Hydrodynamics and kinetics of Vlasov and Liouville equations

V.V. Vedenyapin,
M.A. Negmatov,
N.N. Fimin

Keldysh Institute of Applied Mathematics of RAS

- We describe the derivation of Vlasov-Maxwell equation from classical Lagrangian and a similar derivation of the Vlasov-Poisson-Poisson charged gravitating particles. The last term we use for combination of electrostatic and gravitational forces.
- By using an exact substitution we derive some versions of the equations of the electromagnetic hydrodynamics from Vlasov-Maxwell equations and present them to the Godunov's double-divergence form.
- For them we get generalized Lagrange identity. The Lagrange identity is convenient here as a test to compare different forms of equations.
- We analyzes the steady-state solutions of the Vlasov-Poisson-Poisson equation: their types changes at a certain critical mass $m^2 = e^2/G$ having a clear physical meaning with different behavior of particles - recession or collapse trajectories.

Under Vlasov equation simply imply the following equation for an arbitrary K(x, y) pair interaction potential of particles

$$\frac{\partial F}{\partial t} + \left(v, \frac{\partial F}{\partial x}\right) - \left(\nabla_x \int K(x,y) F(t,v,y) dv dy, \frac{\partial F}{\partial v}\right) = 0.$$

Let us consider the substitution

$$F(t,v,x) = \sum_{i=1}^{N} \rho_i \delta(v - V_i(t)) \delta(x - X_i(t)).$$

Substitution takes place if $X_i(t)$ and $V_i(t)$ satisfy N-body equations of motion

$$\begin{cases} \dot{X}_i = V_i, \\ \dot{V}_i = -\sum_{j=1}^{N} \nabla_1 K(X_i, X_j) \rho_j. \end{cases}$$

Generally, the Vlasov type equations are used with some prefix:

- Vlasov-Poisson Equation (for gravity, electrons, and plasma)
- Vlasov-Maxwell Equation (plasma, the galaxy)
- Vlasov-Einstein Equation

Simplest derivation of the Vlasov-Maxwell equation from classical Lagrangian is highly desirable, it is provides us a firm basis:

- For the classification of equations with the same name;
- To assess their validity;
- The nature of the approximations made by various authors.

Derivation of equations of the Vlasov-Maxwell and Vlasov-Poisson-Poisson:

We start with the usual action of the electromagnetic field, the action of Lorentz-Shwartzchield

$$S_L = S_{VM} = -\sum_\alpha m_\alpha c \sum_q \int_0^T \sqrt{g_{\mu\nu}\dot{X}^\mu_\alpha(q,t)\dot{X}^\nu_\alpha(q,t)}dt \; + \qquad (1)$$

$$+\sum_\alpha \frac{e_\alpha}{c}\sum_q \int_0^T A_\mu(X_\alpha(q,t),t)\dot{X}^\mu_\alpha(q,t)dt +$$

$$+\frac{1}{16\pi c}\int F_{\mu\nu}F^{\mu\nu}d^4x = S_{\mathsf{p}} + S_{\mathsf{p\text{-}f}} + S_{\mathsf{f}}$$

S_{p} – is a particle action, S_{f} – is a field action,
$S_{\mathsf{p\text{-}f}}$ – is a particles-fields action.

We have to seek a variation in a special way:

- first we obtain $\delta\left(S_{\mathsf{p}} + S_{\mathsf{p\text{-}f}}\right) = 0$
- than evolution of fields $\delta\left(S_{\mathsf{p\text{-}f}} + S_{\mathsf{f}}\right) = 0.$

However, for particles, we proceed to distribution functions.

$$\delta S_{\mathsf{p}} = mc^2 \sum_q \frac{1}{c^2} \int \frac{\dot{x}_i \delta \dot{x}_i}{\sqrt{1 - v^2/c^2}}\, dt = m \sum \int \frac{d}{dt} \left(\frac{\dot{x}_i}{\sqrt{1 - v^2/c^2}} \right) \delta \dot{x}^i \, dt.$$

$$\delta S_{\mathsf{p\text{-}f}} = \frac{e}{c} \sum_q \int \left[c \frac{\partial A_0}{\partial x^i} \delta x^i + \frac{\partial A_i}{\partial x^j} \dot{x}^i \delta x^j - \left(\frac{d}{dt} A_i \right) \delta x^i \right] \, dt.$$

and from $\delta(S_{\mathsf{p}} + S_{\mathsf{p\text{-}f}}) = 0$ we have:

$$\frac{dp_{\alpha i}}{dt} = e_\alpha \left(-\frac{1}{c} \frac{\partial A_i}{\partial t} - \frac{\partial A_0}{\partial x^i} - \frac{1}{c} F_{ij} \dot{x}^j_\alpha \right), \tag{2}$$

$$p_{\alpha i} = \frac{\partial L_p}{\partial x^i_\alpha} = \frac{m_\alpha \dot{x}_{\alpha i}}{\sqrt{1 - \dot{x}^2_\alpha / c^2}}, \;\; F_{ij} = \frac{\partial A_i}{\partial x^j} - \frac{\partial A_j}{\partial x^i}$$

- The equation for the distribution function is obtained as the equation of translation along the trajectories of the resulting dynamic system of charges in the field.

- It is seen that is convenient to take the distribution function of the momentum (instead of velocity). It should express velocity through momentums:

$$p_i = \frac{mv_i}{\sqrt{1 - v^2/c^2}} \Rightarrow p^2 = \frac{m^2 v^2}{1 - v^2/c^2}.$$

$$1 - \frac{v^2}{c^2 = \gamma^{-2}}, \quad \gamma^{-2} = 1 + \frac{p^2}{(m^2 c^2}, \quad v_i = \frac{p_i}{(\gamma m)}.$$

Hence we can find the equation for the distribution function $f_\alpha(x, p, t)$:

$$\frac{\partial f_\alpha}{\partial t} + \left(v_\alpha, \frac{\partial f_\alpha}{\partial x} \right) + \frac{e_\alpha}{c} \left(-\frac{\partial A_i}{\partial t} - c\frac{\partial A_0}{\partial x} - F_{ij} v_\alpha^j \right) \frac{\partial f_\alpha}{\partial p_i} = 0. \qquad (3)$$

Fields equations.

