Symplectic non-squeezing for the discrete nonlinear Schrödinger equation

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Joint work with Alexandre Sukhov

Gromov's Non-Squeezing Theorem

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- What is complex here? ... Only notation.
- Gromov's proof is based on complex analysis, namely on J-complex (pseudoholomorphic) curves.

Infinite-dimensional versions

Flows of Hamiltonian PDEs are symplectic transformations. Non-squeezing property is of great interest. There are many results for specific PDEs.

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- Kuksin (1994-95) proved a general non-squeezing result for symplectomorphisms of the form $F = I +$ compact.
- Bourgain (1994-95) proved the result for cubic NLS. Consider time t flow $F : u(0) \mapsto u(t)$ of the equation

$$iu_t + u_{xx} + |u|^p u = 0, \quad x \in \mathbb{R}/\mathbb{Z}, t > 0.$$

Then F is a symplectic transformation of $L^2(0, 1)$, $0 < p \leq 2$. Bourgain proved the non-squeezing property for $p = 2$. For other values of p the question is open.

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We prove a non-squeezing result for a symplectic transformation F of the Hilbert space assuming that the derivative F' is bounded in Hilbert scales. We apply our result to discrete nonlinear Schrödinger equations.

Hilbert scales

Let $\mathbb{H}$ be a complex Hilbert space with fixed orthonormal basis $(e_n)_{n=1}^{\infty}$. Let $(\theta_n)_{n=1}^{\infty}$ be a sequence of positive numbers such that $\theta_n \to \infty$ as $n \to \infty$, for example, $\theta_n = n$.

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For $s \in \mathbb{R}$ we define $\mathbb{H}_s$ as a Hilbert space with the following norm:

$$\|x\|_s^2 = \sum |x_n|^2 \theta_n^{2s}, \qquad x = \sum x_n e_n.$$

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The family $(\mathbb{H}_s)$ is called the Hilbert scale corresponding to the basis (e_n) and sequence (θ_n). We have $\mathbb{H}_0 = \mathbb{H}$. For $s > r$, the space $\mathbb{H}_s$ is dense in $\mathbb{H}_r$, and the inclusion $\mathbb{H}_s \subset \mathbb{H}_r$ is compact.

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Example. $\mathbb{H} = L^2(0,1)$ with the standard Fourier basis, $\theta_n = (1 + n^2)^{1/2}$, $n \in \mathbb{Z}$. Then $\mathbb{H}_s$ is the standard Sobolev space.

Main Result

Let $\mathbb{B}(r) = \mathbb{B}^\infty(r)$ be the ball of radius r in $\mathbb{H}$.

Theorem

Let $r, R > 0$. Let $F : \mathbb{B}(r) \to \mathbb{D}(R) \times \mathbb{H}$ be a symplectic embedding of class C^1. Suppose there is $s_0 > 0$ such that for every $|s| < s_0$ the derivative $F'(z)$ is bounded in $\mathbb{H}_s$ uniformly in $z \in \mathbb{B}(r)$. Then $r \leq R$.

Discrete non-linear Schrödinger equation

Consider the following system of equations

$$iu_n' + f(|u_n|^2)u_n + \sum_k a_{nk}u_k = 0. \tag{1}$$

Here $u(t) = (u_n(t))_{n\in\mathbb{Z}}$, $u_n(t) \in \mathbb{C}$, $t \geq 0$.

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We assume that $f : \mathbb{R}_+ \to \mathbb{R}$ and its derivative are continuous on the positive reals, furthermore, $\lim_{x\to 0} f(x) = \lim_{x\to 0}[xf'(x)] = 0$. For example, one can take $f(x) = x^p$ with real $p > 0$. The hypotheses on the function f are imposed in order for the flow of (1) to be C^1 smooth.

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Here $A = (a_{nk})$ is an infinite matrix independent of t. Furthermore, A is a hermitian matrix, that is, $a_{nk} = \overline{a_{kn}}$. For simplicity we also assume that the entries a_{nk} are uniformly bounded and there exists $m > 0$ such that $a_{nk} = 0$ if $|n-k| > m$.

