Iterative computational identification of a spacewise dependent source in parabolic equations

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Outlook

Inverse problems

The mathematical modeling of many applied problems of science and engineering leads to the need for the numerical solution of inverse problems. The inverse problems for partial differential equations are particularly noteworthy.
Inverse problems are formulated as non-classical problems for partial differential equations. They are often classified as ill-posed (conditionally well-posed) problems. In the theoretical study, the fundamental questions of uniqueness of the solution and its stability are primarily considered.

Alifanov OM. Inverse Heat Transfer Problems. Springer; 2011.

Lavrent'ev MM, Romanov VG, Shishatskii SP. Ill-posed Problems of Mathematical Physics and Analysis. American Mathematical Society; 1986.

Coefficient inverse problems

Coefficient inverse problems related to identifying coefficient and/or the right-hand side of an equation with use of some additional information is of interest among inverse problems for partial differential equations. When considering non-stationary problems, tasks of recovering the dependence of the right-hand side on time or spatial variables can be usually treated as independent. These tasks relate to a class of linear inverse problems, which sufficiently simplifies their study. Only in some cases we have linear inverse problems – identification of the right-hand side of equation, Other coefficient inverse problems are nonlinear, that significantly complicated their study.

Isakov V. Inverse Problems for Partial Differential Equations. Springer; 206.

Prilepko AI, Orlovsky DG, Vasin IA. Methods for Solving Inverse Problems in Mathematical Physics. Marcel Dekker, Inc; 2000.

Additional conditions

The task of identifying the dependence of the right-hand side on spatial variables is one of the most important problems. Additional conditions are often formulated using the solution at the final moment of time — final overdetermination. In more general case the overdetermination condition is stated as some time integral average — integral overdetermination.
The existence and uniqueness of the solution to such an inverse problem and well-posedness of this problem in various functional classes are examined in the many works.

Rundell W. Applicable Analysis. 1980;10(3):231–242.

Prilepko AI, Solov'ev VV. Differential Equations. 1987;23(11):1971–1980.

Isakov V. Communications on Pure and Applied Mathematics. 1991; 44(2):185–209.

Prilepko AI, Kostin AB. Sbornik: Mathematics. 1993;75(2):473–490.

Computational algorithms

In the numerical solution of inverse problem the main focus is on the development of stable computational algorithms that take into account the peculiar properties of inverse problems. Inverse problems for partial differential equations can be formulated as optimal control problems. Computational algorithms are based on using gradient iterative methods for corresponding residual functional. The implementation of such approaches relates to the solution of initial-boundary problems for the original parabolic equation and its conjugate.

Vogel CR. Computational Methods for Inverse Problems. Society for Industrial and Applied Mathematics; 2002.

Samarskii AA, Vabishchevich PN. Numerical Methods for Solving Inverse Problems of Mathematical Physics. De Gruyter; 2007.

Lions JL. Optimal Control of Systems Governed by Partial Differential Equations. Springer; 1971.

Inverse problem with final overdetermination

For the required right-hand side of a parabolic equation, which does not depend on time, an inverse problem with final overdetermination can be formulated as a boundary problem for evolution equation of the second order. In this case, we can use standard computational algorithms for the solution of stationary boundary value problems. Such direct computational algorithms based on finite-difference approximation is described in the SV book (section 6.4). In the XYJ work the identification problem is numerically solved on the basis of transition to a evolutionary problem for the derivative of the solution with respect to time, peculiarity of which is the non-local boundary condition.

Samarskii AA, Vabishchevich PN. Numerical Methods for Solving Inverse Problems of Mathematical Physics. De Gruyter; 2007.

Xiangtuan X, Yaomei Y, Junxia W. In: Journal of Physics: Conference Series; Vol. 290. IOP Publishing; 2011. p. 012017.

This work

We construct special iterative methods for approximate solution of identification problem of a spacewise dependence of the source in a parabolic equations. They fully take into account considered inverse problems features, which relate to their evolutionary character. These methods are based on the numerical solution of the standard Cauchy problems on each iteration. The first method is based on the iterative refinement of initial condition for time derivative of the solution. The second method relates to the iterative refinement of the dependence of the right-hand side on the spatial variables. Such approach have been used before.