- We use the distribution function instead of density:

$$\delta S_{\text{p-f}} = \sum \frac{e_\alpha}{c^2} \int \delta A_\mu(x) v^\mu_\alpha f_\alpha(x,p) d^3p d^4x,$$

$$\delta S_{\text{f}} = \frac{1}{16\pi c} 2 \int \delta F_{\mu\nu} F^{\mu\nu} d^4x = \frac{1}{8\pi c} 2 \int \delta A_\mu \partial_\mu F^{\mu\nu} d^4x.$$

- If $\delta(S_{\text{p-f}} + S_{\text{f}}) = 0$ then:

$$\partial_\mu F^{\mu\nu} = -\frac{4\pi}{c} \sum_\alpha e_\alpha \int v^\mu_\alpha f_\alpha(x,p) d^3p. \tag{4}$$

- System of equations (3),(4) is Vlasov-Maxwell with some small adjustments: we have explicite expression of velocities over momentum.
- Similarly, we can derive the system of equations of the Vlasov-Poisson with gravitation in the nonrelativistic case.

- The Lagrangian of electrostatics derived from the general Lagrangian, and gravitation part is derived by analogy with electrostatics.
- So, in the nonrelativistic case:

$$\sqrt{1-\frac{\dot{x}_\alpha^2}{c^2}} \approx 1-\frac{\dot{x}_\alpha^2}{2c^2}$$

- Particles action: $S_{\mathsf{p}} = -\sum_{\alpha,q} \int m_\alpha c^2 + \sum_{\alpha,q} \int \frac{m_\alpha \dot{x}_\alpha^2(q,t)}{2}$.
- Particle-fields action (electrostatic): $S^e_{\mathsf{p\text{-}f}} = -\sum_\alpha e_\alpha \int \varphi(x,t) f_\alpha(x,p,t) dxdpdt$.
- Particle-fields action (gravity): $S^g_{\mathsf{p\text{-}f}} = -\sum_\alpha m_\alpha \int U(x,t) f_\alpha(x,p,t) dxdpdt$.
- Fields action (electrostatic): $S^e_{\mathsf{f}} = \frac{1}{8\pi} \int (\nabla\varphi)^2 dxdt$.
- Fields action (gravity): $S^g_{\mathsf{f}} = -\frac{1}{8\pi G} \int (\nabla U)^2 dxdt$

- Lagrangian $S = S_{\mathsf{p}} + S^{e}_{\mathsf{p\text{-}f}} + S^{g}_{\mathsf{p\text{-}f}} + S^{e}_{\mathsf{f}} + S^{g}_{\mathsf{f}}$

$$S = \sum_{\alpha,q} \int \frac{m_\alpha \dot{x}^2_\alpha(q,t)}{2} - \sum_\alpha e_\alpha \int \varphi(x,t) f_\alpha(x,p,t)\,dxdpdt -$$

$$- \sum_\alpha m_\alpha \int U(x,t) f_\alpha(x,p,t)\,dxdpdt + \frac{1}{8\pi}\int (\nabla\varphi)^2\,dxdt - \frac{1}{8\pi G}\int (\nabla U)^2\,dxdt.$$

- Varying this expression as before we obtain a system of Vlasov-Poisson-Poisson plasma with gravitation:

$$\frac{\partial f_\alpha}{\partial t} + \left(\frac{p}{m_\alpha}, \frac{\partial f_\alpha}{\partial x}\right) - \left(m_\alpha \frac{\partial U}{\partial x} + e_\alpha \frac{\partial \varphi}{\partial x}, \frac{\partial f_\alpha}{\partial p_i}\right) = 0,$$

$$\Delta U = 4\pi G \sum_\alpha m_\alpha \int f_\alpha(x,p,t)\,dp,$$

$$\Delta \varphi = -4\pi \sum_\alpha e_\alpha \int f_\alpha(x,p,t)\,dp.$$

- As shown, a complete system of Vlasov-Maxwell equations is obtained by varying the action of electro-magnetism with the transition to the distribution function:

$$\frac{\partial f_\alpha}{\partial t} + \left(v_\alpha, \frac{\partial f_\alpha}{\partial x}\right) + e_\alpha \left(-\frac{1}{c}\frac{\partial A_i}{\partial t} - \frac{\partial A_0}{\partial x} - \frac{1}{c} F_{ij} v_\alpha^j\right) \frac{\partial f_\alpha}{\partial p_i} = 0, \quad (5)$$

$$\frac{\partial F^{\mu\nu}}{\partial x_\nu} = -\frac{4\pi}{c} \sum_\alpha e_\alpha \int v_\alpha^\mu f_\alpha(x,p,t) dp,\ F_{\mu\nu} = \frac{\partial A_\mu}{\partial x_\nu} - \frac{\partial A_\nu}{\partial x_\mu},\ (\mu,\ \nu\ : 1, \ldots, 4),$$

$$E_i = -\frac{1}{c}\frac{\partial A_i}{\partial t} - \frac{\partial A_0}{\partial x^i},\ [v_\alpha, H] = -F_{ij} v_\alpha^j,\ v_\alpha = \frac{p}{m_\alpha \gamma_\alpha}, \gamma_\alpha = \sqrt{1 + \frac{p^2}{m_\alpha^2 c^2}}.$$

- Lagrange's Identity is defined as the second time derivative of the moment of inertia through the kinetic and potential energy.
- Following to *V. V. Kozlov, "The generalized Vlasov kinetic equation", Russian Math. Surveys, 63:4 (2008)* we show that the Lagrange Identity can be extended to the case of Vlasov-Maxwell equations.

- Let us introduce the moment of inertia of the particles respect to origin of coordinates:

$$I(t) = \sum_{\alpha} \int f_{\alpha}(t, x, p) x^2 d^3p d^3x,$$

$$T(t) = \frac{1}{2} \sum_{\alpha} \int f_{\alpha}(t, x, p) v_{\alpha}^2 d^3p d^3x,$$

$$\Pi = \sum_{\alpha} \int \frac{e_{\alpha}}{\gamma_{\alpha} m_{\alpha}} \left(x, E + \frac{1}{c} \left[v_{\alpha}, H \right] \right) f_{\alpha} d^3p d^3x -$$

$$- \sum_{\alpha} \int \frac{e_{\alpha}}{\gamma_{\alpha}^3 m_{\alpha}^3 c^2} (p, x)(p, E) f_{\alpha} d^3p d^3x.$$

Lagrange Identity is valid as:

$$\ddot{I} = 4T - 2\Pi. \tag{6}$$

- Prove: From (5) we have

$$\ddot{I} = -2\sum_{\alpha}\int\left(\frac{\partial f_\alpha}{\partial x}, v_\alpha\right)(v_\alpha, x)d^3pd^3x -$$

$$-2\sum_{\alpha}\int(v_\alpha, x)e_\alpha\left(E + \frac{1}{c}[v_\alpha, H]\right)\frac{\partial f_\alpha}{\partial p_i}d^3pd^3x.$$