Discrete non-linear Schrödinger equation

The equation (1) with $f(x) = x$ is called the discrete self-trapping equation. The special case with $a_{nk} = 1$ if $|n - k| = 1$ and $a_{nk} = 0$ otherwise, is the discrete nonlinear (cubic) Schrödinger equation:

$$iu_n' + |u_n|^2 u_n + u_{n-1} + u_{n+1} = 0.$$

There are other discretizations of the Schrödinger equation, in particular, the Ablowitz-Ladik model that can be treated in a similar way.

Discrete non-linear Schrödinger equation

The equation (1) can be written in the Hamiltonian form:

$$u_n' = i\frac{\partial H}{\partial \overline{u_n}}.$$

The Hamiltonian H is given by

$$H = \sum_n F(|u_n|^2) + \sum_{n,k} a_{nk}\overline{u_n}u_k,$$

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The equation (1) preserves the $l^2(\mathbb{Z})$ norm $\|u\|_{l^2} = (\sum_n |u_n|^2)^{1/2}$. Hence, the flow $u(0) \mapsto u(t)$ of (1) is globally defined on $l^2(\mathbb{Z})$ and preserves the standard symplectic form $\omega = (i/2)\sum_n du_n \wedge d\overline{u_n}$.

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We verify that our main result applies to (1), hence, the non-squeezing property holds for the flow of (1).

Holomorphic discs

The proof is based on (pseudo) holomorphic discs.
A holomorphic disc $z : \mathbb{D} \to \mathbb{H}$, $\zeta \mapsto z(\zeta)$ satisfies the Cauchy-Riemann equation

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Change coordinates by a non-holomorphic diffeomorphism $w = F(z)$. Then the equation for a holomorphic disc will turn into

$$w_{\overline{\zeta}} = A(w)\overline{w}_{\overline{\zeta}}.$$

Here

$$A = Q\overline{P}^{-1}, \quad P = F_z, \quad Q = F_{\overline{z}}.$$

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Furthermore, if F satisfies the hypotheses of the main theorem, then there is $0 < a < 1$ and $s_1 > 0$ such that for all $z \in \mathbb{B}(r)$ and $0 \le s \le s_1$ we have $\|A(F(z))\|_s < a$.

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Let A be an operator valued function on $\mathbb{H}$. We now don't assume that A is obtained as above, but we do assume that $\|A\| < 1$. We call maps $z : \mathbb{D} \to \mathbb{H}$ satisfying the equation $z_{\overline{\zeta}} = A(z)\overline{z}_{\overline{\zeta}}$ pseudo-holomorphic or *A*-complex discs.

Pseudo-holomorphic discs in a cylinder

Theorem (A)

Let $\Sigma = \mathbb{D} \times \mathbb{H}$*. Let* A *be a continuous operator-valued function on* $\mathbb{H}$ *such that* $A(z) = 0$ *for* $z \notin \Sigma$*.*

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Then for some $p > 2$, for every point $z_0 \in \Sigma$ there exists an A-complex disc $f \in W^{1,p}(\mathbb{D}, \mathbb{H})$ such that
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and $\mathrm{Area}(f) = \pi$.

Here $\mathrm{Area}(f) = \int_{\mathbb{D}} f^*\omega$.

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Let $A = Q\overline{P}^{-1}$, $P = F_z$, $Q = F_{\overline{z}}$.
Then $\|A\|_s < a$, $0 < s < s_1$. Extend A to $\mathbb{H}$ satisfying the hypotheses of Theorem (A).

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Hence $r \leq 1$ contrary to the assumption. The proof is complete.

Attempting to prove Theorem (A)

Notation:

$\zeta \in \overline{\mathbb{D}}$, $(z, w) \in \mathbb{C} \times \mathbb{H} = \mathbb{H}$,

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$|\zeta| = 1 \quad \Rightarrow |z(\zeta)| = 1$.

