\|A\|_s < a$, $0 < s < s_1$. Extend $A$ to $\mathbb{H}$ satisfying the hypotheses of Theorem (A).
Proof of Non-Squeezing

Then there exist an $A$-complex disc satisfying the conclusions of Theorem (A), in particular $f(0) = F(0)$ and $\text{Area}(f) = \pi$. 

Note that the area of an $A$-complex disc as well as the area of any part of it is positive. Since $\text{Area}(f) = \pi$, we have $\text{Area}(X) \leq \pi$. On the other hand by Lelong's result of 1950, $\text{Area}(X) \geq \pi r^2$. Hence $r \leq 1$ contrary to the assumption. The proof is complete.
Then there exist an $A$-complex disc satisfying the conclusions of Theorem (A), in particular $f(0) = F(0)$ and $\text{Area}(f) = \pi$. Then $X = F^{-1}(f(D))$ is a usual analytic set in $B(r)$. Note that the area of an $A$-complex disc as well as the area of any part of it is positive. Since $\text{Area}(f) = \pi$, we have $\text{Area}(X) \leq \pi$. On the other hand by Lelong's result of 1950, $\text{Area}(X) \geq \pi r^2$. Hence $r \leq 1$ contrary to the assumption. The proof is complete.
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Hence $r \leq 1$ contrary to the assumption. The proof is complete.
Attempting to prove Theorem (A)

Notation:
\[ \zeta \in \mathbb{D}, \ (z, w) \in \mathbb{C} \times \mathbb{H} = \mathbb{H}, \]
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Introduce the triangle

$$\Delta = \{ z \in \mathbb{C} : 0 < \text{Im } z < 1 - |\text{Re } z| \}.$$ 

Note $\text{Area}(\Delta) = 1$, so we will be looking for a disc of area 1.
Recall the Cauchy-Green operator

$$Tf(\zeta) = \frac{1}{2\pi i} \int_{\mathbb{D}} \frac{f(t) \, dt \wedge d\overline{t}}{t - \zeta}.$$  

$T : L^p(\mathbb{D}) \to W^{1,p}(\mathbb{D})$ is bounded for $p > 1$. 

$\partial T u = u$, that is, $T$ solves the $\partial$-problem in $\mathbb{D}$. 

Let $\Phi : \mathbb{D} \to \Delta$ be the conformal map, $\Phi(\pm 1) = \pm 1$, $\Phi(i) = i$. We look for a solution of Cauchy-Riemann equations in the form

$$z = T_2 u + \Phi w = T_1 v + \text{const}.$$ 

The operators $T_1$ and $T_2$ are modified Cauchy-Green operators. They differ from $T$ by holomorphic functions.
Reduction to modified Cauchy-Green operators

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The operators \( T_1 \) and \( T_2 \) are modified Cauchy-Green operators. They differ from \( T \) by holomorphic functions.

\( T_1 \) satisfies \( \text{Re} \left( T_1 u \right) \big|_{\partial D} = 0 \).

\( T_2 u \big|_{\partial D} \) takes values in the lines \( L_j \) parallel to the sides of \( \Delta \).
Let $Q$ be a non-vanishing holomorphic function in $\mathbb{D}$. We define

$$T_Q u(\zeta) = Q(\zeta) \left( T(u/Q)(\zeta) + \zeta^{-1} \overline{T(u/Q)(1/\zeta)} \right)$$