- The first integral in this expression with integrating by parts can be transformed to :

$$2\sum_{\alpha}\int(v_\alpha, v_\alpha)\, f_\alpha d^3pd^3x = 4T.$$

- The second integral can be transformed, if we count :

$$\frac{\partial v_i}{\partial p_j} \text{ where, } v_i = \frac{p_i}{m_\alpha \gamma_\alpha},\ \gamma_\alpha\sqrt{1 + \frac{p^2}{m_\alpha^2 c^2}},$$

$$\frac{\partial v_i}{\partial p_j} = \frac{\delta_{ij}}{\gamma_\alpha m_\alpha} - p_j\frac{p_i}{\gamma_\alpha^3 m_\alpha^3 c^2} \Rightarrow F_{ij}\frac{\partial v_j}{\partial p_i} = 0$$

- Then we can get:

$$-2\sum_{\alpha}\int\frac{\partial v_{\alpha j}}{\partial p_i}x_j e_\alpha\left(-\frac{1}{c}\frac{\partial A_i}{\partial t} - \frac{\partial A_0}{\partial x^i} - \frac{1}{c}F_{ij}v_\alpha^j\right)f_\alpha d^3pd^3x =$$

$$= -2\sum_{\alpha}\int\left(\frac{\delta_{ij}}{\gamma_\alpha m_\alpha} - \frac{p_i p_j}{\gamma_\alpha^3 m_\alpha^3 c^2}\right) x_j e_\alpha \left(-\frac{1}{c}\frac{\partial A_i}{\partial t} - \frac{\partial A_0}{\partial x^i} - \frac{1}{c}F_{ij}v_\alpha^j\right) f_\alpha d^3p d^3x$$

$$= -2\sum_{\alpha}\int \frac{e_\alpha}{\gamma_\alpha m_\alpha}\left(x, E + \frac{1}{c}[v_\alpha, H]\right) f_\alpha d^3p d^3x +$$

$$+2\sum_{\alpha}\int \frac{p_i p_j}{\gamma_\alpha^3 m_\alpha^3 c^2} x_j e_\alpha E_i\_ f_\alpha d^3p d^3x = -2\Pi$$

- We use: $p_i F_{ij} v_\alpha^i = 0$.
- So finaly, the second term is transformed into:

$$+2\sum_{\alpha}\int \frac{e_\alpha}{\gamma_\alpha^3 m_\alpha^3 c^2}(p, x)(p, E) f_\alpha d^3p d^3x.$$

- Lagrange's Identity can be useful in studies of stability.
- Derivation shows that the second term in the functional Π is associated with relativism.

1. Derivation and classificaition of magnetohydrodynamic equations

- First we consider the case of zero temperature, to obtain the corresponding equations of the exact consequences of Vlasov-Maxwell equations by using the following substitution:

$$f_\alpha(t,x,p) = n_\alpha(x,t)\delta(p - P_\alpha(x,t)) \tag{7}$$

- This is the ultimate form of Maxwell distribution when temperature $T_\alpha \to 0$

$$f_\alpha(t,x,p) = \frac{n_\alpha(x,t)}{(2k\pi T_\alpha m_\alpha)^{\frac{3}{2}}}\, e^{-\frac{(p-P_\alpha)^2}{2kT_\alpha m_\alpha}} \xrightarrow[T_\alpha \to 0]{} n_\alpha(x,t)\delta(p - P_\alpha(x,t)).$$

- We obtain the equations of the electromagnetic form of multifluid hydrodynamics. These equations have the form

$$\frac{\partial n_\alpha}{\partial t} + \textbf{div}(v_\alpha(P_\alpha) n_\alpha) = 0; \tag{8}$$

$$\frac{\partial P_\alpha}{\partial t} + v_\alpha^i(P_\alpha)\frac{\partial P_\alpha}{\partial x_i} - e_\alpha\left(E + \frac{1}{c}[v_\alpha(P_\alpha), H]\right) = 0;$$

- These equations are supplemented by Maxwell's equations

$$\nabla \times E - \frac{\partial B}{\partial t} = 0; \ \nabla \cdot H = 0; \tag{9}$$

$$\nabla \cdot E = 4\pi \sum_{\alpha} e_\alpha n_\alpha;$$

$$\nabla \times H - \frac{1}{c}\frac{\partial E}{\partial t} = -\frac{4\pi}{c}\sum_{\alpha} e_\alpha n_\alpha v_\alpha(P_\alpha);$$

- We should notice that these equations are exact consequences of Vlasov-Maxwell equations, so the Lagrange Identity obtained for the system (8) - (9) by substituting (7) in the Lagrange Identity (6):

$$\ddot{I}_2 = 4T_2 - 2\Pi_2, \text{ where } I_2(t) = \sum_{\alpha} \int n_\alpha(x,t) x^2 d^3x, \tag{10}$$

$$T_2(t) = \frac{1}{2}\sum_{\alpha}\int n_\alpha(x,t) v_\alpha^2(P_\alpha(x,t)) d^3x,$$

$$\Pi_2 = \sum_{\alpha}\int \frac{e_\alpha}{\gamma_\alpha m_\alpha} n_\alpha(x,t)\left(x, E + \frac{1}{c}\left[v_\alpha(P_\alpha), H\right]\right) d^3x -$$

$$-\sum_{\alpha}\int \frac{e_\alpha}{\gamma_\alpha^3 m_\alpha^3 c^2} n_a(x,t)(P_\alpha, x)(P_\alpha, E) d^3x.$$

- In the nonrelativistic case for Vlasov-Maxwell equation we have

$$S_{\mathsf{p}} = \sum_{\alpha} \frac{m_\alpha}{2} \sum_{q} \int \dot{x}_\alpha^2(q,t)dt, \;\; v_\alpha(p) = \frac{p}{m_\alpha}.$$

- Lagrange Identity in this case:

$$\ddot{I}_3 = 4T_3 - 2\Pi_3, \;\text{where}\; I_3(t) = \sum_{\alpha} \int f_\alpha(t,x,p) x^2 d^3p d^3x, \qquad (11)$$

$$T_3(t) = \frac{1}{2} \sum_{\alpha} \int f_\alpha(t,x,p) v_\alpha^2 d^3p d^3x,$$

$$\Pi_3 = \sum_{\alpha} \int \frac{e_\alpha}{m_\alpha} \left(x, E + \frac{1}{c}\left[v_\alpha, H\right] \right) f_\alpha d^3p d^3x.$$