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$z(0) = z_0$, $w(0) = w_0$.

Boundary condition:
$|\zeta| = 1 \quad \Rightarrow |z(\zeta)| = 1$.

The boundary condition is non-linear. Most if not all general results assume linear boundary conditions.

Let Δ be a triangle.
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Introduce the triangle

$$\Delta = \{z \in \mathbb{C} : 0 < \operatorname{Im} z < 1 - |\operatorname{Re} z|\}.$$

Note $\operatorname{Area}(\Delta) = 1$, so we will be looking for a disc of area 1.

Reduction to modified Cauchy-Green operators

Recall the Cauchy-Green operator

$$Tf(\zeta) = \frac{1}{2\pi i} \int_{\mathbb{D}} \frac{f(t)\, dt \wedge d\bar{t}}{t - \zeta}.$$

$T : L^p(\mathbb{D}) \to W^{1,p}(\mathbb{D})$ is bounded for $p > 1$.
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T_1 satisfies $\mathrm{Re}\,(T_1 u)|_{b\mathbb{D}} = 0$.
$T_2 u|_{b\mathbb{D}}$ takes values in the lines L_j parallel to the sides of Δ.

Modified Cauchy-Green operators

Let Q be a non-vanishing holomorphic function in $\mathbb{D}$. We define

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\begin{aligned}
T_Q u(\zeta) &= Q(\zeta)\left(T(u/Q)(\zeta) + \zeta^{-1}\overline{T(u/Q)(1/\bar{\zeta})}\right) \\
&= Q(\zeta)\int_{\mathbb{D}}\left(\frac{u(t)}{Q(t)(t-\zeta)} + \frac{\overline{u(t)}}{\overline{Q(t)}(\bar{t}\zeta - 1)}\right)\frac{dt\wedge d\bar{t}}{2\pi i}.
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$T_2 = T_Q$ with $Q(\zeta) = \sigma(\zeta-1)^{1/4}(\zeta+1)^{1/4}(\zeta-i)^{1/2}$, $\sigma = \mathrm{const}$. Then $T_2 u(\gamma_j) \subset L_j$. Here γ_j, $j = 0, 1, 2$, denote the arcs $[-1, 1]$, $[1, i]$, $[i, -1]$ respectively.

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Operators similar to T_2 were introduced by Antoncev and Monakhov for application to problems of gas dynamics.

Operators S_j

Recall that the operator $S = \partial T$ for the whole plane is an isometry of $L^2(\mathbb{C})$. It turns out the operators $S_j = \partial T_j$, $j = 1, 2$, have similar properties.

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The operators S_j extend as bounded operators on $L^p(\mathbb{D}, \mathbb{H})$ for all p close to 2 and have the corresponding properties.

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The Cauchy-Riemann equation $f_{\bar{\zeta}} = A\bar{f}_{\bar{\zeta}}$ turns into the integral equation

$$\begin{pmatrix} u \\ v \end{pmatrix} = A(z, w) \begin{pmatrix} \overline{S_2 u} + \overline{\Phi'} \\ \overline{S_1 v} \end{pmatrix}.$$

How to satisfy $z(\tau) = z_0$

Using the equation

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Here $\Psi : \mathbb{C} \to \overline{\mathbb{D}}$ is a continuous map defined as follows.

$$\Psi(z) = \begin{cases} \Phi^{-1}(z) & \text{if } \; z \in \overline{\Delta}, \\ \Phi^{-1}(b\Delta \cap [z_0, z]) & \text{if } \; z \notin \overline{\Delta}. \end{cases}$$

Existence of solution

We now have the system

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By a priori estimates in $L^p(\mathbb{D}, \mathbb{H}_s)$ for some $p > 2$, we show that the system defines a compact operator. By Schauder principle the system has a solution.

Properties of the solution

Now that all the quantities (z, w, u, v, τ) are defined, we claim they have all the desired properties.

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The proof is complete.

That's All Folks!