Prilepko AI, Kostin AB. On certain inverse problems for parabolic equations with final and integral observation. Sbornik: Mathematics. 1993;75(2):473–490.

Prilepko AI, Kostin AB. Mathematical Notes. 1993;53(1):63–66.

Kostin AB. Sbornik: Mathematics. 2013;204(10):1391–1434.

2D problem

Let $\boldsymbol{x} = (x_1, x_2)$ and Ω be a bounded polygon. The direct problem is formulated as follows. We search $u(\boldsymbol{x}, t)$, $0 \le t \le T$, $T > 0$ such that it is the solution of the parabolic equation of second order:

$$\frac{\partial u}{\partial t} - \operatorname{div}(k(\boldsymbol{x})\operatorname{grad} u) + c(\boldsymbol{x})u = f(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega, \quad 0 < t \le T,$$

with coefficients $0 < k_1 \le k(\boldsymbol{x}) \le k_2$, $c(\boldsymbol{x}) \ge 0$. The boundary conditions are also specified:

$$k(\boldsymbol{x})\frac{\partial u}{\partial n} + \mu(\boldsymbol{x})u = 0, \quad \boldsymbol{x} \in \partial\Omega, \quad 0 < t \le T,$$

where $\mu(\boldsymbol{x}) \ge \mu_1 > 0$, $\boldsymbol{x} \in \partial\Omega$ and n is the normal to Ω. The initial conditions are

$$u(\boldsymbol{x}, 0) = u_0(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega.$$

The formulation presents the direct problem, where the right-hand side, coefficients of the equation as well as the boundary and initial conditions are specified.

Inverse problem

Let us consider the inverse problem, where in equation, the right-hand side $f(\boldsymbol{x})$ is unknown. An additional condition is often formulated as

$$u(\boldsymbol{x}, T) = u_T(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega.$$

In this case, we speak about the final overdetermination.
We assume that the above inverse problem of finding a pair of $u(\boldsymbol{x}, t), \ f(\boldsymbol{x})$ from equations and additional conditions is well-posed. The corresponding conditions for existence and uniqueness of the solution are available in the above-mentioned works.

Bilinear form

In the Hilbert space $H = L_2(\Omega)$, we define the scalar product and norm in the standard way:

$$(u, v) = \int_\Omega u(\boldsymbol{x}) v(\boldsymbol{x}) d\boldsymbol{x}, \quad \|u\| = (u, u)^{1/2}.$$

To solve numerically the problem, we employ finite-element approximations in space. We define the bilinear form

$$a(u, v) = \int_\Omega (k \operatorname{grad} u \operatorname{grad} v + c\, uv)\, d\boldsymbol{x} + \int_{\partial\Omega} \mu\, uv d\boldsymbol{x}.$$

We have

$$a(u, u) \geq \delta \|u\|^2, \quad \delta > 0.$$

Brenner SC, Scott LR. The mathematical theory of finite element methods. Springer; 2008.

Thomée V. Galerkin Finite Element Methods for Parabolic Problems. Berlin: Springer Verlag; 2006.

Finite elements

Define a subspace of finite elements $V^h \subset H^1(\Omega)$. Let $\boldsymbol{x}_i,\ i = 1, 2, ..., M_h$ be triangulation points for the domain Ω. For example, when using Lagrange finite elements of the first order (piece-wise linear approximation) we can define pyramid function $\chi_i(\boldsymbol{x}) \subset V^h,\ i = 1, 2, ..., M_h$, where

$$\chi_i(\boldsymbol{x}_j) = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$

For $v \in V_h$, we have

$$v(\boldsymbol{x}) = \sum_{i=i}^{M_h} v_i \chi_i(\boldsymbol{x}),$$

where $v_i = v(\boldsymbol{x}_i),\ i = 1, 2, ..., M_h$.

Discrete elliptic operator

We define the discrete elliptic operator A as

$$(Ay, v) = a(y, v), \quad \forall\, y, v \in V^h.$$

The operator A acts on a finite dimensional space V^h and

$$A = A^* \geq \delta I, \quad \delta > 0, \tag{1}$$

where I is the identity operator in V^h.