- And for the system (8)-(9):

$$\ddot{I}_4 = 4T_4 - 2\Pi_4, \;\text{where}\; [I_4(t) = \sum_{\alpha} \int n_\alpha(x,t) x^2 d^3x, \qquad (12)$$

$$T_4(t) = \frac{1}{2} \sum_{\alpha} \int n_\alpha(t,x) v_\alpha^2(P_\alpha(x,t)) d^3x,$$

$$\Pi_4 = \sum_{\alpha} \int \frac{n_\alpha(x,t)}{m_\alpha} e_\alpha \left(x, E + \frac{1}{c}\left[v_\alpha(P_\alpha), H\right] \right) d^3x.$$

Generalization of Lagrange's Identity

- There is a generalization of Lagrange's Identity when, instead of an function of x^2 is taken arbitrary function $\varphi(x)$

$$I(t) = \sum_\alpha \int f_\alpha \varphi(x) d^3p d^3x,$$

- For this functional from the Vlasov-Maxwell system (5) we have:

$$\ddot{I} = \sum_\alpha \int \frac{\partial\varphi}{\partial x_i \partial x_j} v_{\alpha i} v_{\alpha j} f_\alpha d^3p d^3x +$$

$$+ \sum_\alpha \int \frac{e_\alpha}{\gamma_\alpha m_\alpha} \left(\frac{\partial \varphi}{\partial x}, E + \frac{1}{c}\left[v_\alpha, H\right] \right) f_\alpha d^3p d^3x -$$

$$- \sum_\alpha \int \frac{e_\alpha}{\gamma_\alpha^3 m_\alpha^3 c^2} (p, \frac{\partial\varphi}{\partial x})(p, E) f_\alpha d^3p d^3x.$$

Two-fluid and regular MHD (or EMHD) with non-zero temperature

- We obtained the EMHD equations of hydrodynamic type from system of kinetic equations, by introducing the following momentums and integrating the Vlasov-Maxwell system:

$$n_\alpha = \int f_\alpha(t,x,p)\,d^3p,\;\; P_{\alpha i} = \frac{1}{n_\alpha}\int p_i f_\alpha(t,x,p)\,d^3p, \qquad (13)$$

$$D_\alpha = \frac{1}{n_\alpha}\int (p - P_\alpha)^2 f_\alpha(t,x,p)\,d^3p,$$

- $n_\alpha(x,t)$ – density numbers of particles of α-sorts,
- $P_{\alpha i}(x,t)$– mathematical expectation of momentum,
- D_α – variance of the momentums of all particles of each kind, which is proportional to the energy of random motion.

$$\frac{\partial n_\alpha}{\partial t} + \frac{\partial}{\partial x}(n_\alpha v_\alpha) = 0, \qquad (14)$$

$$\frac{\partial}{\partial t}(n_\alpha P_\alpha) + \frac{\partial}{\partial x_i}(n_\alpha P_{\alpha i} P_{\alpha j} + \sigma_{\alpha ij}) - n_\alpha e_\alpha\left(\vec{\mathbf{E}} + \frac{1}{c}[\vec{v}_\alpha, \vec{\mathbf{H}}]\right) = 0,$$

$$\sigma_{\alpha ij} = \int (p_i - P_{\alpha i})(p_j - P_{\alpha j}) f_\alpha(t,x,p)\,d^3p$$ – stress tensor,

$$\frac{\partial}{\partial t}(n_\alpha D_\alpha) + \frac{\partial}{\partial x_i} q_i = 0,$$

$$q_i = \int \frac{p_i}{m}(p - P_\alpha)^2 f_\alpha(t,x,p)\,d^3p$$ – heat flow vector.

- This is a presise system of equations (14), but it is not closed. To close it, we must add the collision integral, or (from interaction with the environment) add a linear collision integral. This means that higher-order momentums are determined through the lower with the Maxwell distribution.

$$f_\alpha(t,x,p) = \frac{n_\alpha(x,t)}{(2k\pi T_\alpha m_\alpha)^{\frac{3}{2}}} e^{-\frac{(p-P_\alpha)^2}{2kT_\alpha m_\alpha}},$$

- It turns out that: $\sigma_{\alpha ij} = \delta_{ij} k n_\alpha T_\alpha, \ \ D_\alpha = 3kT_\alpha.$
- More briefly those equations can be written in the Godunov's form, for this we introduce Godunov's function:

$$G^\alpha(\beta^\alpha_\mu) = \int f^0_\alpha(\beta^\alpha_\mu) d^3p, \quad \mu = (0,...,4), \tag{15}$$

$$f^0_\alpha(\beta^\alpha_\mu) = \exp[\beta^\alpha_0 + \beta^\alpha_1 p_1 + \beta^\alpha_2 p_2 + \beta^\alpha_3 p_3 + \beta^\alpha_4 p^2],$$

Compare

$f^0_\alpha(\beta^\alpha_\mu) = \exp\left[\beta_0 - \frac{\beta_1^2+\beta_2^2+\beta_3^2}{4\beta_4}\right] \exp\left[\beta_4\left(p + \frac{\beta}{2\beta_4}\right)^2\right], \ \beta = (\beta_1, \beta_2, \beta_3)$

with $f_\alpha(t,x,P) = \frac{n_\alpha(x,t)}{(2k\pi T_\alpha m_\alpha)^{\frac{3}{2}}} \exp\left[-\frac{(p-P_\alpha)^2}{2kT_\alpha m_\alpha}\right]$

We obtain β_μ in terms of thermodynamic variables:

$$\beta_0^\alpha = \ln n_\alpha - \frac{3}{2}\ln(2\pi k T_\alpha m_\alpha) - \frac{P_\alpha^2}{2kT_\alpha}, \quad \beta_1^\alpha = \frac{P_{\alpha 1}}{kT_\alpha},$$

$$\beta_2^\alpha = \frac{P_{\alpha 2}}{kT_\alpha}, \quad \beta_3^\alpha = \frac{P_{\alpha 3}}{kT_\alpha}, \quad \beta_4^\alpha = -\frac{1}{2kT_\alpha m_\alpha},$$