$$= Q(\zeta) \int_{\mathbb{D}} \left( \frac{u(t)}{Q(t)(t - \zeta)} + \frac{\overline{u(t)}}{Q(t)(\overline{t}\zeta - 1)} \right) \frac{dt \wedge d\overline{t}}{2\pi i}.$$
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$$T_1 f = T_Q f + 2i \text{Im} \ T f(1) \text{ with } Q(\zeta) = \zeta - 1. \text{ Then } \Re (T_1 u)|_{b\mathbb{D}} = 0 \text{ (Vekua).}$$
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$T_2 = T_Q$ with $Q(\zeta) = \sigma(\zeta - 1)^{1/4}(\zeta + 1)^{1/4}(\zeta - i)^{1/2}$, $\sigma = \text{const}$. Then $T_2 u(\gamma_j) \subset L_j$. Here $\gamma_j$, $j = 0, 1, 2$, denote the arcs $[-1, 1]$, $[1, i]$, $[i, -1]$ respectively.
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Operators similar to $T_2$ were introduced by Antoncev and Monakhov for application to problems of gas dynamics.
Recall that the operator $S = \partial T$ for the whole plane is an isometry of $L^2(\mathbb{C})$. It turns out the operators $S_j = \partial T_j$, $j = 1, 2$, have similar properties.
 Operators $S_j$

Recall that the operator $S = \partial T$ for the whole plane is an isometry of $L^2(\mathbb{C})$. It turns out the operators $S_j = \partial T_j, j = 1, 2,$ have similar properties.

Lemma

$S_j : L^p(\mathbb{D}) \rightarrow L^p(\mathbb{D})$ is bounded for $p$ close to 2.

$\|S_j\|_{L^2(\mathbb{D})} = 1.$
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The second assertion looks like a fluke. It follows because the boundary values of $T_j u$ do not bound a positive area.  
The operators $S_j$ extend as bounded operators on $L^p(\mathbb{D}, \mathbb{H})$ for all $p$ close to 2 and have the corresponding properties.
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\end{align*}
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The Cauchy-Riemann equation \( f_{\bar{z}} = A f_{\bar{z}} \) turns into the integral equation

\[
\begin{pmatrix}
   u \\
   v
\end{pmatrix} = A(z, w) \left( \frac{S_2 u + \Phi'}{S_1 v} \right).
\]
Using the equation

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we now rewrite the condition \( z(\tau) = z_0 \) in the form

\[ \tau = \Psi(z_0 - T_2 u(\tau)). \]
How to satisfy $z(\tau) = z_0$

Using the equation

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we now rewrite the condition $z(\tau) = z_0$ in the form

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Here $\Psi : \mathbb{C} \rightarrow \overline{D}$ is a continuous map defined as follows.

$$\Psi(z) = \begin{cases} 
\Phi^{-1}(z) & \text{if } z \in \overline{\Delta}, \\
\Phi^{-1}(b\Delta \cap [z_0, z]) & \text{if } z \notin \overline{\Delta}.
\end{cases}$$
Existence of solution

We now have the system

\[ z = T_2 u + \Phi \]
\[ w = T_1 v - T_1 v(\tau) + w_0 \]
\[ \begin{pmatrix} u \\ v \end{pmatrix} = A(z, w) \begin{pmatrix} S_2 u + \Phi' \\ S_1 v \end{pmatrix} \]
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\end{align*}
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By a priori estimates in \(L^p(\mathbb{D}, \mathbb{H}_s)\) for some \(p > 2\), we show that the system defines a compact operator. By Schauder principle the system has a solution.
Properties of the solution

Now that all the quantities \((z, w, u, v, \tau)\) are defined, we claim they have all the desired properties.

1. \(\tau \in \mathbb{D}\) (not on the boundary). It follows by the boundary conditions of \(T_2\).
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- \(z(\overline{\mathbb{D}}) \subset \overline{\mathbb{D}}\) by maximum principle because \(z\) is holomorphic at \(\zeta\) if \(z(\zeta) \notin \overline{\mathbb{D}}\).
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- \(z(b\mathbb{D}) \subset b\Delta\) and \(\text{deg}(z|_{b\mathbb{D}} : b\mathbb{D} \to b\Delta) = 1\) by the boundary conditions of \(T_2\).
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- \(\text{Area}(f) = 1\) by the boundary conditions of \(T_1\) and \(T_2\). Indeed, \(\text{Area}(f) = \text{Area}(z) + \text{Area}(w)\). \(\text{Area}(z) = 1\) by the previous item. \(\text{Area}(w) = 0\) because every component of \(w\) takes values on a real line.
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The proof is complete.
That's All Folks!