Semidiscrete inverse problem

For the problem, we put into the correspondence the operator equation for $w(t) \in V^h$:

$$\frac{dw}{dt} + Aw = \varphi, \quad 0 < t \leq T,$$

$$w(0) = \phi,$$

where $\varphi = Pf$, $\phi = Pu_0$ with P denoting L_2-projection onto V^h. When considering the inverse problem assume

$$w(T) = \psi,$$

where $\psi = Pu_T$.

Equation of the second order

For the numerical solution of the inverse problem with finding $w(t), \varphi$ the simplest approach is to eliminate variable φ [0, 0]. Differentiating equation on time, we obtain

$$\frac{d^2 w}{dt^2} + A \frac{dw}{dt} = 0, \quad 0 < t \leq T.$$

Further, we consider the boundary value problem. The correctness of such problem, the computational algorithm and examples of the numerical solution are presented in SV. The weakness of such approach is caused by the computational complexity of the numerical solution of the boundary problem. We practically lose the evolutionary character of original problem and must store data in each time step.

Samarskii AA, Vabishchevich PN. Numerical Methods for Solving Inverse Problems of Mathematical Physics. De Gruyter; 2007.

Non-local boundary conditions

The second approach (see, for example, XYJ) is based on considering the time derivative. Let $v = \frac{dw}{dt}$, then equation can be written as

$$\frac{dv}{dt} + Av = 0, \quad 0 < t \leq T.$$

We formulate non-local boundary conditions. We have

$$v(0) + Aw(0) = \varphi,$$
$$v(T) + Aw(T) = \varphi.$$

Taking into account boundary conditions yields

$$v(T) - v(0) = \chi, \quad \chi = A(\phi - \psi).$$

Xiangtuan X, Yaomei Y, Junxia W. In: Journal of Physics: Conference Series; Vol. 290. IOP Publishing; 2011. p. 012017.

Iterative process

For numerical solution of the problem we use the simplest iterative refinement of the initial condition for equation. The iterative process is organized as follows. The new approximation $k+1$ is found by solving the Cauchy problem:

$$v^{k+1}(0) = v^k(T) - \chi,$$

$$\frac{dv^{k+1}}{dt} + Av^{k+1} = 0, \quad 0 < t \le T, \quad k = 0, 1, ...,$$

with some given $v^0(0)$. The desired right-hand side of equation is determined using $v^{k+1}(0)$, for example, from the equality

$$\varphi^{k+1} = \phi + v^{k+1}(0).$$

Convergence of iterative process - 1/2

We consider the problem for error $z^{k+1}(t) = v^{k+1}(t) - v(t)$:

$$z^{k+1}(0) = z^k(T),$$

$$\frac{dz^{k+1}}{dt} + Az^{k+1} = 0, \quad 0 < t \le T, \quad k = 0, 1, ...,$$

with given $z^0(0)$. Multiplying equation for z^k in V^h by z^k, we obtain

$$\left(\frac{dz^k}{dt}, z^k \right) + (Az^k, z^k) = 0.$$

Taking into account

$$\left(\frac{dz^k}{dt}, z^k \right) = \|z^k\| \frac{d}{dt} \|z^k\|,$$

yields

$$\frac{d}{dt} \|z^k\| + \delta \|z^k\| \le 0.$$

Convergence of iterative process - 2/2

Thus,

$$\|z^k(t)\| \leq \exp(-\delta t)\|z^k(0)\|.$$

We have

$$\|z^{k+1}(0)\| = \|z^k(T)\| \leq \exp(-\delta T)\|z^k(0)\|.$$

This gives the desired estimate

$$\|v^{k+1}(0) - v(0)\| \leq \varrho \, \|v^k(0) - v(0)\|, \quad \varrho = \exp(-\delta T),$$

for the convergence of the iterative process with linear speed $\varrho < 1$. For the right-hand side we have

$$\|\varphi^{k+1} - \varphi\| \leq \varrho \, \|v^k(0) - v(0)\|.$$

Two-level scheme

For computational implementation of proposed algorithm the time approximation deserves special attention. Let us define a uniform grid in time

$$t_n = n\tau, \quad n = 0, 1, ..., N, \quad \tau N = T$$

and denote $y_n = y(t_n), \; t_n = n\tau$.
For the numerical solution of the problem we used fully implicit two-level scheme, when

$$\frac{v_{n+1} - v_n}{\tau} + A v_{n+1} = 0, \quad n = 0, 1, ..., N-1,$$

$$v_N - v_0 = \chi.$$

Samarskii AA. The theory of difference schemes. New York: Marcel Dekker; 2001.