If we define the vector

$$K_\mu^\alpha = \left(0, F_1^\alpha n_\alpha, F_2^\alpha n_\alpha, F_3^\alpha n_\alpha, -2F_i^\alpha G_{\beta_i}^\alpha\right),$$

$$F^\alpha = e_\alpha\left(E + [v_\alpha, H]\right), \quad i = (1,2,3),$$

the system (14) can be written in the form of Godunov:

$$\frac{\partial G_{\beta_\mu}^\alpha}{\partial t} + \frac{\partial G_{\beta_\mu \beta_i}^\alpha}{\partial x_i} + K_\mu^\alpha = 0, \text{ here } G_{\beta_\mu}^\alpha = \frac{\partial G^\alpha}{\partial \beta_\mu}. \tag{16}$$

A generalization of Lagrange's Identity in this case has the following remarkable representation:

$$I(t) = \sum_\alpha \int f_\alpha^0(\beta_\mu^\alpha)\phi(x) d^3p d^3x,$$

$$\ddot{I} = \sum_\alpha \int \frac{\partial \phi}{\partial x_i \partial x_j} G_{\beta_i}^\alpha G_{\beta_j}^\alpha d^3x - \int \frac{\partial \phi}{\partial x_i} G^\alpha F_i^\alpha d^3x.$$

1. Steady-state solutions and critical mass value

- As shown, a complete system of Vlasov-Poisson-Poisson can be obtained from Lagrangian of the electrostatic plus gravitation (in nonrelativistic case) with the transition to the distribution function.

$$\frac{\partial f_\alpha}{\partial t}+\left(\frac{p}{m_\alpha},\frac{\partial f_\alpha}{\partial x}\right)-\left(m_\alpha\frac{\partial U}{\partial x}+e_\alpha\frac{\partial \varphi}{\partial x},\frac{\partial f_\alpha}{\partial p_i}\right)=0, \qquad (17)$$

$$\Delta U=4\pi G\sum_\alpha m_\alpha\int f_\alpha(x,p,t)dp,$$

$$\Delta\varphi=-4\pi\sum_\alpha e_\alpha\int f_\alpha(x,p,t)dp.$$

- Now we investigate the possible stationary solutions for (17).
- Assume that the distribution functions f_α are different functions of energy and are as follows:

$$f_\alpha=g_\alpha\left(\frac{p^2}{2m_\alpha}+m_\alpha U+e_\alpha\varphi\right).$$

- g_α – are arbitrary nonnegative function

- In this case we obtain a system of nonlinear elliptic equations for potentials

$$\Delta U = V(U,\varphi),\ V(U,\varphi) = 4\pi G \sum_{\alpha=1}^{N} m_\alpha \int g_\alpha(\frac{p^2}{2m_\alpha} + m_\alpha U + e_\alpha \varphi) d^3p,$$

$$\Delta \varphi = \Psi(U,\varphi),\ \Psi(U,\varphi) = -4\pi \sum_{\alpha=1}^{N} e_\alpha \int g_\alpha(\frac{p^2}{2m_\alpha} + m_\alpha U + e_\alpha \varphi) d^3p.$$

- We investigate this system of equations. Let's start with the simplest case of one type of particles, when $N = 1$ and

$$\Delta U = 4\pi G m \int g(\frac{p^2}{2m} + mU + e\varphi) d^3p, \tag{18}$$

$$\Delta \varphi = -4\pi e \int g(\frac{p^2}{2m} + mU + e\varphi) d^3p.$$

- Therefore the system can be rewritten as:

$$\Delta\left(mU+e\varphi\right)=\left(Gm^2-e^2\right)\int g\left(\frac{p^2}{2m}+mU+e\varphi\right)d^3p, \tag{19}$$

$$\Delta\left(eU+Gm\varphi\right)=0.$$

- It turns out that the conditions for the solvability of the first equation, depends on the sign of expression Gm^2-e^2.
- If this value is positive, the boundary problem is correct, otherwise there are global solutions. Thus, the value of the mass $m=\sqrt{\frac{e^2}{G}}$ – is critical.
- When $m>\sqrt{\frac{e^2}{G}}$, the gravitational force stronger than the electrostatic forces. If e – is an electron charge, then this mass is $m\approx 10^{-12}$ grams.

Conclusion:

- We considered the derivation of the Vlasov-Maxwell from classical Lagrangian of electrodynamics and the Lagrange Identity.
- This derivation is a convenient alternative to the methods of the BBGKY hierarchy [3] and the microscopic solutions methods [4-9], because it is simplier, can be used for important case of Vlasov Maxwell where other methods does not work, and give us classification of different types of equations of Vlasov type.
- We propose a derivation of MHD- and EMHD-type equations, for which variety only increases, and it allows us to monitor for the nature of the approximations made. We present this equations to the remarkable Godunov's double divergence form.
- We also examined the derivation of the Vlasov-Poisson-Poisson plasma with gravitation. Study of stationary solutions of these equations in the cases, where the distribution function is an arbitrary function of the energy integral show us that in this case the problem reduces to the elliptic system of nonlinear equations with different behavior.

References

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Thank you for your attention!

For the equations of ideal incompressible fluid, V.I.Arnold proved a theorem about the structure of stationary solutions, based on the existence of two commuting vector fields.
V. I. Arnol'd, "On the topology of three-dimensional steady flows of an ideal fluid", J. Appl. Math. Mech., 30 (1966), PP 223–226)

This construction was generalized by V.V.Kozlov for the case of compressible fluid.
V.V. Kozlov, "Notes on steady vortex motions of continuous medium", J. Appl. Math. Mech., 47:2 (1983), 288–289

We explore the possibility of such structures for the case of the Vlasov-Poisson and Vlasov-Maxwell equations.

Consider more general case: Vlasov-Poisson-Poisson equations with electrostatics and with gravity:

$$
\begin{aligned}
&\frac{\partial f_\alpha}{\partial t} + \left(\frac{p}{m_\alpha}, \frac{\partial f_\alpha}{\partial x}\right) + \left(-m_\alpha \frac{\partial U}{\partial x} - e_\alpha \frac{\partial \varphi}{\partial x}, \frac{\partial f_\alpha}{\partial p}\right) = 0,\\
&\Delta U = 4\pi G \sum_\alpha m_\alpha \int f_\alpha(t,x,p)dp, \\
&\Delta \varphi = -4\pi \sum_\alpha e_\alpha \int f_\alpha(t,x,p)dp.
\end{aligned}
\tag{1.1}
$$

The first equation is the equation of collective motion of the particles (Liouville equation) for the ordinary Newton equations of motion:

$$
\begin{cases}
\dot{x} = \dfrac{p}{m_\alpha}, \\
\dot{p} = -m_\alpha \dfrac{\partial U}{\partial x} - e_\alpha \dfrac{\partial \varphi}{\partial x}.
\end{cases}
$$

“hydrodynamic” substitution gives us the exact solutions

$$f_\alpha(t,x,p) = n_\alpha(x,t)\delta(p - P_\alpha(x,t)) \tag{1.2}$$

for the system (1.1), if n_α and P_α is determined by the system of equations

$$\begin{aligned}
&\frac{\partial n_\alpha}{\partial t} + \frac{1}{m_\alpha}\textbf{div}(n_\alpha P_\alpha) = 0,\\
&\frac{\partial P_\alpha}{\partial t} + \frac{1}{m_\alpha}P_{\alpha i}\frac{\partial P_\alpha}{\partial x_i} = -m_\alpha\frac{\partial U}{\partial x} - e_\alpha\frac{\partial \varphi}{\partial x},\\
&\Delta U = 4\pi G\sum_\alpha m_\alpha n_\alpha(x,t),\\
&\Delta\varphi = -4\pi\sum_\alpha e_\alpha n_\alpha(x,t).
\end{aligned} \tag{1.3}$$

If we rewrite the second equation in the Gromeka form:

$$\frac{\partial P_\alpha}{\partial t} + \frac{1}{m_\alpha}P_{\alpha i}\left(\frac{\partial P_\alpha}{\partial x_i} - \frac{\partial P_{\alpha i}}{\partial x}\right) = -m_\alpha\nabla U - e_\alpha\nabla\varphi - \frac{1}{2m_\alpha}\nabla(P_\alpha^2) \tag{1.4}$$

one can see a gradient from the "Bernoulli's integral" in the right side.

Let R^α be the matrix R^α_{ij}

$$R^\alpha_{ij} = \frac{\partial P_{\alpha i}}{\partial x_j} - \frac{\partial P_{\alpha j}}{\partial x_i} \text{ curl of momentum } P_\alpha.$$

Taking curl of momentum from the equation (1.4), give us the equation

$$\frac{\partial R_\alpha}{\partial t} + \mathbf{rot}[R^\alpha \times P_\alpha] = 0.$$

In the steady-state case we have

$$\mathbf{rot}[R^\alpha \times P^\alpha] = 0. \tag{1.5}$$

V.I.Arnold and V.V.Kozlov: If the continuity equation $\mathbf{div}(n_\alpha P_\alpha) = 0$ is satisfied then vector fields $\frac{R^\alpha}{n_\alpha}$ and P^α commute. Then a surface formed by those two vector fields is either plane, or cilinder or torus.

$$\left[\frac{R^\alpha}{n_\alpha}, P^\alpha\right] = 0. \tag{1.6}$$

For the Vlasov-Maxwell system of equations we have two difficulties

- The Lorentz force does not have the form of the gradient.
- Momentum and velocity of the particle are of relativistic dependence. Nevertheless those difficulties could be overcome.

The system of Vlasov-Maxwell equations has the following form:

$$\frac{\partial f_\alpha}{\partial t} + \left(v_\alpha(p), \frac{\partial f_\alpha}{\partial x} \right) + e_\alpha \left(E + \frac{1}{c}[v_\alpha(p) \times H], \frac{\partial f_\alpha}{\partial p} \right) = 0,$$
$$\mathbf{rot} H - \frac{1}{c}\frac{\partial E}{\partial t} = -\frac{4\pi}{c}\sum_\alpha e_\alpha \int v_\alpha(p) f_\alpha(t,x,p) d^3p, \qquad (2.1)$$
$$\mathbf{div} E = 4\pi \sum_\alpha e_\alpha \int f_\alpha(t,x,p) d^3p, \mathbf{rot} E = \frac{\partial H}{\partial t}, \mathbf{div} H = 0.$$

Here $v_\alpha(p) = \frac{p}{m_\alpha}\frac{1}{\sqrt{1+\frac{p^2}{m_\alpha^2 c^2}}}$.

Electro-Magnetic-Hydro-Dynamic (EMHD) equations obtained from the system (2.1) by substituting

$$f_\alpha(t,x,p) = n_\alpha(x,t)\delta(p - Q_\alpha(x,t))$$

has the form

$$\begin{aligned}
&\frac{\partial n_\alpha}{\partial t} + \mathbf{div}(n_\alpha v_\alpha(P_\alpha)) = 0,\\
&\frac{\partial Q_\alpha}{\partial t} + v_\alpha^i(Q_\alpha)\frac{\partial Q_\alpha}{\partial x_i} = e_\alpha\left(E + \frac{1}{c}[v_\alpha(P_\alpha)\times H]\right),\\
&\mathbf{rot}E = \frac{\partial H}{\partial t}, \mathbf{div}E = 4\pi\sum_\alpha e_\alpha n_\alpha.\\
&\mathbf{div}H = 0, \mathbf{rot}H - \frac{1}{c}\frac{\partial E}{\partial t} = -\frac{4\pi}{c}\sum_\alpha e_\alpha n_\alpha v_\alpha(P_\alpha),
\end{aligned} \tag{2.2}$$

We transform the second equation of system (2.2) to Gromeka form

$$v_\alpha^i(Q_\alpha)\frac{\partial Q_{\alpha i}}{\partial x} = \nabla K(Q_\alpha),$$

$$\text{here } K(Q_\alpha) = \sum_i \int v_\alpha^i(Q)d(Q_i) = \sum_i \frac{1}{m_\alpha}\int \frac{p_i dp_i}{\sqrt{1+\frac{p^2}{m_\alpha^2 c^2}}} =$$

$$= \frac{(m_\alpha c)^2}{m_\alpha}\int \frac{d(\frac{p^2}{(m_\alpha c)^2}+1)}{2\sqrt{1+\frac{p^2}{(m_\alpha c)^2}}} = \frac{(m_\alpha c)^2}{m_\alpha}\sqrt{1+\frac{p^2}{(m_\alpha c)^2}}.$$

In the relativistic case this term will have a gradient form – so we have overcome the second of difficulties.

However, the Lorentz force do not have a gradient form, so we convert it by combining with a similar member $[v^\alpha \times \textbf{rot } Q_\alpha]$ and moving to the left side $[v^\alpha \times (\textbf{ rot } Q_\alpha - \frac{e_\alpha}{c} H)]$.
If we take the curl of both sides then we get

$$\textbf{rot}[v_\alpha \times (\textbf{rot} Q_\alpha - \frac{e_\alpha}{c} H)] = 0. \tag{2.3}$$

But we have the equation of continuity - the first of equations (2.2)

$$\textbf{div}(n_\alpha v_\alpha(p)) = 0.$$

Hence by the theorem of Arnold Kozlov follows that the vector fields

$$v_\alpha, \quad \frac{\textbf{rot} Q_\alpha - \frac{e_\alpha}{c} H}{n_\alpha}$$

commute:

$$\left[v_\alpha(Q_\alpha), \frac{\textbf{rot} Q_\alpha - \frac{e_\alpha}{c} H}{n_\alpha} \right] = 0.$$

Here $Q_\alpha - \frac{e_\alpha}{c} A$ – generalized momentum of the electromagnetic field.
We have the same result but for other fields.