Iterative process

The grid problem is solved using the following iterative process:

$$v_0^{k+1} = v_N^k - \chi,$$

$$\frac{v_{n+1}^{k+1} - v_n^{k+1}}{\tau} + A v_{n+1}^{k+1} = 0, \quad n = 0, 1, ..., N-1, \quad k = 0, 1, ...,$$

where

$$\varphi^{k+1} = \phi + v_0^{k+1}.$$

Study of the iterative process - 1/3

The study of the iterative process is conducted using the same approach as for the iterative process for the semidiscrete inverse problem. Let now $z_{n+1}^{k+1} = v_{n+1}^{k+1} - v_{n+1}$, then

$$z_0^{k+1} = z_N^k,$$

$$\frac{z_{n+1}^{k+1} - z_n^{k+1}}{\tau} + A z_{n+1}^{k+1} = 0, \quad n = 0, 1, ..., N-1, \quad k = 0, 1, ...$$

The key moment of our consideration consists in finding an estimate norm of the solution over time.

Study of the iterative process - 2/3

We multiply equation for z_{n+1}^k in V^h by τz_{n+1}^k and obtain

$$\|z_{n+1}^k\|^2 + \tau(Az_{n+1}^k, z_{n+1}^k) = (z_n^k, z_{n+1}^k).$$

We have

$$(1+\tau\delta)\|z_{n+1}^k\| \le \|z_n^k\|, \quad n = 0, 1, ..., N-1,$$

$$\|z_n^k\| \le (1+\tau\delta)^{-n}\|z_0^k\|, \quad n = 1, 2, ..., N.$$

A priori estimate allows us to obtain

$$\|z_0^{k+1}\| = \|z_N^k\| \le (1+\tau\delta)^{-N}\|z_0^k\|.$$

Thereby

$$\|v_0^{k+1} - v_0\| \le \bar{\varrho}\,\|v_0^k - v_0\|, \quad \bar{\varrho} = (1+\tau\delta)^{-N},$$

which provides the convergence of the iterative process ($\bar{\varrho} < 1$).

Study of the iterative process - 3/3

The convergence of the right-hand side is ensured by the estimate

$$\|\varphi^{k+1} - \varphi\| \leq \bar{\varrho}\, \|v_0^k - v_0\|.$$

This allow us to formulate the following main assertion.

Theorem

The iterative process for the numerical solution of the problem converges linearly with speed $\bar{\varrho} < 1$.

Iterative refinement of the right-hand side

When studying the correctness of the inverse problem the constructive method of iterative refinement of the right-hand side is often used.
We consider the possibility of using this approach for the approximate solution of the problem.

Prilepko AI, Kostin AB. On certain inverse problems for parabolic equations with final and integral observation. Sbornik: Mathematics. 1993;75(2):473–490.

Prilepko AI, Kostin AB. Mathematical Notes. 1993;53(1):63–66.

Kostin AB. Sbornik: Mathematics. 2013;204(10):1391–1434.

Iterative process

In the new iterative step the right-hand side is determined for $t = T$:

$$\varphi^{k+1} = \frac{dw^k}{dt}(T) + A\psi, \quad k = 0, 1, ...,$$

with some given initial assumption φ^0. Then, the Cauchy problem is solved:

$$\frac{dw^{k+1}}{dt} + Aw^{k+1} = \varphi^{k+1}, \quad 0 < t \le T,$$

$$w^{k+1}(0) = \phi.$$

Time discretization

The time discretization is again formulated in the basis of implicit approximation. Formally, we define the solution of the problem on expanded grid:

$$t_n = n\tau, \quad n = -1, 0, ..., N, \quad \tau N = T.$$

We come to the problem

$$\frac{w_{n+1} - w_n}{\tau} + A w_{n+1} = \varphi, \quad n = -1, 0, ..., N-1$$

$$w_0 = \phi,$$

$$w_N = \psi.$$

Iterative process

For the numerical solution of the problem the grid analogue of the iterative process for the semidiscrete inverse problem is applied:

$$\varphi^{k+1} = \frac{w_N^k - w_{N-1}^k}{\tau} + A\psi, \quad k = 0, 1, ...$$

To approximate equation the implicit difference scheme is used

$$\frac{w_{n+1}^{k+1} - w_n^{k+1}}{\tau} + A w_{n+1}^{k+1} = \varphi^{k+1}, \quad n = -1, 0, ..., N-1,$$

under condition

$$w_0^{k+1} = \phi.$$

Main result

Theorem

The iterative process for the numerical solution of the problem converges linearly with speed

$$\bar{\varrho} = (1 + \tau\delta)^{-N},$$

and the estimate

$$\|\varphi^{k+1} - \varphi\| \leq \bar{\varrho}\, \|\varphi^k - \varphi\|.$$

is valid.

Integral overdetermination

When considering inverse problem of identifying the right-hand side of parabolic equation, the integral overdetermination is often used instead of the final overdetermination. In this case, the following condition is involved

$$\int_0^T \omega(t) u(\boldsymbol{x}, t) dt = u_T(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega,$$

where $\omega(t)$ — given function and

$$\omega(t) \geq 0, \quad \int_0^T \omega(t) dt = 1.$$

For the numerical solution of the inverse problem iterative process considered above can be used.

Iterative refinement

Integrating equation with weight $\omega(t)$ over t from 0 to T, we obtain

$$\int_0^T \omega(t) \frac{dw}{dt}(t) dt + A\psi = \varphi.$$

The iterative refinement of the right-hand side:

$$\varphi^{k+1} = \int_0^T \omega(t) \frac{dw^k}{dt}(t) dt + A\psi, \quad k = 0, 1, \dots .$$

We have

$$\|\varphi^{k+1} - \varphi\| \leq \varrho \, \|\varphi^k - \varphi\|, \quad k = 0, 1, ...,$$

at that

$$\varrho = \int_0^T \omega(t) \exp(-\delta t) dt.$$

More general problems

We have investigated the iterative methods for the numerical solution of the inverse problem, when the right-hand side does not depend on time. In more general case, the problem of identifying multiplicative right-hand side, when the dependence of the right-hand side on time is known and the dependence on spatial variables is unknown, is stated.
We consider the following equation

$$\frac{\partial u}{\partial t} - \operatorname{div}(k(\boldsymbol{x})\operatorname{grad} u) + c(\boldsymbol{x})u = \beta(t)f(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega, \quad 0 < t \leq T,$$

where $\beta(t)$ — some given function. The inverse problem of finding the pair $u(\boldsymbol{x}, t)$, $f(\boldsymbol{x})$.
We assume that

$$\beta(t) > 0, \quad \frac{d\beta}{dt} \geq 0, \quad 0 \leq t \leq T, \quad \beta(T) = 1.$$

2D model problem

We consider model problem, when

$$\frac{\partial u}{\partial t} - \operatorname{div} \operatorname{grad} u + cu = f(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega, \quad 0 < t \leq T,$$

$$\frac{\partial u}{\partial n} = 0, \quad \boldsymbol{x} \in \partial\Omega, \quad 0 < t \leq T,$$

$$u(\boldsymbol{x}, 0) = 0, \quad \boldsymbol{x} \in \Omega.$$

The forward problem is solved in the unit square

$$\Omega = \{\boldsymbol{x} = (x_1, x_2) \mid 0 < x_1 < 1, \ 0 < x_2 < 1\}$$

with given right-hand side $f(\boldsymbol{x})$ and

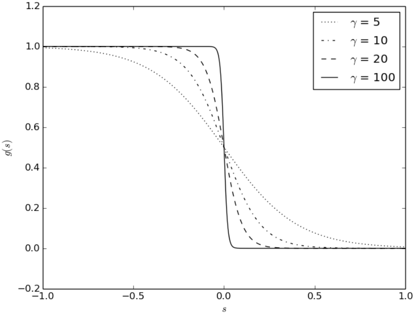
$$u_T(\boldsymbol{x}) = u(\boldsymbol{x}, T).$$

The inverse problem is solved when $u_T(\boldsymbol{x})$ is known, but we need to find $f(\boldsymbol{x})$.