In N-layered case, we have:

$$f_\alpha(t,x,p) = \sum_{m=1}^{N} n_{\alpha\mu}\delta(p - Q_\alpha^\mu(x,t)).$$

In the continuum-layer version we have:

$$f_\alpha(t,x,p) = \int n_\alpha(\mu;x,t)\delta(p - Q_\alpha(\mu;x,t))d\mu. \tag{3.1}$$

The equations obtained here

$$\begin{aligned} &\frac{\partial n_\alpha(\mu;x,t)}{\partial t} + \mathbf{div}(n_\alpha v_\alpha(\mu;x,t)) = 0,\\ &\frac{\partial Q_\alpha(\mu)}{\partial t} + v_\alpha^i \frac{\partial Q_\alpha(\mu)}{\partial x_i} = e_\alpha(E + \frac{1}{c}[v_\alpha \times H]),\\ &\mathbf{div} E = 4\pi \sum_\alpha e_\alpha \int n_\alpha(\mu;x,t)d\mu,\\ &\mathbf{rot} H - \frac{1}{c}\frac{\partial E}{\partial t} = -\frac{4\pi}{c}\sum_\alpha e_\alpha \int n_\alpha v_\alpha d\mu,\\ &\mathbf{rot} E - \frac{\partial H}{\partial t} = 0, \mathbf{div} H = 0. \end{aligned} \tag{3.2}$$

For each μ , we have equation of the form (2.2) and its equation of continuity with the same conclusions for each layer μ .

We use the analogy between the Vlasov and Liouville equations.
What does the hydrodynamic substitution give for the Liouville equation?
Consider an arbitrary system of nonlinear equations

$$\dot{\mathbf{x}} = \mathbf{G}(\mathbf{x}), \mathbf{x} \in R^n, \mathbf{G}(\mathbf{x}) \in R^n. \tag{4.1}$$

And its Lioville or continuity equation

$$\frac{\partial f(\mathbf{x}, t)}{\partial t} + \frac{\partial}{\partial x_i}(\rho G_i) = 0, \tag{4.2}$$

We can arbitrarily divide the variables $\mathbf{x}$ to "coordinates" and "impulses" or momenta $\mathbf{x} = \{\mathbf{q}, \mathbf{p}\}, \mathbf{q} \in R^k, \mathbf{p} \in R^{n-k}$, here $k :\ 1 \leq k \leq n-1$.
"Hydrodynamic" substitution:

$$f(\mathbf{x}, t) = f(\mathbf{q}, \mathbf{p}, t) = \rho(\mathbf{q}, t)\delta(\mathbf{p} - \mathbf{Q}(\mathbf{q}, t)).$$

So for ρ and $\mathbf{Q}$ we have

$$\begin{cases} \dfrac{\partial \rho(\mathbf{q},t)}{\partial t} + \dfrac{\partial}{\partial q_i}(\rho V_i) = 0, \\ \dfrac{\partial \mathbf{Q}(\mathbf{q},t)}{\partial t} + V_i \dfrac{\partial \mathbf{Q}}{\partial q_i} = F. \end{cases} \tag{4.3}$$

Here we determinate **V** and **F** by rewriting the system of the equations (4.1)

$$\begin{cases} \dot{\mathbf{q}} = \mathbf{v}(\mathbf{q},\mathbf{p}), \\ \dot{\mathbf{p}} = \mathrm{g}(\mathbf{q},\mathbf{p}). \end{cases} \tag{4.4}$$

Then

$$\mathbf{V}(\mathbf{q},t) = \mathbf{v}(\mathbf{q},\mathbf{Q}(t,\mathbf{q})), \quad \mathbf{F}(\mathbf{q},t) = \mathbf{g}(\mathbf{q},\mathbf{Q}(t,\mathbf{q})).$$

To prove this we rewrite the Liouville equation (4.2) as

$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial q_i}(v_i f) + \frac{\partial}{\partial p_j}(\mathrm{g}_j f) = 0. \tag{4.5}$$

The easiest way to obtain equation (4.3) - using "moments method"

We integrate (4.5) over the :

$$\frac{\partial \rho}{\partial t}+\int \frac{\partial}{\partial q_i}(v_i f)d\mathbf{p}^{n-k}+\int \frac{\partial}{\partial p_j}(\mathrm{g}_j f)d\mathbf{p}^{n-k}=0. \tag{4.6}$$

The third term is equal to zero for f decreasing at infinity, while the second takes the form of divergence

$$\frac{\partial}{\partial q_i}\int \left(v_i(\mathbf{q},\mathbf{p})\rho(\mathbf{q},t)\delta(\mathbf{p}-\mathbf{Q}(\mathbf{q},t))\right)d\mathbf{p}=\frac{\partial}{\partial q_i}(V_i\rho).$$

So we got the first equation (4.3). To get the second, we multiply (4.5) by $\mathbf{p}$ and integrate using the fact that: $\int \mathbf{p}\delta(\mathbf{p}-\mathbf{Q})d\mathbf{p}=Q$:

$$\frac{\partial(\mathbf{Q}\rho)}{\partial t}+\int \mathbf{p}\frac{\partial}{\partial q_i}(v_i f)d\mathbf{p}+\int \mathbf{p}\frac{\partial(\mathrm{g}_j f)}{\partial p_j}d\mathbf{p}=0.$$

The second term is transformed by putting differentiation over the $\mathbf{q}$ before integral:

$$\frac{\partial}{\partial q_i}\int \mathbf{p}\rho v_i\delta(\mathbf{p}-\mathbf{Q})d\mathbf{p}^{n-k}=\frac{\partial}{\partial q_i}\left(\rho v_i(\mathbf{q})\mathbf{Q}\right).$$

The third term is transformed by integrating by parts:

$$\int \mathbf{p}\frac{\partial}{\partial p_j}(\mathrm{g}_j f)d\mathbf{p} = -\int \rho\mathbf{g}(x,\mathbf{p})\delta(\rho - \mathbf{Q})d\rho = -\rho\mathbf{g}(\mathbf{q},\mathbf{Q}(\mathbf{q},t)) = -\rho\mathbf{F}(\mathbf{q},t).$$

So we get the system of equations, which differs from (4.3)

$$\begin{cases} \dfrac{\partial\rho}{\partial t} + \dfrac{\partial}{\partial q_j}(\rho V_j) = 0, \\ \dfrac{\partial(\mathbf{Q}\rho)}{\partial t} + \dfrac{\partial}{\partial q_i}(\rho V_i\mathbf{Q}) = \rho\mathbf{F}. \end{cases} \tag{4.7}$$

However, taking into account the continuity equation, the second equation in (4.7) is equivalent to the second equation in (4.3).

Suppose now that system (4.1) is Hamiltonian, and in (4.4) $v = \frac{\partial H}{\partial p}, \mathrm{g} = -\frac{\partial H}{\partial q}$. The system (4.4) takes the usual Hamiltonian form:

$$\begin{cases} \dot{q} = \dfrac{\partial H}{\partial p} \\ \dot{p} = -\dfrac{\partial H}{\partial q} \end{cases} \tag{4.8}$$

If $k = n - k$):

$$\frac{\partial Q}{\partial t} + V_i \frac{\partial Q}{\partial x_i} - V_i \frac{\partial Q_i}{\partial x} = F - V_i \frac{\partial Q_i}{\partial x}.$$

This equation is identical to that V.V.Kozlov get by another method, for wich it was not clear where from continuity equations comes. At the right we have gradient

$$F - V_i \frac{\partial Q_i}{\partial x} = -\frac{\partial H}{\partial q}(x,p)\bigg|_{p=Q(x,t)} - \frac{\partial H}{\partial p}(x,p)\bigg|_{p=Q(x,t)} \frac{\partial Q}{\partial x}$$

V.V.Kozlov: substitution $\frac{\partial S}{\partial x}$ give us the following

$$\frac{\partial}{\partial x}\left(\frac{\partial S}{\partial t}+H\left((x,\frac{\partial S}{\partial x}\right)\right)=0.$$

For $S(x,t)$ obtained the Hamilton-Jacobi equation after "calibration" $S+\mathrm{g}(t)\to S$ on a function of time. So we have generalized Kozlov conclusions for nonhamiltonian case. It seems that Hamilton-Jacobi method could be applied in nonhamiltonian situation. Our following goal – apply this to the Vlasov-Poisson and to Vlasov-Maxwell equations. For Vlasov-Poisson-Poisson one gets.
Let for(1.3) $P_\alpha=\frac{\partial S}{\partial x}$. We get:

$$\begin{cases} \dfrac{\partial n_\alpha}{\partial t}+\dfrac{1}{m_\alpha}\dfrac{\partial}{\partial x_i}\left(n_\alpha\dfrac{\partial S_\alpha}{\partial x_i}\right)=0 \\ \dfrac{\partial S}{\partial t}+\dfrac{1}{2m_\alpha}\left(\nabla S_\alpha,\nabla S_\alpha\right)=-m_\alpha U(x)-e_\alpha\varphi(x) \\ \Delta U=4\pi G\sum\limits_\alpha m_\alpha n_\alpha \\ \Delta\varphi=-4\pi\sum\limits_\alpha e_\alpha n_\alpha \end{cases} \tag{4.9}$$

Vlassov-Maxwell case does not pass due to the fact that the Lorentz force do not have a gradient form. But for Vlasov-Poisson or Vlasov-Poisson–Poisson equation we can make it even for the case of a chain of hydrodynamic equation or continuum, that one can get by the substitution:

$$f_\alpha(t,x,p) = \int n_\alpha(\mu;x,t)\delta(p - Q_\alpha(\mu;x,t))d\mu. \tag{4.10}$$

$$\begin{cases} \dfrac{\partial n_\alpha(\mu;x,t)}{\partial t} + \dfrac{1}{m_\alpha}\,\mathbf{div}\,(n_\alpha(\mu;x,t)P_\alpha(\mu;x,t)) = 0 \\ \dfrac{\partial P_\alpha(\mu;x,t)}{\partial t} + \dfrac{1}{m_\alpha}P_{\alpha i}\dfrac{\partial P_\alpha}{\partial x_i} = -m_\alpha\dfrac{\partial U}{\partial x} - e_\alpha\dfrac{\partial \varphi}{\partial x} \\ \Delta u = 4\pi G\displaystyle\sum_\alpha m_\alpha \int n_\alpha(\mu;x,t)d\mu \\ \Delta\varphi = -4\pi\displaystyle\sum_\alpha e_\alpha \int n_\alpha(\mu;x,t)d\mu \end{cases} \tag{4.11}$$

This is the analogue of the chain of Benny, "Benny continuum" for (1.1). Substitution

$$P_\alpha(\mu; x, t) = \frac{\partial S_\alpha}{\partial x}(\mu; x, t)$$

passes, and using the Gromeka form we obtain an analogue of the Hamilton-Jacobi equations similar to (4.1)

$$\begin{cases} \dfrac{\partial n_\alpha(\mu; x, t)}{\partial t} + \dfrac{1}{m_\alpha} \, \textbf{div} \left(n_\alpha(\mu; x, t) \dfrac{\partial S}{\partial x}(\mu; x, t) \right) = 0 \\ \dfrac{\partial S_\alpha(\mu; x, t)}{\partial t} + \dfrac{1}{2m_\alpha} \left(\nabla S_\alpha, \nabla S_\alpha \right) = -m_\alpha U(x) - e_\alpha \varphi(x) \\ \Delta u = 4\pi G \displaystyle\sum_\alpha m_\alpha \int n_\alpha(\mu; x, t) d\mu \\ \Delta \varphi = -4\pi \displaystyle\sum_\alpha e_\alpha \int n_\alpha(\mu; x, t) d\mu \end{cases} \tag{4.12}$$

- We have considered the analogy between the equations of Liouville and Vlasov equation with the mutual enrichment.
- For the Liouville equation, we have a short path to the Hamilton – Jacobi equation using hydrodynamic substitution with generalization to non-Hamiltonian case.
- For the Vlasov equation, we obtain an equation of Hamilton-Jacobi equation for the Vlasov-Poisson equation. (not for the Vlasov-Maxwell).
- In both cases, one can get two commuting Arnold- Kozlov fields.
- It is advisable, to verify (even in the examples of three-dimensional Hamiltonian systems, one, two, three or more Newtonian (Coulomb) attractive centers) the effectiveness of this method.