Right-hand side

The right-hand side is taken as

$$f(\boldsymbol{x}) = \frac{1}{1 + \exp(\gamma(x_1 - x_2))}.$$



Function $g(s) = (1 + \exp(\gamma s))^{-1}$ at different values γ

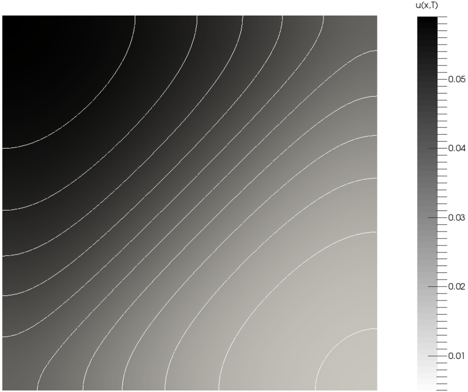
Quasi-real computational experiment

The forward problem is solved within the first quasi-real computational experiment. The solution of this problem at the finite time (the function $u(\boldsymbol{x}, T)$) is used as input data for the inverse problem.

We perform the evaluation of the effect of computational errors on the basis of calculations on different time grids, when using the input data derived from the solution of the forward problem on more detailed time grid and with a more accurate approximations in time.

For the base case we set $c = 10$, $T = 0.1$, $\gamma = 10$. When solving the forward problem we use the Crank-Nicolson scheme for time discretization, the time step is $\tau = 1 \cdot 10^{-4}$. The uniform mesh with the division into 50 intervals in each direction is used, the Lagrangian finite elements of the second degree are applied.

The solution at the finite time



The solution of the forward problem $u_T(\boldsymbol{x}) = u(\boldsymbol{x}, T)$

Error of the approximate solution

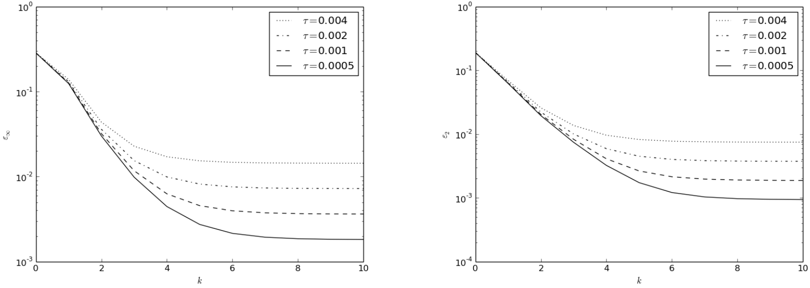
The inverse problem is solved using fully implicit scheme. The error of the approximate solution of the problem of identification on a single iteration is evaluated as follows

$$\varepsilon_\infty(k) = \max_{\boldsymbol{x} \in \Omega} |\varphi^k(\boldsymbol{x}) - f(\boldsymbol{x})|,$$

$$\varepsilon_2(k) = \|\varphi^k(\boldsymbol{x}) - f(\boldsymbol{x})\|,$$

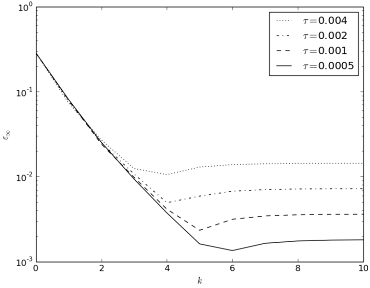
where $\varphi(\boldsymbol{x})$ — the approximate solution, and $f(\boldsymbol{x})$ — the exact solution of the inverse problem.

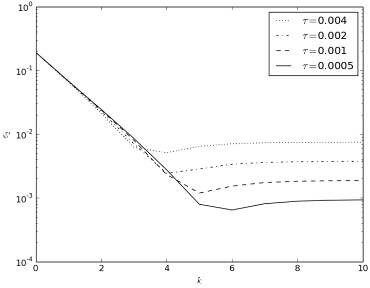
Convergence of iterations - 1/2



The iterative process for non-local problem

Convergence of iterations - 2/2





Iterative process for identifying the right-hand side